Math 313 Final KEY

Winter 2011

section 003

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Serious Instructions: Write your name very clearly on this exam. In this booklet, write your mathematics clearly, legibly, in big fonts, and, most important, "have a point", i.e. make your work logically and even pedagogically acceptable. (Other human beings not already understanding 313 should be able to learn from your exam.) To avoid excessive erasing, first put your ideas together on scratch paper, then commit the logically acceptable fraction of your scratchings to this exam booklet. More is not necessarily better: say what you mean and mean what you say.

Instructions for those who want their psychology to be optimal for an assessment[1]: a) you should communicate in complete sentences, 2) you should write on your own paper and d) you should be neat as possible.

NOTE: Almost none of the problems below are worth 25 points. That's funny.

[1] In case of an overview of this document by an administrator, my students have learned of research indicating that humorous instruction may increase capacity on exams. This claim is similar to the following: "Three grams of soluble fiber daily from whole grain oat foods, like Honey Nut Cheerios, in a diet low in saturated fat and cholesterol, may reduce the risk of heart disease." So there.

1. Put the following matrix in reduced row-echelon form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 4 & -2 & 1 & 4 \end{bmatrix}. \tag{1}$$

15pts

Solution

The row reduction may proceed as follows:

$$\begin{aligned} &\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 4 & -2 & 1 & 4 \end{bmatrix}^{\substack{R2-R1 \\ R3-4R1}} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & -6 & -3 & 0 \end{bmatrix}^{\substack{R2/(-2) \\ R3/(-3)}} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}^{R3-2R2} \\ &\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}^{R1-R3} \sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}^{R1-R2} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}. \end{aligned} \tag{2}$$

In whichever way the row reduction proceeds, the row echelon form is unique—the last matrix indicated in (2) is ***the*** answer.

2. Solve the following system of 3 equations and 3 unknowns by performing ***Gauss-Jordan*** elimination on the relevant augmented matrix. 2 points for doing this, ***3*** points for checking you answer works.

$$\begin{aligned} x+y+z &= 1 \\ x-y+z &= -1 \\ 4x-2y+z &= 4 \end{aligned} \tag{3}$$

15pts

Solution

The augmented matrix is the matrix in (1), Gauss-Jordan elimination giving the last matrix in (2), which is code for the equations/statement of the solution

$$\begin{aligned} x \quad\quad &= 2 \\ y \quad &= 1 \\ z &= -2 \end{aligned} \tag{4}$$

Check: Clearly

$$\begin{aligned} 2+1+(-2) &= \ \ 1, \\ 2-1+(-2) &= -1, \text{ and} \\ 4\cdot 2-2\cdot 1+(-2) &= \ \ 4, \end{aligned} \tag{5}$$

so our answer indicated in (4) solves (3).

3. Determine whether

$$\mathbf{x} \in Span\{\mathbf{u},\mathbf{v},\mathbf{w}\} \tag{6}$$

where

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}. \tag{7}$$

Of course explain why or why not for full credit.

<u>15pts</u>

<u>Solution</u>

In problem 2) we showed that

$$\mathbf{x} = 2\mathbf{u} + 1\mathbf{v} - 2\mathbf{w} \in Span\{\mathbf{u},\mathbf{v},\mathbf{w}\}, \tag{8}$$

that is

$$\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{9}$$

so in fact (6) holds.

4. Find the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{10}$$

by row reducing $\left[A|I\right]$ to $\left[I|A^{-1}\right]$. Assume the parameters $a, b, c,$ and d do not take on any special values, nor have a special relationship among them, i.e., row reduce naively, without worrying about any divisions by hidden zeros.

<u>15 pts</u>

<u>Solution</u>

The naïve row reduction mentioned may proceed as follows:

$$\begin{aligned}
\left[A|I\right] &= \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]^{aR2-cR1} \sim \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right]^{(ad-bc)R1-bR2} \\
&\sim \left[\begin{array}{cc|cc} a(ad-bc) & 0 & ad & -ab \\ 0 & ad-bc & -c & a \end{array}\right]^{R1/a} \sim \left[\begin{array}{cc|cc} ad-bc & 0 & d & -b \\ 0 & ad-bc & -c & a \end{array}\right]^{R1/a} \\
&\sim \left[\begin{array}{cc|c} 1 & 0 & \multirow{2}{*}{$\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$} \\ 0 & 1 & \end{array}\right]^{R1/a} = \left[I|A^{-1}\right].
\end{aligned} \tag{11}$$

5. For the given set of objects, together with the indicated notions of addition $\oplus$ and scalar multiplication $\odot$, determine whether each of the ten vector space axioms holds: the set of objects is real triples (x, y, z), where

$$(x, y, z) \oplus (x', y', z') := (x + x', y + y', z + z'), \quad k \odot (x, y, z) := (kx, y, z). \tag{12}$$

Recall that for vector space $(V, \mathbb{R}, \oplus, \odot)$ the ten axioms are

$$
\begin{array}{ll}
1. & \forall \mathbf{u},\mathbf{v}\in V,\ \mathbf{u}\oplus\mathbf{v}\in V \\
2. & \forall \mathbf{u},\mathbf{v}\in V,\ \mathbf{u}\oplus\mathbf{v}=\mathbf{v}\oplus\mathbf{u} \\
3. & \forall \mathbf{u},\mathbf{v},\mathbf{w}\in V,\ \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})=(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} \\
4. & \exists \mathbf{z}\in V \text{ s.t } \forall \mathbf{u}\in V,\ \mathbf{u}\oplus\mathbf{z}=\mathbf{u} \\
5. & \forall \mathbf{u}\in V\ \exists \mathbf{u}_{-}\in V \text{ s.t } \mathbf{u}\oplus\mathbf{u}_{-}=\mathbf{z} \\
6. & \forall k\in\mathbb{R},\forall \mathbf{u}\in V,\ k\odot\mathbf{u}\in V \\
7. & \forall k\in\mathbb{R},\forall \mathbf{u},\mathbf{v}\in V,\ k\odot(\mathbf{u}\oplus\mathbf{v})=(k\odot\mathbf{u})\oplus(k\odot\mathbf{v}) \\
8. & \forall k,m\in\mathbb{R},\forall \mathbf{u}\in V,\ (k+m)\odot\mathbf{u}=(k\odot\mathbf{u})\oplus(m\odot\mathbf{u}) \\
9. & \forall k,m\in\mathbb{R},\forall \mathbf{u}\in V,\ k\odot(m\odot\mathbf{u})=(km)\odot\mathbf{u} \\
10. & \forall \mathbf{u}\in V,\ 1\odot\mathbf{u}=\mathbf{u}
\end{array}
\tag{13}
$$

20pts

Solution

Closure axioms 1) and 6) hold because sums and products of real numbers give real numbers, and because on the right hand sides of equation (12) the objects are again triples of those real numbers. 2) through 5) will also hold, since they reference only vector addition, which in (12) is the standard notion (giving the relevant fraction of the 10 axioms as *theorems*). For axiom 7) we have

$$
\begin{aligned}
k\odot\big((x,y,z)\oplus(x',y',z')\big) &= k\odot(x+x',y+y',z+z')=\big(k(x+x'),y+y',z+z'\big) \\
&=(kx+kx',y+y',z+z')=(kx,y,z)\oplus(kx',y',z') \\
&=\big(k\odot(x,y,z)\big)\oplus\big(k\odot(x',y',z')\big),
\end{aligned}
\tag{14}
$$

as required. But for axiom 8) we have the generally distinct results

$$
\begin{aligned}
(k+m)\odot(x,y,z)&=\big((k+m)x,y,z\big)=(kx+mx,y,z),\ \text{and} \\
\big(k\odot(x,y,z)\big)\oplus\big(m\odot(x,y,z)\big)&=(kx,y,z)\oplus(mx,y,z)=(kx+mx,y+y,z+z) \\
&=(kx+mx,2y,2z),
\end{aligned}
\tag{15}
$$

so that this axiom does not hold. Axioms 9) and 10) hold in an obvious way—essentially because the scalar multiplication is normal in the one slot it affects.

6. Consider the following set S of vectors in $\mathbb{R}^4$. Explain why S is linearly dependent without doing any calculations. Next, give a basis for the subspace $W=\operatorname{Span} S$ and use this basis for W to express one of the vectors in S as a linear

combination of ***others*** in S. (No fair saying a vector is 1 times itself.) Finally, what is the dimension of $W = \mathrm{Span}\ S$?

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\} \tag{16}$$

You may use the fact that

$$A := \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 3 & 3 & 1 \\ 2 & 2 & 4 & 2 & 2 \\ 0 & 1 & -1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 3 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} =: B \tag{17}$$

where the tilde ($\sim$) indicates “row equivalent to”.

<u>20pts</u>

<u>Solution</u>

S is 5 vectors from $\mathbb{R}^4$, so since $5 > 4 = \dim \mathbb{R}^4$, theorem, S is dependent. Next, since B is in reduced row echelon form, its pivot columns (clearly) define a basis for its column space, all other columns (clearly) linear combinations then of these special columns. These pivot columns are its first, second and fifth. And since, theorem, row reduction does not alter the linear relationships among columns of a matrix, the associated columns of A are a basis for the column space of A: the first, second and fifth columns of A give a basis for the column space of A. So since these columns are the first, second and fifth elements of S, it must be that the first, second and fifth elements of S are a basis for $W = \mathrm{Span}\ S$: a basis for $W = \mathrm{Span}\ S$ is the set

$$S' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}. \tag{18}$$

Here we see then that $\dim W = \dim \mathrm{Span}\ S = \dim \mathrm{Span}\ S' = |S'| = 3$, which answers the last question.

With the theory just presented, we have that since

$$\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5\end{bmatrix} := \begin{bmatrix} 1 & 0 & 3 & 3 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

gives

$$\mathbf{b}_3 = (3)\mathbf{b}_1 + (-1)\mathbf{b}_2 = 3\mathbf{b}_1 - \mathbf{b}_2, \quad \mathbf{b}_4 = (3)\mathbf{b}_1 + (-2)\mathbf{b}_2 = 3\mathbf{b}_1 - 2\mathbf{b}_2, \tag{20}$$

it must be that

$$\begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \tag{21}$$

either one of which two statements answering then the second question.

7. Determine bases for both the image and kernel of (left multiplication by) A, where

$$A := \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 3 & 3 & 1 \\ 2 & 2 & 4 & 2 & 2 \\ 0 & 1 & -1 & -2 & 1 \end{bmatrix}. \tag{22}$$

<u>20 pts</u>

<u>Solution</u>

Since, by definition of matrix multiplication (from the left),

$$\begin{aligned} A\mathbf{x} &= \left[\mathbf{c}_1 \middle| \ldots \middle| \mathbf{c}_n\right]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{c}_1 + \ldots x_n\mathbf{c}_n \in \operatorname{Span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\} \\ &:= \{x_1\mathbf{c}_1 + \ldots x_n\mathbf{c}_n : x_1, \ldots, x_n \in \mathbb{R}\}, \end{aligned} \tag{23}$$

then clearly the image of $\mathbf{x} \mapsto A\mathbf{x}$ is the column space of A. So from the previous problem we have that a basis for the image of (left multiplication by) A is

$$S' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}. \tag{24}$$

Now the kernel of (left multiplication by) A is unaltered by row reduction (row reduction does not change the solution space of a system of equations—which is why we use it to solve them), so the kernel of A is the kernel of B in (17), the latter exposed by realizing that the structure of B there dictates that for $\mathbf{x} = (x_1, \dots, x_5)^T \in \ker B$ we have

$$\begin{aligned} x_1 + 3x_3 + 3x_4 = 0 = x_2 - x_3 - 2x_4 = 0 = x_5 \\ \Leftrightarrow \\ x_1 = -3x_3 - 3x_4 \\ x_2 = x_3 + 2x_4 \\ x_5 = 0, \end{aligned} \tag{25}$$

i.e.,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3x_3 - 3x_4 \\ x_3 + 2x_4 \\ x_3 \\ x_4 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in \operatorname{Span} \left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} =: \operatorname{Span} S'', \tag{26}$$

where then S'', which is clearly linearly independent, is a basis for the kernel A.

8. Let $T : V \to W$ be linear, V and W finite dimensional vector spaces. Recall

$$\operatorname{Im} T := \{ \mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V \} \tag{27}$$

is a subspace of W (hence nonempty, closed under linear combination). Assuming $\operatorname{Im} T \neq \{\mathbf{z}_W\}$ ($\mathbf{z}_W$ denotes the additive identity in W), we get $\dim \operatorname{Im} T \geq 1$ (and $\dim \operatorname{Im} T \leq \dim W < \infty$), and by previous theorem get that $\operatorname{Im} T$ has a basis $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ with, as indicated, $m \geq 1$ (and $m \leq \dim W < \infty$). (Note $m \geq 1$ means here that the set is not empty.) Since each of these $\mathbf{w}$'s is in the image of T, as per (27), each one of them is a "T of something": there exist $\mathbf{v}_1, \dots, \mathbf{v}_m \in V$, *necessarily distinct*, such that $(\mathbf{w}_1, \dots, \mathbf{w}_m) = (T(\mathbf{v}_1), \dots, T(\mathbf{v}_m))$. So $\{\mathbf{w}_1, \dots, \mathbf{w}_m\} = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_m)\}$ and

$\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_m)\}$ is a basis for $\operatorname{Im}T$, hence linearly independent, etc. Show that the set $\{\mathbf{v}_1,\ldots,\mathbf{v}_m\}$ is also linearly independent.

15pts

Solution

Since a basis $\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_m)\}$ is independent, we have

$$k_1T(\mathbf{v}_1)+\ldots+k_mT(\mathbf{v}_m)=\mathbf{z}_W \Rightarrow k_1=\ldots=k_m=0. \tag{28}$$

We are hoping that this implication implies the implication

$$k_1\mathbf{v}_1+\ldots+k_m\mathbf{v}_m=\mathbf{z}_V \Rightarrow k_1=\ldots=k_m=0 \tag{29}$$

so that $\{\mathbf{v}_1,\ldots,\mathbf{v}_m\}$ is also independent. So we start on the left-hand side of (29) and see if we can pass to the right hand side of (29) using (28) and the linearity of T. This is no problem:

$$\begin{aligned} k_1\mathbf{v}_1+\ldots+k_m\mathbf{v}_m=\mathbf{z}_V &\overset{\substack{\text{do the same thing to}\\ \text{both sides of an equation}}}{\Rightarrow} T(k_1\mathbf{v}_1+\ldots+k_m\mathbf{v}_m)=T(\mathbf{z}_V)\overset{T\text{ linear}}{=}\mathbf{z}_W \\ &\overset{T\text{ linear}}{\Leftrightarrow} k_1T(\mathbf{v}_1)+\ldots+k_mT(\mathbf{v}_m)=\mathbf{z}_W \\ &\Rightarrow k_1=\ldots=k_m=0 \end{aligned} \tag{30}$$

the last step by the given independence statement (28).

9. As above assume $T:V\to W$ is linear, V and W finite dimensional vector spaces. Recall

$$\ker T:=\{\mathbf{v}\in V:T(\mathbf{v})=\mathbf{z}_W\} \tag{31}$$

is a subspace of V, hence $\dim\ker T\le\dim V<\infty$, i.e., the kernel of T is finite dimensional (with dimension no bigger than that of V). Now, similar to the last problem, assume $\ker T\ne\{\mathbf{z}_V\}$ ($\mathbf{z}_V$ denotes the additive identity in V), so that there is a basis for $\ker T$: assume a basis $\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ for the kernel of T, with, as indicated, $n\ge1$ (and $n\le\dim V<\infty$). (Note $n\ge1$ means here that the set is not empty.) Forgetting these kernel ideas for a moment, and using the basis $\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_m)\}$ of $\operatorname{Im}T$ introduced in the previous problem, we see that for every $\mathbf{v}\in V$ there are scalars $a_1,\ldots,a_m$ such that

$$T(\mathbf{v}) = a_1 T(\mathbf{v}_1) + \ldots + a_m T(\mathbf{v}_m). \tag{32}$$

Using T's linearity and the vector space axioms we see that (32) is equivalent to

$$T\left(\mathbf{v} - (a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m)\right) = \mathbf{z}_W. \tag{33}$$

So then it must be that

$$V = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_n\}. \tag{34}$$

Why? Because (33) says that

$$\mathbf{v} - (a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m) \in \ker T := \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{z}_W\}, \tag{35}$$

and since we're assuming that $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for $\ker T$, we have $\ker T = \operatorname{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$. So (35) (and previous statements) says that for any $\mathbf{v} \in V$ there are scalars $a_1, \ldots, a_m$ (giving (32)) and scalars $c_1, \ldots, c_n$ such that

$$\mathbf{v} - (a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m) = c_1\mathbf{u}_1 + \ldots + c_n\mathbf{u}_n, \tag{36}$$

i.e., using the vector space axioms (allowing for algebra to be performed), for any $\mathbf{v} \in V$ there are scalars $a_1, \ldots, a_m$ and scalars $c_1, \ldots, c_n$ such that

$$\mathbf{v} = a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m + c_1\mathbf{u}_1 + \ldots + c_n\mathbf{u}_n \in \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_n\}. \tag{37}$$

Consequently $V \subset \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_n\}$. We then easily get $\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_n\} \subset V$, so that $V \subset \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_n\} \subset V \Leftrightarrow V = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_n\}$, since a) $\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_n \in V$ and since b) V is closed under linear combination. Cool. We've just established (34). Now I leave you to show that i) $\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_n\}$ in (34) is ***linearly independent***, hence leave you to show that ii)

$$\dim V = \dim \operatorname{Im} T + \dim \ker T. \tag{38}$$

[(38) is the "dimension theorem", which can be used in subsequent problems. It works even if one or more of the indicated dimensions are zero, which we precluded in deriving it. It even works if $\dim V = 0$. (In that particular case, since $\dim \operatorname{Im} T, \dim \ker T \geq 0$, we must have $\dim \operatorname{Im} T = \dim \ker T = 0 = \dim V$, the former statement giving that T is the zero map—it kills "everything"—and this despite the fact that the second statement says that it *only* kills zero.) You can thank me later for handing this theorem to you—instead of asking that you remember it.]

20pts

Solution

We first want to show that $\{\mathbf{v}_1,\ldots,\mathbf{v}_m,\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ is linearly independent, i.e., show that

$$a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V\Rightarrow a_1=\ldots=a_m=c_1=\ldots=c_n=0. \qquad (39)$$

So we start on the left hand side of (39) and try to find a path to the right hand side, given that a) $\{\mathbf{v}_1,\ldots,\mathbf{v}_m\}$ is linearly independent, b) $\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ is linearly independent and spans the kernel of T, and c) that $\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_m)\}$ is linearly independent and spans the image of T. Perhaps we will only use some of these facts at this late stage. Let's see.

Applying linear $T:V\to W$ to each side of the vector equation in (39) we get

$$\begin{gathered}a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V\Rightarrow\\ a_1T(\mathbf{v}_1)+\ldots+a_mT(\mathbf{v}_m)+c_1T(\mathbf{u}_1)+\ldots+c_nT(\mathbf{u}_n)=T(\mathbf{z}_V)=\mathbf{z}_W\end{gathered} \qquad (40)$$

Then using that each of the $\mathbf{u}$'s is in the kernel of T, that linear combinations of the additive identity $\mathbf{z}_W$ give $\mathbf{z}_W$, which "does nothing" to anything in W, we get (40) becomes

$$\begin{gathered}a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V\Rightarrow\\ a_1T(\mathbf{v}_1)+\ldots+a_mT(\mathbf{v}_m)=\mathbf{z}_W.\end{gathered} \qquad (41)$$

But since $\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_m)\}$ is linearly independent, we have

$$\begin{gathered}a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V\Rightarrow\\ a_1T(\mathbf{v}_1)+\ldots+a_mT(\mathbf{v}_m)=\mathbf{z}_W\Rightarrow a_1=\ldots=a_m=0,\end{gathered} \qquad (42)$$

i.e., in short,

$$a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V\Rightarrow a_1=\ldots=a_m=0. \qquad (43)$$

But since a statement implies itself, we could also write this as

$$
\begin{aligned}
&a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V \Rightarrow \\
&a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V, a_1=\ldots=a_m=0 \Rightarrow \\
&c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V, a_1=\ldots=a_m=0,
\end{aligned}
\tag{44}
$$

the last implication following by the fact that the zero multiple of anything in V is $\mathbf{z}_V$, that sums of $\mathbf{z}_V$ give $\mathbf{z}_V$, and/or that $\mathbf{z}_V$ does nothing to anything in V. But then since $\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ is independent, we then have

$$
\begin{aligned}
a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V \Rightarrow c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V, a_1=\ldots=a_m=0 \\
\Rightarrow a_1=\ldots=a_m=0=c_1=\ldots=c_m,
\end{aligned}
\tag{45}
$$

i.e.,

$$
a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V \Rightarrow a_1=\ldots=a_m=0=c_1=\ldots=c_m \tag{46}
$$

which is the same as (39).

So now with $\{\mathbf{v}_1,\ldots,\mathbf{v}_m,\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ linearly independent (which implies all nonempty subsets are independent, including the important ones we've thought about recently) and with (34), we have

$$
\begin{aligned}
\dim V &= \dim \operatorname{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_m,\mathbf{u}_1,\ldots,\mathbf{u}_n\} = \left|\{\mathbf{v}_1,\ldots,\mathbf{v}_m,\mathbf{u}_1,\ldots,\mathbf{u}_n\}\right| = m+n \\
&= \left|\{\mathbf{v}_1,\ldots,\mathbf{v}_m\}\right| + \left|\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}\right| = \left|\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_m)\}\right| + \left|\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}\right| \\
&= \dim \operatorname{Span}\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_m)\} + \dim \operatorname{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_n\} \\
&= \dim \operatorname{Im} T + \dim \ker T,
\end{aligned}
\tag{47}
$$

which is the desired (38), i.e. the dimension theorem. In (47) we also used $\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_m)\}$ is linearly independent.

10. Explain why it is that if A is a (real) $m \times n$ matrix with $n > m$, then the kernel of A, i.e.

$$
\ker A := \left\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0} \in \mathbb{R}^m\right\}, \tag{48}
$$

can't just be only the zero vector. (That is, show that $\ker A \neq \left\{\mathbf{0} \in \mathbb{R}^n\right\}$). Hint: use the dimension theorem. (Aren't you glad I reminded you of that?)

15pts

Solution

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ of the previous problem be defined by $T(\mathbf{x}) = A\mathbf{x}$, so that T is a linear map from a finite dimensional vector space to another one, and so that $\ker A = \ker T$. The dimension theorem (38) in this context says that

$$n = \dim V = \dim \operatorname{Im} T + \dim \ker T = \dim \operatorname{Im} T + \dim \ker A \le m + \dim \ker A. \tag{49}$$

Here we also used that $\operatorname{Im} T$ is a subspace of $\mathbb{R}^m$, so that $\dim \operatorname{Im} T \le \dim \mathbb{R}^m = m$. So then with $n > m \Leftrightarrow n - m \ge 1$ [since these are (nonnegative) integers], from (49) we have

$$\dim \ker A \ge n - m \ge 1 \Rightarrow \dim \ker A \ge 1 \tag{50}$$

and $\ker A$ can't be $\{\mathbf{0} \in \mathbb{R}^n\}$ (which we say has dimension 0).

11. The eigenvalues of

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & 2 \\ -1 & -3 & 5 \end{bmatrix} \tag{51}$$

are $\lambda = 1, 2,$ and 3. Compute the eigenspaces associated to each of these eigenvalues. (Recall eigenspaces are subspaces, hence specified as the span of a basis.)

15pts

Solution

We have

$$E_1(A) = \ker(1I - A) = \ker\begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & -2 \\ 1 & 3 & -4 \end{bmatrix} = \ker\begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & -2 \\ 0 & 2 & -2 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\},$$
$$E_2(A) = \ker(2I - A) = \ker\begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & -2 \\ 1 & 3 & -3 \end{bmatrix} = \ker\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\},$$
$$E_3(A) = \ker(3I - A) = \ker\begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -2 \\ 1 & 3 & -2 \end{bmatrix} = \ker\begin{bmatrix} 2 & 2 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} == \ker\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right\}. \tag{52}$$

12. For k any positive integer, compute

$$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & 2 \\ -1 & -3 & 5 \end{bmatrix}^k . \tag{53}$$

You may use that

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}, \tag{54}$$

i.e., you may use that the two matrices in (54) are row equivalent.

<u>15pts</u>

<u>Solution</u>

If square matrix A is diagonalizable, then we can write

$$\begin{aligned} A^k &= \left(SDS^{-1}\right)^k = SD^kS^{-1} \\ &= S\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}^k S^{-1} = S\begin{bmatrix} \lambda_1{}^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n{}^k \end{bmatrix} S^{-1} \end{aligned} \tag{55}$$

where S's columns are eigenvectors of A. A will certainly be diagonalizable if all the eigenvalues $\lambda_1,\ldots,\lambda_n$ are distinct, which, by the previous problem, is what occurs for the matrix of (53). And by the results of that previous problem, we can take

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \Leftrightarrow S^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \tag{56}$$

where we used (54). So then by (55) we have

$$\begin{aligned} A^k &= S\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}^k S^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 3^k \\ 1 & 2^k & 3^k \\ 1 & 2^k & 2\cdot 3^k \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1-3^k & 3^k-1 \\ 1-2^k & 1+2^k-3^k & 3^k-1 \\ 1-2^k & 1+2^k-2\cdot 3^k & 2\cdot 3^k-1 \end{bmatrix}. \end{aligned} \tag{57}$$

13. Prove that for a real $m\times n$ matrix A, we must have $\ker A^TA = \ker A$. To do this one shows that $\ker A^TA \supset \ker A$ (easy), and then shows that $\ker A^TA \subset \ker A$ (harder). For the easy first part, one simply notes that the definitions

$$\begin{aligned} \ker A &:= \left\{\mathbf{x}\in\mathbb{R}^n : A\mathbf{x} = \mathbf{0}_m \in \mathbb{R}^m\right\} \\ \ker A^TA &:= \left\{\mathbf{x}\in\mathbb{R}^n : \left(A^TA\right)\mathbf{x} = \mathbf{0}_n \in \mathbb{R}^n\right\} \end{aligned} \tag{58}$$

make it so that

$$\mathbf{x} \in \ker A \Rightarrow \mathbf{x} \in \ker A^T A, \tag{59}$$

giving $\ker A \subset \ker A^T A$, and this because clearly

$$A\mathbf{x} = \mathbf{0}_m \Rightarrow \left(A^T A\right)\mathbf{x} = A^T\left(A\mathbf{x}\right) = A^T\left(\mathbf{0}_m\right) = \mathbf{0}_n. \tag{60}$$

For the harder part $\ker A^T A \subset \ker A$, we want to somehow get that

$$\mathbf{x} \in \ker A^T A \Rightarrow \mathbf{x} \in \ker A, \tag{61}$$

i.e., we need to somehow show the converse of (60), i.e., we must somehow show that

$$\left(A^T A\right)\mathbf{x} = \mathbf{0}_n \Rightarrow A\mathbf{x} = \mathbf{0}_m. \tag{62}$$

So I leave it to you to prove (62), i.e., I leave it to you to show that you can start on the left of (62) and find a way to get to the right of (62). Hint: note that

$$\left(A^T A\right)\mathbf{x} = \mathbf{0}_n \Rightarrow \mathbf{x}\cdot\left(A^T A\right)\mathbf{x} = \mathbf{x}\cdot\mathbf{0}_n = 0, \tag{63}$$

and then note that

$$\mathbf{x}\cdot\left(A^T A\right)\mathbf{x} = \mathbf{x}^T\left(\left(A^T A\right)\mathbf{x}\right) = \left(\mathbf{x}^T A^T\right)\left(A\mathbf{x}\right) = ... \tag{64}$$

(If I go any further here I will have proved the whole thing for you.)

15pts

Solution

Following the hint, we find

$$\begin{aligned} A^T A\mathbf{x} = \mathbf{0}_n &\Rightarrow 0 = \mathbf{x}\cdot\mathbf{0}_n = \mathbf{x}\cdot\left(A^T A\right)\mathbf{x} = \mathbf{x}^T\left(\left(A^T A\right)\mathbf{x}\right) = \left(\mathbf{x}^T A^T\right)\left(A\mathbf{x}\right) = \left(A\mathbf{x}\right)^T A\mathbf{x} \\ &= A\mathbf{x}\cdot A\mathbf{x} = \left\|A\mathbf{x}\right\|^2 \Leftrightarrow A\mathbf{x} = \mathbf{0}_m. \end{aligned} \tag{65}$$

14. Find all least squares solutions $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of the inconsistent system

$$\begin{aligned} x_1 + x_2 &= 7 \\ 2x_1 + x_2 &= -4 \\ -x_1 + 3x_2 &= -1. \end{aligned} \tag{66}$$

15pts

Solution

Note that the system is in fact inconsistent because

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 2 & 1 & -4 \\ -1 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & -1 & -18 \\ 0 & 4 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 18 \\ 0 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 18 \\ 0 & 0 & -33 \end{array}\right], \tag{67}$$

the latter the augmented matrix for the obviously inconsistent system

$$\begin{aligned} x_1 + x_2 &= 7 \\ x_2 &= 18 \\ 0 &= -33. \end{aligned} \tag{68}$$

So we write (66) as

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x} = \mathbf{b} = \begin{bmatrix} 7 \\ -4 \\ -1 \end{bmatrix} \tag{69}$$

and instead of (69) solve the (always consistent) normal equations

$$\begin{aligned} &\begin{bmatrix} 6x_1 \\ 11x_2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \\ &A^T A\mathbf{x} = A^T\mathbf{b} \\ &= \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{aligned} \tag{70}$$

the one and only solution to which obviously being

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{71}$$

15. Here is a quadratic form:

$$Q(x_1, x_2) = 14x_1^{\ 2} - 4x_1x_2 + 11x_2^{\ 2}. \tag{72}$$

Find new variables

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^T\mathbf{x} = \begin{bmatrix} P_{11} & P_{21} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{73}$$

with P an orthogonal matrix, such that in terms of the new variables the form in (72) is diagonal. Write out your form with the old variables x_1, x_2, but in the diagonalized form, i.e., write out that

$$Q(x_1, x_2) = 14x_1^{\ 2} - 4x_1x_2 + 11x_2^{\ 2} = \lambda_1 y_1^{\ 2} + \lambda_2 y_2^{\ 2} = \lambda_1\left(P_{11}x_1 + P_{21}x_2\right)^2 + \lambda_2\left(P_{12}x_1 + P_{22}x_2\right)^2. \tag{74}$$

Here you get 5 of the 10 points only by checking that

$$14x_1^{\ 2} - 4x_1x_2 + 11x_2^{\ 2} \equiv \lambda_1\left(P_{11}x_1 + P_{21}x_2\right)^2 + \lambda_2\left(P_{12}x_1 + P_{22}x_2\right)^2. \tag{75}$$

Is the form (72)(74) positive definite? Positive semidefinite? Indefinite? Negative semidefinite? Negative definite? Explain.

20pts

Solution

Write (72) as

$$Q(\mathbf{x}) = Q(x_1, x_2) = 14x_1^{\ 2} - 4x_1x_2 + 11x_2^{\ 2} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 14 & -2 \\ -2 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^T A\mathbf{x}. \tag{76}$$

Inserting the inverse of (73) into (76), which, with orthogonality is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P\mathbf{y} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{77}$$

we get

$$q(\mathbf{y}) := Q(P\mathbf{y}) = \left(P\mathbf{y}\right)^T A\left(P\mathbf{y}\right) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y}$$
$$= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \lambda_1 {y_1}^2 + \lambda_2 {y_2}^2, \tag{78}$$

provided $P^T AP = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, i.e., provided orthogonal matrix P diagonalizes A. To find the eigenvalues and then orthogonal matrix P we note that

$$0 = P_A(\lambda) := \det\left(\lambda I - A\right) = \det \begin{bmatrix} \lambda - 14 & 2 \\ 2 & \lambda - 11 \end{bmatrix} = \lambda^2 - 25\lambda + 14\cdot 11 - 4$$
$$= \lambda^2 - 25\lambda + 2\left(7\cdot 11 - 2\right) = \lambda^2 - 25\lambda + 2\cdot 3\cdot 5\cdot 5 = \lambda^2 - \left(10+15\right)\lambda + 10\cdot 15 \tag{79}$$
$$= \left(\lambda - 10\right)\left(\lambda - 15\right),$$

and then that

$$E_A\left(10\right) = \ker\left(10I - A\right) = \ker \begin{bmatrix} 10-14 & 2 \\ 2 & 10-11 \end{bmatrix} = \ker \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} = \ker \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\},$$
$$E_A\left(15\right) = \ker\left(15I - A\right) = \ker \begin{bmatrix} 15-14 & 2 \\ 2 & 15-11 \end{bmatrix} = \ker \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \ker \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}, \tag{80}$$

so that with transformation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P\mathbf{y} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$
$$\Leftrightarrow \tag{81}$$
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^T\mathbf{x} = \begin{bmatrix} P_{11} & P_{21} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

and with result (78), form (72) can be rewritten as

$$Q(\mathbf{x}) = q(P^T\mathbf{x}) = \lambda_1 {y_1}^2 + \lambda_2 {y_2}^2 = 10\left(\frac{x_1 + 2x_2}{\sqrt{5}}\right)^2 + 15\left(\frac{-2x_1 + x_2}{\sqrt{5}}\right)^2. \tag{82}$$

We check that (82) is correct by noting that

$$
\begin{aligned}
10\left(\frac{x_1+2x_2}{\sqrt{5}}\right)^2+15\left(\frac{-2x_1+x_2}{\sqrt{5}}\right)^2 &= 2\left(x_1+2x_2\right)^2+3\left(-2x_1+x_2\right)^2 \\
&= 2\left(x_1{}^2+4x_1x_2+4x_2{}^2\right)+3\left(4x_1{}^2-4x_1x_2+x_2{}^2\right) \\
&= \left(2+3(4)\right)x_1{}^2+\left(2(4)+3(-4)\right)x_1x_2+\left(2(4)+3\right)x_2{}^2 \\
&= 14x_1{}^2-4x_1x_2+11x_2{}^2 =: Q(x_1,x_2).
\end{aligned} \tag{83}
$$

From (82) we see the form is positive definite:

$$
0=10\left(\frac{x_1+2x_2}{\sqrt{5}}\right)^2+15\left(\frac{-2x_1+x_2}{\sqrt{5}}\right)^2 \Leftrightarrow \begin{array}{c} x_1+2x_2=0 \\ -2x_1+x_2=0 \end{array} \Leftrightarrow \begin{array}{c} x_1=0 \\ x_2=0 \end{array}. \tag{84}
$$