Math 313 Midterm III KEY
Winter 2011
section 003
Instructor: Scott Glasgow

Write your name very clearly on this exam. In this booklet, write your mathematics clearly, legibly, in big fonts, and, most important, "have a point", i.e. make your work logically and even pedagogically acceptable. (Other human beings not already understanding 313 should be able to learn from your exam.) To avoid excessive erasing, first put your ideas together on scratch paper, then commit the logically acceptable fraction of your scratchings to this exam booklet. More is not necessarily better: say what you mean and mean what you say.

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1) Recall, as in problem 45 in 6.5, that one can always define an innerproduct on a real, finite dimensional vector space V by defining it on a basis in a certain way: Start with a specific basis $\{\mathbf{b}_1,\ldots,\mathbf{b}_n\}$ (for a vector space V with $\dim V = n \geq 1$), and then DECREE that, for $i, j \in \{1,\ldots,n\}$, the innerproduct $\langle , \rangle$ has the properties that

$$\langle \mathbf{b}_i, \mathbf{b}_j \rangle = \delta_{ij} := \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}. \tag{1}$$

Thus we essentially just decree that our favorite basis $\{\mathbf{b}_1,\ldots,\mathbf{b}_n\}$ is orthonormal with respect to the inner product we are trying to define. Recall also that we then extend the definition of the innerproduct to all of V by use of (1) together with the linearity of $\langle , \rangle$: if we want to know the innerproduct $\langle \mathbf{u}, \mathbf{v} \rangle$ of any two vectors $\mathbf{u}$ and $\mathbf{v}$ in V, we simply use that $\{\mathbf{b}_1,\ldots,\mathbf{b}_n\}$ is a basis for real vector space V, hence use that for any such two vectors there is a unique n-tuple of real numbers $\mathbf{c} = (c_1,\ldots,c_n) \in \mathbb{R}^n$ and $\mathbf{d} = (d_1,\ldots,d_n) \in \mathbb{R}^n$ such that

$$\mathbf{u} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n =: \sum_{i=1}^{n} c_i\mathbf{b}_i, \quad \text{and} \qquad \mathbf{v} = d_1\mathbf{b}_1 + \ldots + d_n\mathbf{b}_n =: \sum_{j=1}^{n} d_j\mathbf{b}_j, \tag{2}$$

so that with (1) and linearity we get

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle &= \left\langle \sum_{i=1}^{n} c_i\mathbf{b}_i, \sum_{j=1}^{n} d_j\mathbf{b}_j \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i d_j \langle \mathbf{b}_i, \mathbf{b}_j \rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i d_j \delta_{ij} = \sum_{i=1}^{n} c_i d_i = \\ &= c_1d_1 + \ldots + c_nd_n = [c_1 \ldots c_n]\begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \mathbf{c}\cdot\mathbf{d}, \end{aligned} \tag{3}$$

the last indicating the ordinary innerproduct of two vectors in $\mathbb{R}^n$. So suppose we choose (nonstandard) basis $\{\mathbf{b}_1, \mathbf{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$, yet, as in (1), decree that

$$\begin{aligned} \langle \mathbf{b}_1, \mathbf{b}_1 \rangle &= \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = 1 = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle = \langle \mathbf{b}_2, \mathbf{b}_2 \rangle, \\ \langle \mathbf{b}_1, \mathbf{b}_2 \rangle &= \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle = 0 = \langle \mathbf{b}_2, \mathbf{b}_1 \rangle. \end{aligned} \tag{4}$$

Now compute

$$\langle \mathbf{u}, \mathbf{v} \rangle = \left\langle \begin{bmatrix} \sqrt{3}+1 \\ \sqrt{3} \end{bmatrix}, \begin{bmatrix} \sqrt{3}-1 \\ -1 \end{bmatrix} \right\rangle \tag{5}$$

Hint: think about/use (2) and (3). Note the answer is NOT

$$\begin{bmatrix} 1+\sqrt{3} \\ \sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3}-1 \\ -1 \end{bmatrix} = \left(1+\sqrt{3}\right)\left(\sqrt{3}-1\right) + \left(\sqrt{3}\right)\left(-1\right) = 3-1-\sqrt{3} = 2-\sqrt{3}. \tag{6}$$

15pts

Solution

As in (2) write

$$\begin{aligned} \mathbf{u} &= \begin{bmatrix} 1+\sqrt{3} \\ \sqrt{3} \end{bmatrix} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1+c_2 \\ c_2 \end{bmatrix}, \quad \text{and} \\ \mathbf{v} &= \begin{bmatrix} \sqrt{3}-1 \\ -1 \end{bmatrix} = d_1\mathbf{b}_1 + d_2\mathbf{b}_2 = d_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} d_1+d_2 \\ d_2 \end{bmatrix} \end{aligned} \tag{7}$$

so that clearly $c_2 = \sqrt{3}, c_1 = 1, d_2 = -1, d_1 = \sqrt{3}$, and from (3) get

$$\langle \mathbf{u}, \mathbf{v} \rangle = \left\langle \begin{bmatrix} 1+\sqrt{3} \\ \sqrt{3} \end{bmatrix}, \begin{bmatrix} \sqrt{3}-1 \\ -1 \end{bmatrix} \right\rangle = \mathbf{c} \cdot \mathbf{d} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \cdot \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix} = \sqrt{3} - \sqrt{3} = 0. \tag{8}$$

Thus $\left\{ \begin{bmatrix} 1+\sqrt{3} \\ \sqrt{3} \end{bmatrix}, \begin{bmatrix} \sqrt{3}-1 \\ -1 \end{bmatrix} \right\}$ is another orthogonal basis for $\mathbb{R}^2$ with respect to the innerproduct defined by (4), and an another ortho*normal* basis is $\left\{ \frac{1}{2}\begin{bmatrix} 1+\sqrt{3} \\ \sqrt{3} \end{bmatrix}, \frac{1}{2}\begin{bmatrix} \sqrt{3}-1 \\ -1 \end{bmatrix} \right\}$: note, for example,

$$\begin{aligned} \left\langle \frac{1}{2}\begin{bmatrix} 1+\sqrt{3} \\ \sqrt{3} \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1+\sqrt{3} \\ \sqrt{3} \end{bmatrix} \right\rangle &= \frac{1}{4}\left\langle \begin{bmatrix} 1+\sqrt{3} \\ \sqrt{3} \end{bmatrix}, \begin{bmatrix} 1+\sqrt{3} \\ \sqrt{3} \end{bmatrix} \right\rangle = \frac{1}{4}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \frac{1}{4}\left(1+3\right) \\ &= 1. \end{aligned} \tag{9}$$

2) Find all least squares solutions $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of the inconsistent system

$$\begin{aligned} x_1 &= 1 \\ 0 &= 1 \end{aligned}, \tag{10}$$

i.e. of the system

$$\begin{aligned} 1x_1 + 0x_2 &= 1 \\ 0x_1 + 0x_2 &= 1 \end{aligned} \tag{11}$$

Hint: You might want to look at the Long Tutorial in problem 4. Also, as indicated, there may or may not be more than one solution.

15 points

Solution

Write inconsistent system (11) as

$$\begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{x} = A\mathbf{x} = \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{12}$$

So while (12) can never actually hold, we can in fact minimize $\|A\mathbf{x} - \mathbf{b}\|^2$, and the theory presented in the text (and below in problem 4) shows that any such $\mathbf{x}$ minimizing $\|A\mathbf{x} - \mathbf{b}\|^2$ satisfies the normal equation

$$\begin{aligned} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ = A^T A\mathbf{x} = A^T\mathbf{b} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \end{aligned} \tag{13}$$

which holds iff $x_1 = 1$. Since x_2 is unspecified, the set of $\mathbf{x}$'s minimizing $\|A\mathbf{x} - \mathbf{b}\|^2$ is the set

$$\left\{ \begin{bmatrix} 1 \\ c \end{bmatrix} : c \in \mathbb{R} \right\}. \tag{14}$$

3) Find all least squares solutions $\mathbf{x} = [x_1]$ of the inconsistent system

$$\begin{aligned} x_1 &= 1 \\ 0 &= 1 \end{aligned}, \qquad (15)$$

i.e. of the system

$$\begin{aligned} 1x_1 &= 1 \\ 0x_1 &= 1 \end{aligned} \qquad (16)$$

Hint: You might want to look at the Long Tutorial in the next problem. Also, as indicated, there may or may not be more than one solution. But the (least squares) solution space of this problem will NOT be the same as the last problem.

15 points

Solution

Write inconsistent system (16) as

$$\begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [x_1] = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{x} = A\mathbf{x} = \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \qquad (17)$$

While (17) never has a solution, any $\mathbf{x}$ minimizing $\|A\mathbf{x} - \mathbf{b}\|^2$ satisfies the normal equation

$$\begin{aligned} [x_1] = [1]x_1 = [1]\mathbf{x} = [1 \quad 0] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{x} \\ = A^T A\mathbf{x} = A^T\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [1 \quad 0] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [1], \end{aligned} \qquad (18)$$

which holds iff $x_1 = 1$. There is no other variable to specify, so this is the unique solution of the minimization problem.

4) Long Tutorial: Let A be a real $m \times n$ matrix, with $m > n$. Then the equation

$$A\mathbf{x} = \mathbf{b} \qquad (19)$$

doesn't always have a solution $\mathbf{x} \in \mathbb{R}^n$: for any fixed, real $A = A_{m \times n}$ with $m > n$ there is at least a whole vector space of $\mathbf{b}$'s in $\mathbb{R}^m$ (less the zero vector)—i.e. an entire subspace of $\mathbb{R}^m$ (less the zero vector) from which $\mathbf{b}$'s can be picked—such that (19) has no solution $\mathbf{x} \in \mathbb{R}^n$. To see this realize that the image of A, which is all objects of the form $A\mathbf{x}$, is the column space of A, which is at most $n\,(<m)$ dimensional (since A has n columns), and, so, $\mathbf{x} \mapsto A\mathbf{x}$ cannot be onto $\mathbb{R}^m$. Thus there is at least one $\mathbf{b}$ in $\mathbb{R}^m$ not expressible as $A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$, and for any scalar $\alpha \neq 0$, $A\mathbf{x} = \alpha\mathbf{b}$ won't have a solution either. (If it did have a solution, then we could rewrite $A\mathbf{x} = \alpha\mathbf{b}$ as $A\left(\alpha^{-1}\mathbf{x}\right) = \mathbf{b}$, so that $A\mathbf{x} = \mathbf{b}$ does in fact have a solution, namely $\alpha^{-1}\mathbf{x}$, where $\mathbf{x}$ is a solution of $A\mathbf{x} = \alpha\mathbf{b}$). Anyways…

When (19) doesn't have a solution, we often try to "do the best we can", i.e. try to pick $\mathbf{x}$ so that $A\mathbf{x}$ is as close to $\mathbf{b}$ as possible, i.e. try to pick $\mathbf{x}$ to minimize

$$\begin{aligned} \|A\mathbf{x}-\mathbf{b}\|^2 &= (A\mathbf{x}-\mathbf{b})\cdot(A\mathbf{x}-\mathbf{b}) \overset{\text{fact}}{=} (A\mathbf{x}-\mathbf{b})^T (A\mathbf{x}-\mathbf{b}) = \left((A\mathbf{x})^T - \mathbf{b}^T\right)(A\mathbf{x}-\mathbf{b}) \\ &= \left(\mathbf{x}^T A^T - \mathbf{b}^T\right)(A\mathbf{x}-\mathbf{b}) = \mathbf{x}^T A^T (A\mathbf{x}-\mathbf{b}) - \mathbf{b}^T (A\mathbf{x}-\mathbf{b}) \\ &= \mathbf{x}^T A^T A\mathbf{x} - \mathbf{x}^T A^T \mathbf{b} - \mathbf{b}^T A\mathbf{x} + \mathbf{b}^T\mathbf{b}. \end{aligned} \tag{20}$$

Since $\mathbf{x}^T A^T\mathbf{b}$ is a scalar (actually a 1×1 matrix), $\mathbf{x}^T A^T\mathbf{b} = \left(\mathbf{x}^T A^T\mathbf{b}\right)^T = \mathbf{b}^T A\mathbf{x}$, and (20) simplifies to

$$\|A\mathbf{x}-\mathbf{b}\|^2 = \mathbf{x}^T A^T A\mathbf{x} - 2\mathbf{b}^T A\mathbf{x} + \mathbf{b}^T\mathbf{b}. \tag{21}$$

Now "you have been told" (hopefully you understand it as well) that in order to minimize $\|A\mathbf{x}-\mathbf{b}\|^2$, you need to choose $\mathbf{x} = \mathbf{x}_{\text{least squares}} =: \mathbf{x}_{\text{ls}}$ where $\mathbf{x}_{\text{ls}}$ satisfies the "normal equation" $A^T A\mathbf{x}_{\text{ls}} = A^T\mathbf{b}$, which, "you have been told", always has a solution $\mathbf{x}_{\text{ls}}$. One way to see that a $\mathbf{x}_{\text{ls}}$ satisfying $A^T A\mathbf{x}_{\text{ls}} = A^T\mathbf{b}$ minimizes the right hand side of (21) is to write there that $\mathbf{x} = \mathbf{x} - \mathbf{x}_{\text{ls}} + \mathbf{x}_{\text{ls}} =: \mathbf{y} + \mathbf{x}_{\text{ls}}$, and note that $\mathbf{y} = \mathbf{0} = \mathbf{x} - \mathbf{x}_{\text{ls}}$ certainly minimizes (21) when $A^T A\mathbf{x}_{\text{ls}} = A^T\mathbf{b}$: writing $\mathbf{x} = \mathbf{y} + \mathbf{x}_{\text{ls}}$ in (21) gives (after a lot of algebra involving things like $\mathbf{x}_{\text{ls}}{}^T A^T A\mathbf{y} = \left(\mathbf{x}_{\text{ls}}{}^T A^T A\mathbf{y}\right)^T$ $= \mathbf{y}^T A^T A\mathbf{x}_{\text{ls}} = \mathbf{y}^T A^T\mathbf{b}$, etc.)

$$\|A\mathbf{x}-\mathbf{b}\|^2 = A\mathbf{y}\cdot A\mathbf{y} - \mathbf{b}\cdot\left(A\mathbf{x}_{\text{ls}} - \mathbf{b}\right) = \|A\mathbf{y}\|^2 - \mathbf{b}\cdot\left(A\mathbf{x}_{\text{ls}} - \mathbf{b}\right) \geq 0 - \mathbf{b}\cdot\left(A\mathbf{x}_{\text{ls}} - \mathbf{b}\right), \tag{22}$$

where we used $A\mathbf{y}\cdot A\mathbf{y} = \|A\mathbf{y}\|^2 \geq 0$, equality holding certainly if $\mathbf{y} = \mathbf{0} = \mathbf{x} - \mathbf{x}_{\text{ls}} \iff \mathbf{x} = \mathbf{x}_{\text{ls}}$, where, reminder, $\mathbf{x}_{\text{ls}}$ is such that $A^T A\mathbf{x}_{\text{ls}} = A^T\mathbf{b}$. On the other hand, since (22) holds (with $\mathbf{y} := \mathbf{x} - \mathbf{x}_{\text{ls}}$), to minimize $\|A\mathbf{x}-\mathbf{b}\|^2$ it is only necessary (and certainly sufficient) that $A\mathbf{y} = \mathbf{0}$, i.e. only necessary that

$\mathbf{y} := \mathbf{x} - \mathbf{x}_{\text{ls}} \in \ker A := \left\{ \mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0} \in \mathbb{R}^m \right\}$. So if we find a solution $\mathbf{x}_{\text{ls}}$ to the least squares/normal equation $A^T A\mathbf{x}_{\text{ls}} = A^T\mathbf{b}$, we could add to that solution any solution $\mathbf{v}$ to $A\mathbf{v} = \mathbf{0}$, getting $\mathbf{x} = \mathbf{x}_{\text{ls}} + \mathbf{v}$ also minimizes $\left\| A\mathbf{x} - \mathbf{b} \right\|^2$. (This is also clear from just thinking about $\left\| A\mathbf{x} - \mathbf{b} \right\|^2$ directly:
$\left\| A\mathbf{x} - \mathbf{b} \right\|^2 = \left\| A\left(\mathbf{x}_{\text{ls}} + \mathbf{v}\right) - \mathbf{b} \right\|^2 = \left\| A\mathbf{x}_{\text{ls}} + A\mathbf{v} - \mathbf{b} \right\|^2 = \left\| A\mathbf{x}_{\text{ls}} + \mathbf{0} - \mathbf{b} \right\|^2 = \left\| A\mathbf{x}_{\text{ls}} - \mathbf{b} \right\|^2$.) But note that this ambiguity/potential non-uniqueness in the solution of the minimization problem need not be thought about "separately", since the equation $A^T A\mathbf{x}_{\text{ls}} = A^T\mathbf{b}$ already has the nullspace of A "built into it": if $\mathbf{v} \in \ker A$, and $\mathbf{x} = \mathbf{x}_{\text{ls}}$ solves $A^T A\mathbf{x} = A^T\mathbf{b}$, then so does $\mathbf{x} = \mathbf{x}_{\text{ls}} + \mathbf{v}$, this because then

$$\begin{aligned} A^T A\mathbf{x} &= A^T A\left(\mathbf{x}_{\text{ls}} + \mathbf{v}\right) = A^T A\mathbf{x}_{\text{ls}} + A^T A\mathbf{v} = A^T A\mathbf{x}_{\text{ls}} + A^T \mathbf{0}_{mx1} = A^T A\mathbf{x}_{\text{ls}} + \mathbf{0}_{nx1} = \\ &= A^T A\mathbf{x}_{\text{ls}} = A^T\mathbf{b}. \end{aligned} \tag{23}$$

Here we effectively proved that $\ker A^T A \supset \ker A$. But maybe $\ker A^T A \neq \ker A$, i.e. maybe there is at least one vector $\mathbf{w} \in \ker A^T A$ with $\mathbf{w} \notin \ker A$, so that in fact in considering the solution space of $A^T A\mathbf{x} = A^T\mathbf{b}$ we get "more non-uniqueness" than in considering the solutions space of $A\mathbf{x} = \mathbf{b}$ (when the latter manages to have a solution), which would be a strange set of affairs. (What if, for example, you just decide to never ever solve equations of the form $A\mathbf{x} = \mathbf{b}$ —since you may be frustrated at times in the attempt—but rather decide to always recast such equations into the form $A^T A\mathbf{x} = A^T\mathbf{b}$, since then you can never be frustrated. Something very bad happens if $\ker A^T A$ is actually bigger than $\ker A$, particularly when $A\mathbf{x} = \mathbf{b}$ actually isn't frustrating. Think about it.)

Thankfully, you have been shown that there can be no such $\mathbf{w} \in \ker A^T A$ with $\mathbf{w} \notin \ker A$. The proof went something like this: if $\mathbf{w} \in \ker A^T A$ then $A^T A\mathbf{w} = \mathbf{0}_{nx1}$. Rewrite $A^T A\mathbf{w} = \mathbf{0}_{nx1}$ as $A^T\left(A\mathbf{w}\right) = \mathbf{0}_{nx1}$, so that $A\mathbf{w} \in \ker A^T$. But certainly $A\mathbf{w} \in \text{col}A$, and we then get

$$A\mathbf{w} \in \ker A^T \bigcap \text{col}A = \left(\text{col}A\right)^{\perp} \bigcap \text{col}A = \left\{\mathbf{0}_{mx1}\right\}, \tag{24}$$

i.e. $\mathbf{w} \in \ker A$. Here we used that $\ker A^T$ and $\text{col}A$ are orthogonal compliments, so that since the only vector orthogonal to itself is the zero vector, $\left(\text{col}A\right)^{\perp} \bigcap \text{col}A = \left\{\mathbf{0}_{mx1}\right\}$. The only problem with this proof is that we never actually showed/proved that $\ker A^T$ and $\text{col}A$ are orthogonal compliments, i.e. never showed that $\left(\ker A^T\right)^{\perp} = \text{col}A$ and $\left(\text{col}A\right)^{\perp} = \ker A^T$. The first of these two statements says that the set of all vectors orthogonal to every element in $\ker A^T$ is exactly the column space of A, and the second of these statements says that the set of all vectors orthogonal to every element in the column space of A is exactly the kernel of A^T. The first statement is

$$\left(A^T\mathbf{c}=\mathbf{0}_{nx1}\Rightarrow\mathbf{c}\cdot\mathbf{d}=0\right)\Leftrightarrow\mathbf{d}=A\mathbf{x}\text{ for some }\mathbf{x}\in\mathbb{R}^n \tag{25}$$

while the second statement is

$$\left(\mathbf{x}\in\mathbb{R}^n\Rightarrow\mathbf{c}\cdot A\mathbf{x}\ =0\right)\Leftrightarrow A^T\mathbf{c}=\mathbf{0}_{nx1}. \tag{26}$$

The backward arrows in each of (25) and (26) are easy to prove: if $\mathbf{d}=A\mathbf{x}$ for some $\mathbf{x}\in\mathbb{R}^n$, then, for (25), we have

$$A^T\mathbf{c}=\mathbf{0}_{nx1}\Rightarrow\mathbf{c}\cdot\mathbf{d}=\mathbf{c}\cdot A\mathbf{x}=\mathbf{c}^TA\mathbf{x}=\left(\mathbf{c}^TA\mathbf{x}\right)^T=\mathbf{x}^TA^T\mathbf{c}=\mathbf{x}^T\mathbf{0}_{nx1}=0, \tag{27}$$

and if $A^T\mathbf{c}=\mathbf{0}_{nx1}$, then, for (26) we have

$$\mathbf{x}\in\mathbb{R}^n\Rightarrow\mathbf{c}\cdot A\mathbf{x}\ =\mathbf{c}^TA\mathbf{x}=\left(\mathbf{c}^TA\mathbf{x}\right)^T=\mathbf{x}^TA^T\mathbf{c}=\mathbf{x}^T\mathbf{0}_{nx1}=0. \tag{28}$$

The forward arrow for (25) is sort of hard—5 points extra credit if you can prove that sometime within a week of taking this exam (hint: use an orthogonal basis for $\ker A^T$ and complete to a basis of $\mathbb{R}^m\supset\{\mathbf{d}\}$)—but you can prove the forward arrow in (26). Please do so now. (Hint: $\mathbf{c}\cdot A\mathbf{x}=\mathbf{c}^TA\mathbf{x}=\left(\mathbf{c}^TA\mathbf{x}\right)^T=\mathbf{x}^TA^T\mathbf{c}=\mathbf{x}\cdot A^T\mathbf{c}$, for every $\mathbf{x}\in\mathbb{R}^n$. Now choose $\mathbf{x}$ wisely to get $A^T\mathbf{c}=\mathbf{0}$.)

15 points

Solution

By the hint,

$$0=\mathbf{c}\cdot A\mathbf{x}=\mathbf{x}\cdot A^T\mathbf{c}=A^T\mathbf{c}\cdot A^T\mathbf{c}=\left\|A^T\mathbf{c}\right\|^2\Leftrightarrow A^T\mathbf{c}=\mathbf{0}, \tag{29}$$

where we chose (wisely) $\mathbf{x}=A^T\mathbf{c}$.

5) In the previous problem, we wanted to show $\ker A^TA=\ker A$, the easy part of which was showing $\ker A^TA\supset\ker A$, the hard part of which showing $\ker A^TA\subset\ker A$, i.e. showing $A^TA\mathbf{x}=\mathbf{0}_{nx1}\Rightarrow A\mathbf{x}=\mathbf{0}_{mx1}$. If fact there we punted a bit: we used $\ker A^T$ and $\operatorname{col}A$ are orthogonal compliments, the excruciating part of

which—namely showing $\left(\ker A^T\right)^{\perp} \subset \mathrm{col}A$—I left as an extra credit problem. But there is a rather easy way to get

$$A^T A\mathbf{x} = \mathbf{0}_{nx1} \Rightarrow A\mathbf{x} = \mathbf{0}_{mx1} \tag{30}$$

directly: start on the left of (30) and try to get to the right of (30) by taking the inner product of both sides of the left of (30) with $\mathbf{x}$. Mark, get set, go.

15pts

Solution

Following the hint, we find

$$\begin{aligned} A^T A\mathbf{x} = \mathbf{0}_{nx1} &\Rightarrow 0 = \mathbf{x}\cdot\mathbf{0}_{nx1} = \mathbf{x}\cdot A^T A\mathbf{x} = \mathbf{x}^T A^T A\mathbf{x} = \left(A\mathbf{x}\right)^T A\mathbf{x} = A\mathbf{x}\cdot A\mathbf{x} \\ &= \left\|A\mathbf{x}\right\|^2 \Leftrightarrow A\mathbf{x} = \mathbf{0}_{mx1}. \end{aligned} \tag{31}$$

6) Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be linear. Recall the definition of the matrix $\left[T\right]_{SS}$ for T with respect to basis S (of $\mathbb{R}^n$): for all $\mathbf{v}\in\mathbb{R}^n$ the matrix $\left[T\right]_{SS}$ should satisfy

$$\left(T\left(\mathbf{v}\right)\right)_S = \left[T\right]_{SS}\left(\mathbf{v}\right)_S, \tag{32}$$

where $\left(T\left(\mathbf{v}\right)\right)_S$ and $\left(\mathbf{v}\right)_S$ denote the coordinate vectors of, respectively, $T\left(\mathbf{v}\right)$ and $\mathbf{v}$ with respect to basis S (written as column vectors). Show that if $S = \left\{\mathbf{v}_1,\ldots,\mathbf{v}_n\right\}$, then

$$\left[T\right]_{SS} = \left[T\right]_{SS} I = \left[T\right]_{SS}\left[\left(\mathbf{v}_1\right)_S\middle|\ldots\middle|\left(\mathbf{v}_n\right)_S\right] = \left[\left(T\left(\mathbf{v}_1\right)\right)_S\middle|\ldots\middle|\left(T\left(\mathbf{v}_n\right)\right)_S\right] \tag{33}$$

and then, harder,

$$\left[T\right]_{SS} = \left[\left(T\left(\mathbf{e}_1\right)\right)_S\middle|\ldots\middle|\left(T\left(\mathbf{e}_n\right)\right)_S\right]\left[\left(\mathbf{e}_1\right)_S\middle|\ldots\middle|\left(\mathbf{e}_n\right)_S\right]^{-1}, \tag{34}$$

for standard basis $E = \left\{\mathbf{e}_1,\ldots,\mathbf{e}_n\right\}$ of $\mathbb{R}^n$, and finally

$$\left[\mathbf{v}_1\middle|\ldots\middle|\mathbf{v}_n\right]\left[\left(\mathbf{e}_1\right)_S\middle|\ldots\middle|\left(\mathbf{e}_n\right)_S\right] = \left[\mathbf{e}_1\middle|\ldots\middle|\mathbf{e}_n\right] = I, \tag{35}$$

and

$$\left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right]\left[\left(T(\mathbf{e}_1)\right)_S\Big|\ldots\Big|\left(T(\mathbf{e}_n)\right)_S\right]=\left[T(\mathbf{e}_1)|\ldots|T(\mathbf{e}_n)\right], \tag{36}$$

hence

$$\begin{gathered}\left[(\mathbf{e}_1)_S\Big|\ldots\Big|(\mathbf{e}_n)_S\right]^{-1}=\left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right]\\ \text{and}\\ \left[\left(T(\mathbf{e}_1)\right)_S\Big|\ldots\Big|\left(T(\mathbf{e}_n)\right)_S\right]=\left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right]^{-1}\left[T(\mathbf{e}_1)|\ldots|T(\mathbf{e}_n)\right]\end{gathered} \tag{37}$$

and (34) is actually the very useful statement

$$[T]_{SS}=\left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right]^{-1}\left[T(\mathbf{e}_1)|\ldots|T(\mathbf{e}_n)\right]\left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right]. \tag{38}$$

<u>15pts</u>

<u>Solution</u>

First, since $S=\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$, then clearly $\left[(\mathbf{v}_1)_S\Big|\ldots\Big|(\mathbf{v}_n)_S\right]=\left[\mathbf{e}_1|\ldots|\mathbf{e}_n\right]=I$, and, so, from definition (32) and the way matrix multiplication works, in (33) one finds

$$\begin{aligned}[T]_{SS}&=[T]_{SS}\,I=[T]_{SS}\left[(\mathbf{v}_1)_S\Big|\ldots\Big|(\mathbf{v}_n)_S\right]=\left[[T]_{SS}(\mathbf{v}_1)_S\Big|\ldots\Big|[T]_{SS}(\mathbf{v}_n)_S\right]\\ &=\left[\left(T(\mathbf{v}_1)\right)_S\Big|\ldots\Big|\left(T(\mathbf{v}_n)\right)_S\right]\end{aligned} \tag{39}$$

as advertized. Likewise

$$\begin{gathered}[T]_{SS}\left[(\mathbf{e}_1)_S\Big|\ldots\Big|(\mathbf{e}_n)_S\right]=\left[[T]_{SS}(\mathbf{e}_1)_S\Big|\ldots\Big|[T]_{SS}(\mathbf{e}_n)_S\right]=\left[\left(T(\mathbf{e}_1)\right)_S\Big|\ldots\Big|\left(T(\mathbf{e}_n)\right)_S\right]\\ \Rightarrow\\ [T]_{SS}=\left[\left(T(\mathbf{e}_1)\right)_S\Big|\ldots\Big|\left(T(\mathbf{e}_n)\right)_S\right]\left[(\mathbf{e}_1)_S\Big|\ldots\Big|(\mathbf{e}_n)_S\right]^{-1}\end{gathered} \tag{40}$$

which is (34). For (35) and (36) note that, by definition of coordinate vector,

$$\begin{aligned}
\left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right]\left[(\mathbf{e}_1)_S|\ldots|(\mathbf{e}_n)_S\right] &= \left[\left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right](\mathbf{e}_1)_S|\ldots|\left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right](\mathbf{e}_n)_S\right] \\
&= \left[\mathbf{e}_1|\ldots|\mathbf{e}_n\right] = I, \text{ and} \\
\left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right]\left[(T(\mathbf{e}_1))_S|\ldots|(T(\mathbf{e}_n))_S\right] &= \left[\left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right](T(\mathbf{e}_1))_S|\ldots|\left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right](T(\mathbf{e}_n))_S\right] \\
&= \left[T(\mathbf{e}_1)|\ldots|T(\mathbf{e}_n)\right].
\end{aligned} \tag{41}$$

Hence from (41) we get

$$\begin{aligned}
&\left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right]\left[(\mathbf{e}_1)_S|\ldots|(\mathbf{e}_n)_S\right] = I \Leftrightarrow \left[(\mathbf{e}_1)_S|\ldots|(\mathbf{e}_n)_S\right]^{-1} = \left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right] \text{ and} \\
&\left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right]\left[(T(\mathbf{e}_1))_S|\ldots|(T(\mathbf{e}_n))_S\right] = \left[T(\mathbf{e}_1)|\ldots|T(\mathbf{e}_n)\right] \\
&\Leftrightarrow \\
&\left[(T(\mathbf{e}_1))_S|\ldots|(T(\mathbf{e}_n))_S\right] = \left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right]^{-1}\left[T(\mathbf{e}_1)|\ldots|T(\mathbf{e}_n)\right],
\end{aligned} \tag{42}$$

which is (37), and using this in (40) we get

$$\begin{aligned}
[T]_{SS} &= \left[(T(\mathbf{e}_1))_S|\ldots|(T(\mathbf{e}_n))_S\right]\left[(\mathbf{e}_1)_S|\ldots|(\mathbf{e}_n)_S\right]^{-1} \\
&= \left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right]^{-1}\left[T(\mathbf{e}_1)|\ldots|T(\mathbf{e}_n)\right]\left[\mathbf{v}_1|\ldots|\mathbf{v}_n\right],
\end{aligned} \tag{43}$$

which is (38).

7) In the previous problem, note from (33) that for standard basis $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$, we get

$$[T]_{EE} = \left[(T(\mathbf{e}_1))_E|\ldots|(T(\mathbf{e}_n))_E\right] = \left[T(\mathbf{e}_1)|\ldots|T(\mathbf{e}_n)\right]. \tag{44}$$

So, using the results above, show that $\det T := \det[T]_{SS}$ is well defined, i.e. show you get the same result irrespective of basis $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

<u>15pts</u>

<u>Solution</u>

From (38) we see

$$\begin{aligned}\det[T]_{SS} &= \det\left[\mathbf{v}_1|\dots|\mathbf{v}_n\right]^{-1}\left[T(\mathbf{e}_1)|\dots|T(\mathbf{e}_n)\right]\left[\mathbf{v}_1|\dots|\mathbf{v}_n\right]\\ &= \det\left[\mathbf{v}_1|\dots|\mathbf{v}_n\right]^{-1}\det\left[T(\mathbf{e}_1)|\dots|T(\mathbf{e}_n)\right]\det\left[\mathbf{v}_1|\dots|\mathbf{v}_n\right]\\ &= \det\left[\mathbf{v}_1|\dots|\mathbf{v}_n\right]^{-1}\det\left[\mathbf{v}_1|\dots|\mathbf{v}_n\right]\det\left[T(\mathbf{e}_1)|\dots|T(\mathbf{e}_n)\right]\\ &= \det\left(\left[\mathbf{v}_1|\dots|\mathbf{v}_n\right]^{-1}\left[\mathbf{v}_1|\dots|\mathbf{v}_n\right]\right)\det\left[T(\mathbf{e}_1)|\dots|T(\mathbf{e}_n)\right]\\ &= \det I\det\left[T(\mathbf{e}_1)|\dots|T(\mathbf{e}_n)\right] = \det\left[T(\mathbf{e}_1)|\dots|T(\mathbf{e}_n)\right] = \det[T]_{EE}.\end{aligned} \tag{45}$$

the latter independent of S.

8) Let $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A and C are square matrices. Show

$$\det M = \det A \det C \tag{46}$$

<u>15pts</u>

<u>Solution</u>

We can compute $\det M$ by doing row reduction on it until it is perfectly upper triangular, using then that the determinant of such is the product of the diagonal entries. (Any row echelon form of a square matrix is upper triangular.) In order to get a unique result, do row reduction to reduced row echelon form. So if we get the row reduced matrix of M of the form

$$U_M = \begin{pmatrix} A' & B' \\ 0 & C' \end{pmatrix} \tag{47}$$

with, then, U_M perfectly upper triangular, clearly A' and C' will have to be this way too, and in such case we certainly get

$$\det U_M = \det A' \det C'. \tag{48}$$

Now since A and C can certainly be put into upper triangular forms U_A and U_B by finite sequences of row operations, then there exists elementary matrices $E_1, \dots, E_m$ and $E_1', \dots, E_n'$ such that

$$U_A = E_m \cdots E_1 A, \qquad U_C = E_n' \cdots E_1' C \tag{49}$$

So then note that

$$\begin{pmatrix} I & 0 \\ 0 & E_n{}' \end{pmatrix} \cdots \begin{pmatrix} I & 0 \\ 0 & E_1{}' \end{pmatrix} \begin{pmatrix} E_m & 0 \\ 0 & I' \end{pmatrix} \cdots \begin{pmatrix} E_1 & 0 \\ 0 & I' \end{pmatrix} M = \begin{pmatrix} I & 0 \\ 0 & E_n{}' \end{pmatrix} \cdots \begin{pmatrix} I & 0 \\ 0 & E_1{}' \end{pmatrix} \begin{pmatrix} E_m & 0 \\ 0 & I' \end{pmatrix} \cdots \begin{pmatrix} E_1 & 0 \\ 0 & I' \end{pmatrix} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$
$$= \begin{pmatrix} E_m \cdots E_1 A & E_m \cdots E_1 B \\ 0 & E_n{}' \cdots E_1{}' C \end{pmatrix} = \begin{pmatrix} U_A & E_m \cdots E_1 B \\ 0 & U_C \end{pmatrix}$$
$$= \begin{pmatrix} A' & B' \\ 0 & C' \end{pmatrix} = U_M . \tag{50}$$

where we used (47), specifically its uniqueness. So by (47)/(48) (and a result similar to (47)/(48)) we get

$$\det E_n{}' \cdots E_1{}' \det E_m \cdots E_1 \det M = \det E_n{}' \left(\begin{pmatrix} I & 0 \\ 0 & I' \end{pmatrix} \right) \cdots E_1{}' \begin{pmatrix} I & 0 \\ 0 & I' \end{pmatrix} E_m \left(\begin{pmatrix} I & 0 \\ 0 & I' \end{pmatrix} \right) \cdots E_1 \left(\begin{pmatrix} I & 0 \\ 0 & I' \end{pmatrix} \right) M =$$
$$= \det \begin{pmatrix} I & 0 \\ 0 & E_n{}' \end{pmatrix} \cdots \begin{pmatrix} I & 0 \\ 0 & E_1{}' \end{pmatrix} \begin{pmatrix} E_m & 0 \\ 0 & I' \end{pmatrix} \cdots \begin{pmatrix} E_1 & 0 \\ 0 & I' \end{pmatrix} M$$
$$= \det A' \det C' = \det E_m \cdots E_1 A \det E_n{}' \cdots E_1{}' C$$
$$= \det E_n{}' \cdots E_1{}' \det E_m \cdots E_1 \det A \det C$$
$$\Leftrightarrow$$
$$\det M = \det A \det C. \tag{51}$$

Here we used the lemma that the determinant of a row operation on a matrix is the product of the determinant of the elementary matrix corresponding to that row operation and the determinant of said matrix (and the determinant of the product of matrices is the product of the determinants): for example

$$\det \left(E_n{}' \begin{pmatrix} I & 0 \\ 0 & I' \end{pmatrix} \right) = \det E_n{}' \det \begin{pmatrix} I & 0 \\ 0 & I' \end{pmatrix} = \det E_n{}' \cdot 1 = \det E_n{}' . \tag{52}$$

9) The eigenvalues of

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & 2 \\ -1 & -3 & 5 \end{bmatrix} \tag{53}$$

are $\lambda = 1, 2,$ and 3. Compute the eigenspaces associated to each of these eigenvalues. (Recall eigenspaces are subspaces, hence specified as the span of a basis.)

15pts

Solution

We have

$$
\begin{aligned}
E_1(A) &= \ker(1I - A) = \ker\begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & -2 \\ 1 & 3 & -4 \end{bmatrix} = \ker\begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & -2 \\ 0 & 2 & -2 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}, \\
E_2(A) &= \ker(2I - A) = \ker\begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & -2 \\ 1 & 3 & -3 \end{bmatrix} = \ker\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}, \\
E_3(A) &= \ker(3I - A) = \ker\begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -2 \\ 1 & 3 & -2 \end{bmatrix} = \ker\begin{bmatrix} 2 & 2 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} == \ker\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right\}.
\end{aligned} \tag{54}
$$

10) For k any positive integer, compute

$$
\begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & 2 \\ -1 & -3 & 5 \end{bmatrix}^k . \tag{55}
$$

You may use that

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \tag{56}$$

i.e., that the two matrices in (56) are row equivalent.

15pts

Solution

If square matrix A is diagonalizable, then we can write

$$\begin{aligned} A^k &= \left(SDS^{-1}\right)^k = SD^k S^{-1} \\ &= S\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}^k S^{-1} = S\begin{bmatrix} \lambda_1{}^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n{}^k \end{bmatrix} S^{-1} \end{aligned} \tag{57}$$

where S's columns are eigenvectors of A. A will certainly be diagonalizable of all the eigenvalues $\lambda_1,\ldots,\lambda_n$ are distinct, which, by the previous problem, is what occurs for the matrix of (55). And by the results of that previous problem, we can take

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \Leftrightarrow S^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \tag{58}$$

where we used (56). So then by (57) we have

$$\begin{aligned} A^k &= S\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}^k S^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 3^k \\ 1 & 2^k & 3^k \\ 1 & 2^k & 2\cdot 3^k \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1-3^k & 3^k-1 \\ 1-2^k & 1+2^k-3^k & 3^k-1 \\ 1-2^k & 1+2^k-2\cdot 3^k & 2\cdot 3^k-1 \end{bmatrix}. \end{aligned} \tag{59}$$