Math 313 Midterm III KEY
Spring 2010, June 9
section 001
Instructor: Scott Glasgow

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1) Let $S=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ be a basis for a vector space V. It turns out that $S'=\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ is also a basis for V, where
$$\mathbf{u}_1=\mathbf{v}_1,\ \ \mathbf{u}_2=\mathbf{v}_1+\mathbf{v}_2,\ \ \mathbf{u}_3=\mathbf{v}_1+2\mathbf{v}_2+\mathbf{v}_3. \tag{1}$$
So suppose the coordinate vector $(\mathbf{v})_S$ of a vector $\mathbf{v}\in V$ relative to basis S is given by
$$(\mathbf{v})_S=(a,b,c). \tag{2}$$
What is the coordinate vector $(\mathbf{v})_{S'}$ of $\mathbf{v}$ relative to basis S'?

<u>15pts</u>

<u>Solution</u>

By definition of coordinate vector, (2) holds with $S=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ iff
$$\mathbf{v}=a\mathbf{v}_1+b\mathbf{v}_2+c\mathbf{v}_3. \tag{3}$$
(1) is inverted by noting then that
$$\begin{aligned}&\mathbf{v}_1=\mathbf{u}_1,\ \ \mathbf{v}_2=\mathbf{u}_2-\mathbf{v}_1=-\mathbf{u}_1+\mathbf{u}_2,\\&\mathbf{v}_3=\mathbf{u}_3-\mathbf{v}_1-2\mathbf{v}_2=\mathbf{u}_3-\mathbf{u}_1-2(\mathbf{u}_2-\mathbf{u}_1)=\mathbf{u}_1-2\mathbf{u}_2+\mathbf{u}_3\end{aligned} \tag{4}$$
i.e. (1) implies the inverse transformations
$$\mathbf{v}_1=\mathbf{u}_1,\ \ \mathbf{v}_2=-\mathbf{u}_1+\mathbf{u}_2,\ \ \mathbf{v}_3=\mathbf{u}_1-2\mathbf{u}_2+\mathbf{u}_3. \tag{5}$$
Using (5) in (3) we get
$$\begin{aligned}\mathbf{v}&=a\mathbf{v}_1+b\mathbf{v}_2+c\mathbf{v}_3=a\mathbf{u}_1+b(-\mathbf{u}_1+\mathbf{u}_2)+c(\mathbf{u}_1-2\mathbf{u}_2+\mathbf{u}_3)\\&=(a-b+c)\mathbf{u}_1+(b-2c)\mathbf{u}_2+c\mathbf{u}_3.\end{aligned} \tag{6}$$
Thus, by definition,
$$(\mathbf{v})_{S'}=(a-b+c,b-2c,c). \tag{7}$$

2) For the previous problem, what is the transition matrix $P_{SS'}$ ***<u>from</u>*** basis S' ***<u>to</u>*** basis S? What is the transition matrix $P_{S'S}$ ***<u>from</u>*** basis S ***<u>to</u>*** basis S'? Don't mix these two up. Maybe the easiest thing to do here is use the result of the last problem, but re-inventing can be useful as an independent check.

<u>15pts</u>

<u>Solution</u>

By definition, the matrix $P_{SS'}$ ***<u>from</u>*** basis S' ***<u>to</u>*** basis S behaves as
$$(\mathbf{v})_S=P_{SS'}(\mathbf{v})_{S'}, \tag{8}$$

where the coordinate vectors are obviously then column vectors. For problem 1) (8) is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = P_{SS'} \begin{bmatrix} a-b+c \\ b-2c \\ c \end{bmatrix}. \tag{9}$$

But, by relevant transformations, (9) gives,

$$\begin{aligned} P_{SS'} \begin{bmatrix} a-b+c \\ b-2c \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} &\Leftrightarrow P_{SS'} \begin{bmatrix} a \\ b-2c \\ c \end{bmatrix} = \begin{bmatrix} a+b-c \\ b \\ c \end{bmatrix} \\ &\Leftrightarrow P_{SS'} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b+c \\ b+2c \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \end{aligned} \tag{10}$$

By making in turn choices of (a,b,c) corresponding to standard basis elements (of $\mathbb{R}^3$), on finds that we must have then that the columns of $P_{SS'}$ must be in turn the columns of the 3×3 matrix on the right of (10), i.e. $P_{SS'}$ must be that matrix: one finds

$$P_{SS'} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

To find $P_{S'S}$, we note by definition it behaves as

$$(\mathbf{v})_{S'} = P_{S'S} (\mathbf{v})_S, \tag{12}$$

which for problem 1) is evidently

$$P_{S'S} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a-b+c \\ b-2c \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \tag{13}$$

and logic as above dictates

$$P_{S'S} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{14}$$

One checks the matrices of (11) and (14) are inverses, and that

$$\begin{aligned} &P_{SS'} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad P_{SS'} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \\ &P_{SS'} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \end{aligned} \tag{15}$$

which patently agrees with (1). (And this indicates an alternate, perhaps better approach.)

3) Find a basis for the nullspace of A:

$$A=\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}. \tag{16}$$

What is the nullity of A?

15pts

Solution

$$\begin{aligned}
&\mathrm{Nul}\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix} \\
&= \mathrm{Nul}\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \left\{ \left. \begin{pmatrix} -x_3-2x_4-x_5 \\ -x_3-x_4-2x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \right| x_3, x_4, x_5 \in \mathbb{R} \right\} = \left\{ x_3\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5\begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right| \left. x_3, x_4, x_5 \in \mathbb{R} \right\} \\
&=: \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} =: \mathrm{Span}\, S.
\end{aligned} \tag{17}$$

S is a basis for the nullspace, perhaps the most natural one in some respects. Since this vector space is evidently 3 dimensional, the nullity of the matrix A is 3.

4) Determine bases for both the row and column spaces of the matrix A of problem 3). What is the rank of A? For all parts of this question, make sure you write something that is correct regardless of the matrix, i.e. indicate an explanation of why your answer is correct. (There are accidental ways to get parts of this problem right; I am hoping to eliminate accidents.)

15pts

Solution

In the solution of problem 3) we see that the matrix A is row-equivalent to the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{18}$$

So since row-equivalent matrices have the same row space, it follows that

$$\mathrm{Row}A = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}, \tag{19}$$

which indicates a basis for the row space. And since the first two columns of the matrix in (18) are clearly a basis for its column space, then, theorem, the first two columns of A are a basis for the column space of A:

$$\mathrm{Col}A = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \\ 3 \end{pmatrix} \right\}. \tag{20}$$

Since $\mathrm{Range}A = \mathrm{Col}A$ is evidently 2-dimensional, the rank of A is 2. (When we write $\mathrm{Range}A = \mathrm{Col}A$, we are thinking of A as a linear mapping—left multiplication by A is the relevant linear mapping. Of course, theorem, we also have $\dim \mathrm{Col}A = \dim \mathrm{Row}A$, so that $\mathrm{rank}A = \dim \mathrm{Range}A = \dim \mathrm{Col}A = \dim \mathrm{Row}A$.)

5) Let $W = \mathrm{Row}A$, A the matrix of problem 3). W is a subspace of the vector space $V = \mathbb{R}^5$. Find a basis for the orthogonal complement $W^\perp$ of W.

15pts

Solution

Since, theorem, $\mathrm{Row}A$ and $\mathrm{Nul}A$ are orthogonal complements, we have

$$W^\perp = (\mathrm{Row}A)^\perp = \mathrm{Nul}A = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} =: \mathrm{Span}\, S. \tag{21}$$

So S is a basis for $W^\perp = (\mathrm{Row}A)^\perp$. (From (19), one certainly sees that this space is orthogonal to $\mathrm{Row}A$—and vice versa—and that, as per theorem, the dimensions of these spaces add up to that of the ambient space $V = \mathbb{R}^5 = \mathrm{Dom}A$:

$$\mathrm{Rank}A\left(\overset{\text{theorem}}{=} \dim \mathrm{Row}A\right) + \mathrm{Nullity}A\left(\overset{\text{definition}}{=} \dim \mathrm{Nul}A\right) = \#\text{ of columns of } A\left(= \dim \mathrm{Dom}A\right)$$

.)

6) Using the Gram-Schmidt process, find an orthogonal basis for $\mathrm{Col}A$, A the matrix of problem 3). Don't worry about finding an orthonormal basis—I don't care whether or not you normalize. Also, you might consider eliminating fractions to make life easier, as per the discussion in class.

   5 points here are given for CHECKING YOUR ANSWER: if you find your answer is incorrect, but state this, you will at least get 5 points (if you are correct in this assessment of error).

15pts

Solution

Since by (20)

$$\mathrm{Col}A = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \\ 3 \end{pmatrix} \right\} =: \mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2\} \tag{22}$$

we can take the basis to be $S=\{\mathbf{v}_1,\mathbf{v}_2\}$ where $\mathbf{v}_1=\mathbf{u}_1$ and then

$$\mathbf{v}_2 \propto \mathbf{u}_2-\frac{\langle \mathbf{v}_1,\mathbf{u}_2\rangle}{\langle \mathbf{v}_1,\mathbf{v}_1\rangle}\mathbf{v}_1=\mathbf{u}_2-\frac{\langle \mathbf{u}_1,\mathbf{u}_2\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1, \tag{23}$$

for then

$$\langle \mathbf{v}_1,\mathbf{v}_2\rangle \propto \left\langle \mathbf{u}_1,\mathbf{u}_2-\frac{\langle \mathbf{u}_1,\mathbf{u}_2\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1\right\rangle=\langle \mathbf{u}_1,\mathbf{u}_2\rangle-\frac{\langle \mathbf{u}_1,\mathbf{u}_2\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\langle \mathbf{u}_1,\mathbf{u}_1\rangle=0 \tag{24}$$

as desired for any inner product $\langle\ ,\ \rangle$, and we get $\mathbf{v}_2=\mathbf{0}$ from (23) (with nonzero proportionality indicated in (23) by $\propto$) iff

$$\mathbf{u}_2=\frac{\langle \mathbf{u}_1,\mathbf{u}_2\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1\in \text{Span}\{\mathbf{u}_1\}, \tag{25}$$

i.e. iff $\{\mathbf{u}_1,\mathbf{u}_2\}$ wasn't a basis in the first place. (Here I am reminding of why Gram-Schmidt works, at least in the case of a 2-dimensional subspace.) In any event, for the case at hand, we have (23) gives

$$\mathbf{v}_2 \propto \mathbf{u}_2-\frac{\langle \mathbf{u}_1,\mathbf{u}_2\rangle}{\langle \mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1=\mathbf{u}_2-\frac{\mathbf{u}_1\bullet\mathbf{u}_2}{\mathbf{u}_1\bullet\mathbf{u}_1}\mathbf{u}_1=\begin{pmatrix}4\\-2\\0\\3\end{pmatrix}-\frac{\begin{pmatrix}1\\3\\-1\\2\end{pmatrix}\bullet\begin{pmatrix}4\\-2\\0\\3\end{pmatrix}}{\begin{pmatrix}1\\3\\-1\\2\end{pmatrix}\bullet\begin{pmatrix}1\\3\\-1\\2\end{pmatrix}}\begin{pmatrix}1\\3\\-1\\2\end{pmatrix} \tag{26}$$
$$=\begin{pmatrix}4\\-2\\0\\3\end{pmatrix}-\frac{4-6+6}{1+9+1+4}\begin{pmatrix}1\\3\\-1\\2\end{pmatrix}=\begin{pmatrix}4\\-2\\0\\3\end{pmatrix}-\frac{4}{15}\begin{pmatrix}1\\3\\-1\\2\end{pmatrix}\propto 15\begin{pmatrix}4\\-2\\0\\3\end{pmatrix}-4\begin{pmatrix}1\\3\\-1\\2\end{pmatrix}=\begin{pmatrix}56\\-42\\4\\37\end{pmatrix}=:\mathbf{v}_2,$$

so that an orthogonal basis is

$$S=\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{pmatrix}1\\3\\-1\\2\end{pmatrix},\begin{pmatrix}56\\-42\\4\\37\end{pmatrix}\right\}. \tag{27}$$

One could check that

$$\mathbf{v}_1 \bullet \mathbf{v}_2 = 56 - 126 - 4 + 74 = 0 \tag{28}$$

and that

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 56 \\ 3 & -2 & 3 & -42 \\ -1 & 0 & -1 & 4 \\ 2 & 3 & 2 & 37 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4 & 1 & 56 \\ 0 & -14 & 0 & -210 \\ 0 & 4 & 0 & 60 \\ 0 & -5 & 0 & -75 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4 & 1 & 56 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4 & 1 & 56 \\ 0 & -4 & 0 & -60 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{29}$$

which confirms that each element of the new basis is in the span of the old.

7) Verify that

$$S = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ -2 \end{pmatrix} \right\} \tag{30}$$

is an orthogonal basis for a certain 2-dimensional subspace of $\mathbb{R}^4$ (namely $\mathrm{Span}S$). ***<u>Using this fact</u>***, find the coordinate vector $(\mathbf{w})_S$ of the vector $\mathbf{w} \in \mathrm{Span}S$, where

$$\mathbf{w} = \begin{pmatrix} 1 \\ 5 \\ -9 \\ 10 \end{pmatrix} \tag{31}$$

<u>15pts</u>

Solution

The basis is orthogonal since the Euclidean inner product of the two vectors is $1+6-3-4=0$. (And the set is clearly linearly independent by all accounts, hence a basis for its span.) By theorem then we have

$$\mathbf{w}=\frac{\langle \mathbf{v}_1,\mathbf{w}\rangle}{\langle \mathbf{v}_1,\mathbf{v}_1\rangle}\mathbf{v}_1+\frac{\langle \mathbf{v}_2,\mathbf{w}\rangle}{\langle \mathbf{v}_2,\mathbf{v}_2\rangle}\mathbf{v}_2 \qquad (32)$$

so that

$$(\mathbf{w})_S=\left(\frac{\langle \mathbf{v}_1,\mathbf{w}\rangle}{\langle \mathbf{v}_1,\mathbf{v}_1\rangle},\frac{\langle \mathbf{v}_2,\mathbf{w}\rangle}{\langle \mathbf{v}_2,\mathbf{v}_2\rangle}\right)^T=\left(\frac{\mathbf{v}_1\bullet\mathbf{w}}{\mathbf{v}_1\bullet\mathbf{v}_1},\frac{\mathbf{v}_2\bullet\mathbf{w}}{\mathbf{v}_2\bullet\mathbf{v}_2}\right)^T=\left(\frac{\begin{pmatrix}1\\3\\-1\\2\end{pmatrix}\bullet\begin{pmatrix}1\\5\\-9\\10\end{pmatrix}}{\begin{pmatrix}1\\3\\-1\\2\end{pmatrix}\bullet\begin{pmatrix}1\\3\\-1\\2\end{pmatrix}},\frac{\begin{pmatrix}1\\2\\3\\-2\end{pmatrix}\bullet\begin{pmatrix}1\\5\\-9\\10\end{pmatrix}}{\begin{pmatrix}1\\2\\3\\-2\end{pmatrix}\bullet\begin{pmatrix}1\\2\\3\\-2\end{pmatrix}}\right)^T \qquad (33)$$

$$=\left(\frac{1+15+9+20}{1+9+1+4},\frac{1+10-27-20}{1+4+9+4}\right)^T=\left(\frac{45}{15},\frac{-36}{18}\right)^T=(3,-2)^T.$$

8) Find the least squares solution $\mathbf{x}$ to the inconsistent system $A\mathbf{x}=\mathbf{b}$, where

$$A=\begin{bmatrix}1&1\\3&2\\-1&3\\2&-2\end{bmatrix},\ \mathbf{b}=\begin{pmatrix}0\\5\\-8\\11\end{pmatrix}. \qquad (34)$$

15pts

Solution

The least squares system $A^TA\mathbf{x}=A^T\mathbf{b}$ is gotten by noting that

$$A^T A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 1 & 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1+9+1+4 & 1+6-3-4 \\ 1+6-3-4 & 1+4+9+4 \end{bmatrix} = \begin{bmatrix} 15 & 0 \\ 0 & 18 \end{bmatrix},$$
$$A^T \mathbf{b} = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 1 & 2 & 3 & -2 \end{bmatrix} \begin{pmatrix} 0 \\ 5 \\ -8 \\ 11 \end{pmatrix} = \begin{bmatrix} 15+8+22 \\ 10-24-22 \end{bmatrix} = \begin{bmatrix} 45 \\ -36 \end{bmatrix}, \tag{35}$$

so that clearly the least squares solution is given by

$$\mathbf{x} = \begin{bmatrix} 1/15 & 0 \\ 0 & 1/18 \end{bmatrix} \begin{bmatrix} 45 \\ -36 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}. \tag{36}$$

9) $\langle , \rangle$ defined by

$$\langle p, q \rangle = \int_0^2 p(x)q(x)dx \tag{37}$$

is an inner product on $P_2 = \mathrm{Span} S = \mathrm{Span}\{1, x, x^2\} =: \mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where S is the standard basis for P_2. Find a basis for P_2 that is orthogonal with respect to $\langle , \rangle$ by applying Gram-Schmidt to the standard basis. 5 points are allocated for cheching that the polynomials you generate are orthogonal. (If they're not, go back and fix them.) Avoid fractions!

<u>15pts</u>

<u>Solution</u>

We get an orthogonal basis $S' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ by choosing

$$\mathbf{v}_1 = \mathbf{u}_1 = 1,$$

$$\mathbf{v}_2 \propto \mathbf{u}_2 - \frac{\langle \mathbf{v}_1, \mathbf{u}_2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 = x - \frac{\langle 1, x \rangle}{\langle 1, 1 \rangle} 1 = x - \frac{\int_0^2 1 \cdot x dx}{\int_0^2 1 \cdot 1 dx} 1 = x - \frac{2}{2} 1 = x - 1 =: \mathbf{v}_2,$$

$$\mathbf{v}_3 \propto \mathbf{u}_3 - \frac{\langle \mathbf{v}_1, \mathbf{u}_3 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{v}_2, \mathbf{u}_3 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 = x^2 - \frac{\langle 1, x^2 \rangle}{\langle 1, 1 \rangle} 1 - \frac{\langle x-1, x^2 \rangle}{\langle x-1, x-1 \rangle} (x-1)$$

$$= x^2 - \frac{\int_0^2 1 \cdot x^2 dx}{\int_0^2 1 \cdot 1 dx} 1 - \frac{\int_0^2 (x-1) \cdot x^2 dx}{\int_0^2 (x-1)^2 dx} (x-1) = x^2 - \frac{\frac{8}{3}}{2} 1 - \frac{\frac{16}{4} - \frac{8}{3}}{\left. \frac{(x-1)^3}{3} \right|_0^2} (x-1)$$

$$= x^2 - \frac{4}{3} - \frac{\frac{4}{3}}{\frac{2}{3}} (x-1) = x^2 - \frac{4}{3} - 2(x-1) \propto 3x^2 - 6x + 2 =: \mathbf{v}_3. \tag{38}$$

So an orthogonal basis is $S' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{1, x-1, 3x^2 - 6x + 2\}$.

10) Find the linear function $q(x)$ closest to the quadratic polynomial $p(x) = 3x^2 - 3x + 4$, where the notion of distance is given (indirectly) by the innerproduct in (37), i.e. where the notion of distance is given (directly) by

$$d^2(p, q) = \int_0^2 (p(x) - q(x))^2 \, dx. \tag{39}$$

15pts

Solution

Since $p(x) = 3x^2 - 3x + 4 = 3x + 2 + 1 \cdot (3x^2 - 6x + 2)$, and since $\mathbf{v}_3 := 3x^2 - 6x + 2 \in (\mathrm{Span}\{1, x\})^\perp = (\mathrm{Span}\{1, x-1\})^\perp$ with respect to innerproduct (37), the answer is, theorem,

$$q(x) = 3x + 2. \tag{40}$$

Alternatively, we have, theorem (from the orthogonal basis of the previous problem)

$$
\begin{aligned}
q(x) &= \frac{\langle \mathbf{v}_1, p\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1(x) + \frac{\langle \mathbf{v}_2, p\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\mathbf{v}_2(x) = \frac{\langle 1, 3x^2-3x+4\rangle}{2}\cdot 1 + \frac{\langle x-1, 3x^2-3x+4\rangle}{\frac{2}{3}}\cdot(x-1) \\
&= \frac{1}{2}\int_0^2 \left(3x^2-3x+4\right)dx + \frac{3}{2}\left(\int_0^2 (x-1)\left(3x^2-3x+4\right)dx\right)(x-1) \\
&= \frac{1}{2}\left(8-\frac{3}{2}\cdot 4+8\right) + \frac{3}{2}\left(\int_0^2 \left(3x^3-6x^2+7x-4\right)dx\right)(x-1) \qquad (41) \\
&= 5+\frac{3}{2}\left(\frac{3}{4}\cdot 16-2\cdot 8+\frac{7}{2}\cdot 4-8\right)(x-1) \\
&= 5+3(x-1) = 3x+2,
\end{aligned}
$$

as advertised.