Math 313 Midterm II KEY
Winter 2011
section 003
Instructor: Scott Glasgow

Write your name very clearly on this exam. In this booklet, write your mathematics clearly, legibly, in big fonts, and, most important, "have a point", i.e. make your work logically and even pedagogically acceptable. (Other human beings not already understanding 313 should be able to learn from your exam.) To avoid excessive erasing, first put your ideas together on scratch paper, then commit the logically acceptable fraction of your scratchings to this exam booklet. More is not necessarily better: say what you mean and mean what you say.

Honor Code: After I have learned of the contents of this exam by any means, I will not disclose to anyone any of these contents by any means until after the exam has closed. My signature below indicates I accept this obligation.

Signature:

(*Exams without this signature will not be graded.*)

1) Consider the following set S of vectors in $\mathbb{R}^4$. Explain why S is linearly dependent without doing any calculations. Next, give a basis for the subspace $W = \text{Span}\, S$ and use this basis for W to express one of the vectors in S as a linear combination of ***others*** in S. (No fair saying a vector is 1 times itself.) Finally, what is the dimension of $W = \text{Span}\, S$?

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\} \tag{1}$$

You may use the fact that

$$A := \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 3 & 3 & 1 \\ 2 & 2 & 4 & 2 & 2 \\ 0 & 1 & -1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 3 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} =: B, \tag{2}$$

where the tilde ($\sim$) indicates "row equivalent to".

<u>15pts</u>

<u>Solution</u>

S is 5 vectors from $\mathbb{R}^4$, so since $5 > 4 = \dim \mathbb{R}^4$, theorem, S is dependent. Next, since B is in reduced row echelon form, its pivot columns (clearly) define a basis for its column space, all other columns linear combinations then of these special columns. These pivot columns are its first, second and fifth. And since, theorem, row reduction does not alter the linear relationships among columns of a matrix, the associated columns of A are a basis for the column space of A: the first, second and fifth columns of A give a basis for the column space of A. So since these columns are the first, second and fifth elements of S, it must be that the first, second and fifth elements of S are a basis for $W = \text{Span}\, S$: a basis for $W = \text{Span}\, S$ is the set

$$S' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}. \tag{3}$$

Here we see then that $\dim W = \dim \text{Span}\, S = \dim \text{Span}\, S' = |S'| = 3$, which answers the last question.

With the theory just presented, we have that since

$$\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5\end{bmatrix} := \begin{bmatrix} 1 & 0 & 3 & 3 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

gives

$$\mathbf{b}_3 = (3)\mathbf{b}_1 + (-1)\mathbf{b}_2 = 3\mathbf{b}_1 - \mathbf{b}_2, \;\; \mathbf{b}_4 = (3)\mathbf{b}_1 + (-2)\mathbf{b}_2 = 3\mathbf{b}_1 - 2\mathbf{b}_2, \tag{5}$$

it must be that

$$\begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \;\; \begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \tag{6}$$

either one of which two statements answering then the second question.

2) Determine bases for both the image and kernel of (left multiplication by) A, where

$$A := \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 3 & 3 & 1 \\ 2 & 2 & 4 & 2 & 2 \\ 0 & 1 & -1 & -2 & 1 \end{bmatrix}. \tag{7}$$

<u>15 points</u>

<u>Solution</u>

Since, by definition of matrix multiplication (from the left),

$$\begin{aligned} A\mathbf{x} = \left[\mathbf{c}_1 \middle| \ldots \middle| \mathbf{c}_n\right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} &= x_1\mathbf{c}_1 + \ldots x_n\mathbf{c}_n \in \operatorname{Span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\} \\ &:= \{x_1\mathbf{c}_1 + \ldots x_n\mathbf{c}_n : x_1, \ldots, x_n \in \mathbb{R}\}, \end{aligned} \tag{8}$$

then clearly the image of $\mathbf{x} \mapsto A\mathbf{x}$ is the column space of A. So from the previous problem we have that a basis for the image of (left multiplication by) A is

$$S' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}. \tag{9}$$

Now the kernel of (left multiplication by) A is unaltered by row reduction (row reduction does not change the solution space of a system of equations—which is why we use it to solve them), so the kernel of A is the kernel of B in problem 1), the latter exposed by realizing that the structure of B there dictates that for $\mathbf{x} = (x_1, \ldots, x_5)^T \in \ker B$ we have

$$\begin{gathered} x_1 + 3x_3 + 3x_4 = 0 = x_2 - x_3 - 2x_4 = 0 = x_5 \\ \Leftrightarrow \\ \begin{aligned} x_1 &= -3x_3 - 3x_4 \\ x_2 &= x_3 + 2x_4 \\ x_5 &= 0, \end{aligned} \end{gathered} \tag{10}$$

i.e.,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3x_3 - 3x_4 \\ x_3 + 2x_4 \\ x_3 \\ x_4 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in \operatorname{Span} \left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} =: \operatorname{Span} S'', \tag{11}$$

where then S'', which is clearly linearly independent, is a basis for the kernel A.

3) Find the standard matrix for the following linear operator on $\mathbb{R}^3$: A rotation of 180° counter clockwise about the z axis, followed by a rotation of 90° counter clockwise about the y axis, followed by a rotation of 270° counter clockwise about the x axis.

15pts

Solution

By theorem we have that for linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$[T] = \left[T(\hat{x}) \middle| T(\hat{y}) \middle| T(\hat{z}) \right] = \left[T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \middle| T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] \tag{12}$$

Now $\hat{x} = (1,0,0)^T$ is sent to $-\hat{x} = (-1,0,0)^T$ by the 180° counter clockwise rotation about the z axis, and the rotation of 90° counter clockwise about the y axis sends it to $\hat{z} = (0,0,1)^T$, and then the rotation of 270° counter clockwise about the x axis sends this to $\hat{y} = (0,1,0)^T$. Similarly, $\hat{y} = (0,1,0)^T$ is changed to $-\hat{y} = (0,-1,0)^T$ by the 180° counter clockwise rotation about the z axis, and the latter is unchanged by a rotation about the y axis, which then goes to $\hat{z} = (0,0,1)^T$ via a rotation of 270° counter clockwise about the x axis. Finally $\hat{z} = (0,0,1)^T$ is unchanged by a rotation about its axis, which then is changed to $\hat{x} = (1,0,0)^T$ by a rotation of 90° counter clockwise about the y axis, which then is fixed by any rotation by that axis. Thus,

$$[T] = \left[T(\hat{x}) \middle| T(\hat{y}) \middle| T(\hat{z})\right] = \left[\hat{y} \middle| \hat{z} \middle| \hat{x}\right] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{13}$$

If these operations are composed in the opposite order, one would get the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix},$$

which is incorrect.

4) Let $T : V \to W$ be a linear mapping from real vector space V to real vector space W. DENOTE THE ADDITION IN V by $\oplus_V$, and DENOTE THE ADDITION IN W by $\oplus_W$ (since they are not in general the same). Likewise, DENOTE SCALAR MULTIPLICATION IN V as in $k \odot_V \mathbf{v}$, and DENOTE SCALAR MULTIPLICATION IN W as in $k \odot_W \mathbf{w}$ (since they are not in general the same). With this way of denoting things, carefully write down the implication defining the linearity of T. That is, fill in the "blanks" in the following: $T : V \to W$ is "linear" iff

$$S_1 \Rightarrow S_2. \tag{14}$$

(Carefully/correctly fill in statement S_1 and statement S_2 above, using the two distinct notations for addition and scalar multiplication in the two different real vector spaces V and W. If we weren't being too terribly careful about distinguishing notions of addition and notions of scalar multiplication we would simply write (14) as

$$\mathbf{u}, \mathbf{v} \in V, \alpha, \beta \in \mathbb{R} \Rightarrow T(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v}). \tag{15}$$

So modify (15) somewhat to be more careful about these things.)

15pts

Solution

Linearity for a map between real vector spaces is

$$\mathbf{u},\mathbf{v}\in V,\alpha,\beta\in\mathbb{R}\Rightarrow T\left(\left(\alpha\odot_V\mathbf{u}\right)\oplus_V\left(\beta\odot_V\mathbf{v}\right)\right)=\left(\alpha\odot_W T\left(\mathbf{u}\right)\right)\oplus_W\left(\beta\odot_W T\left(\mathbf{v}\right)\right). \quad (16)$$

Here we have been careful to indicate order of operations as well: $\oplus_V$, for example, must be done AFTER the indicated scalar multiplications are formed.

5) Let $T:V\to W$ be defined by formula

$$T\left(\underline{x}\right):=\underline{e^x}, \quad (17)$$

where $V=\mathbb{R}=\mathbb{R}^1$(the vector space of real numbers, addition ordinary, scalar multiplication ordinary: $\underline{x}\oplus_V\underline{y}:=\underline{x+y}$, $+$on the right the ordinary "grade school" thing, $\alpha\odot_V\underline{x}:=\underline{\alpha x}:=\underline{\alpha\times x}$, $\times$on the right the ordinary grade school thing), but where $W=\left\{\underline{w}:w>0\right\}$ (the set of positive numbers) and addition and scalar multiplication are defined (strangely) in W by $\underline{v}\oplus_W\underline{w}:=\underline{vw}:=\underline{v\times w}$ ($\times$indicates ordinary multiplication) and by $\alpha\odot_W\underline{w}:=\underline{w^\alpha}$ (which is raising w to power α). (We showed this set and notions of addition and scalar multiplication make a vector space.) Show that T defined by (17) is linear given the interesting/strange notion of addition and scalar multiplication in W (and the ordinary one in V). [This is interesting because (17) defines a ***nonlinear*** map between V and itself.] Note that the previous problem sets you up to think about this carefully.

15pts

Solution

Linearity is (16), which we attempt to show: by definition of V, $\mathbf{u},\mathbf{v}\in V$ iff $\mathbf{u}=\underline{x},\mathbf{v}=\underline{y}$ for $x,y\in\mathbb{R}$. And by definitions of addition and scalar multiplication in the various vector spaces, together with defining formula (17), we have

$x, y \in \mathbb{R}, \alpha, \beta \in \mathbb{R} \Rightarrow (\alpha \odot_V \underline{x}) \oplus_V (\beta \odot_V \underline{y}) \in V = \mathbb{R}^1$ (by vector space closure under these operations), so that $T\left((\alpha \odot_V \underline{x}) \oplus_V (\beta \odot_V \underline{y})\right)$ is well defined, and then specifically implies

$$T\left((\alpha \odot_V \underline{x}) \oplus_V (\beta \odot_V \underline{y})\right) \overset{\text{scalar multiplication}}{:=} T\left(\underline{\alpha x} \oplus_V \underline{\beta y}\right) \overset{\text{vector addition}}{:=} T\left(\underline{\alpha x + \beta y}\right)$$
$$\overset{\text{definition of } T}{:=} \underline{\exp(\alpha x + \beta y)} \overset{\text{properties of the exponential}}{=} \underline{(\exp(x))^\alpha (\exp(y))^\beta} \overset{\text{vector addition}}{=:} \underline{(\exp(x))^\alpha} \oplus_W \underline{(\exp(y))^\beta}$$
$$\overset{\text{scalar multiplication}}{=:} \left(\alpha \odot_W \underline{\exp(x)}\right) \oplus_W \left(\beta \odot_W \underline{\exp(y)}\right) \overset{\text{definition of } T}{=:} \left(\alpha \odot_W T(\underline{x})\right) \oplus_W \left(\beta \odot_W T(\underline{y})\right), \tag{18}$$

i.e.,

$$\mathbf{u}, \mathbf{v} \in V, \alpha, \beta \in \mathbb{R} \Rightarrow T\left((\alpha \odot_V \mathbf{u}) \oplus_V (\beta \odot_V \mathbf{v})\right) = \left(\alpha \odot_W T(\mathbf{u})\right) \oplus_W \left(\beta \odot_W T(\underline{\mathbf{v}})\right), \tag{19}$$

which is linearity (16).

6) Let $T : V \to W$ be linear (V and W finite dimensional vector spaces). Recall

$$\operatorname{Im} T := \left\{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\right\} \tag{20}$$

is a subspace of W (hence nonempty, closed under linear combination). Assuming $\operatorname{Im} T \neq \{\mathbf{z}_W\}$ ($\mathbf{z}_W$ denotes the additive identity in W), we get $\dim \operatorname{Im} T \geq 1$ (and $\dim \operatorname{Im} T \leq \dim W < \infty$), and by previous theorem get that $\operatorname{Im} T$ has a basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ with, as indicated, $m \geq 1$ (and $m \leq \dim W < \infty$). (Note $m \geq 1$ means here that the set is not empty.) Since each of these $\mathbf{w}$'s is in the image of T, as per (20), each one of them is a "T of something": there exist $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ such that $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\} = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_m)\}$. So $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_m)\}$ is a basis for $\operatorname{Im} T$, hence linearly independent, etc. Show that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is also linearly independent.

15pts

Solution

Since a basis $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_m)\}$ is independent, we certainly have

$$k_1 T(\mathbf{v}_1) + \ldots + k_m T(\mathbf{v}_m) = \mathbf{z}_W \Rightarrow k_1 = \ldots = k_m = 0. \tag{21}$$

We are hoping that this implication implies the implication

$$k_1\mathbf{v}_1+\ldots+k_m\mathbf{v}_m=\mathbf{z}_V \Rightarrow k_1=\ldots=k_m=0\,, \tag{22}$$

so that $\{\mathbf{v}_1,\ldots,\mathbf{v}_m\}$ is also independent. So we start on the left-hand side of (22) and see if we can pass to the right hand side of (22) using (21) and the linearity of T. This is no problem:

$$\begin{aligned} k_1\mathbf{v}_1+\ldots+k_m\mathbf{v}_m=\mathbf{z}_V \quad &\overset{\substack{\text{do the same thing to}\\ \text{both sides of an equation}}}{\Rightarrow} \quad T\left(k_1\mathbf{v}_1+\ldots+k_m\mathbf{v}_m\right)=T\left(\mathbf{z}_V\right)\overset{T\text{ linear}}{=}\mathbf{z}_W \\ &\overset{T\text{ linear}}{\Leftrightarrow} k_1T\left(\mathbf{v}_1\right)+\ldots+k_mT\left(\mathbf{v}_m\right)=\mathbf{z}_W \\ &\Rightarrow k_1=\ldots=k_m=0 \end{aligned} \tag{23}$$

the last step by the given independence statement (21).

7) As above assume $T:V\to W$ is linear, V and W finite dimensional vector spaces. Recall

$$\ker T:=\left\{\mathbf{v}\in V:T\left(\mathbf{v}\right)=\mathbf{z}_W\right\} \tag{24}$$

is a subspace of V, hence $\dim\ker T\le\dim V<\infty$, i.e the kernel of T is finite dimensional (and no bigger than that of V). Now, similar to the last problem, assume $\ker T\ne\{\mathbf{z}_V\}$ ($\mathbf{z}_V$ denotes the additive identity in V), so that there is a basis for $\ker T$: assume a basis $\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ for the kernel of T, with, as indicated, $n\ge1$(and $n\le\dim V<\infty$). (Note $n\ge1$means here that the set is not empty.) Forgetting these kernel ideas for a moment, and using the basis $\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_m)\}$ of $\operatorname{Im}T$ introduced in the previous problem, we see that for every $\mathbf{v}\in V$ there are scalars $a_1,\ldots,a_m$ such that

$$T\left(\mathbf{v}\right)=a_1T\left(\mathbf{v}_1\right)+\ldots+a_mT\left(\mathbf{v}_m\right). \tag{25}$$

Using T's linearity and the vector space axioms we see that (25) is equivalent to

$$T\left(\mathbf{v}-\left(a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m\right)\right)=\mathbf{z}_W. \tag{26}$$

So then show that

$$V=\operatorname{Span}\left\{\mathbf{v}_1,\ldots,\mathbf{v}_m,\mathbf{u}_1,\ldots,\mathbf{u}_n\right\}. \tag{27}$$

Finally, show that $\{\mathbf{v}_1,\ldots,\mathbf{v}_m,\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ is linearly independent, hence show that

$$\dim V=\dim\operatorname{Im}T+\dim\ker T\,. \tag{28}$$

[(28) is the “dimension theorem”, which can be used in subsequent problems. It works even if one or more of the indicated dimensions are zero, which we precluded in deriving it. It even works if $\dim V = 0$, but in that particular case, since $\dim \operatorname{Im} T, \dim \ker T \geq 0$, we must have $\dim \operatorname{Im} T = \dim \ker T = 0 = \dim V$, the former statement giving that T is the zero map (it kills “everything”), and this despite the fact that the second statement says that it *only* kills zero. You can thank me later for handing this theorem to you—instead of asking that you remember it.)

15pts

Solution

We want to show (27), which says both a) for every $\mathbf{v} \in V$ there are scalars $a_1, \ldots, a_m, c_1, \ldots, c_n$ such that

$$\mathbf{v} = a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m + c_1\mathbf{u}_1 + \ldots + c_n\mathbf{u}_n, \tag{29}$$

(so that

$$\begin{aligned} &\mathbf{v} \in V \Rightarrow \mathbf{v} \in \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_n\} \\ &\quad := \{a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m + c_1\mathbf{u}_1 + \ldots + c_n\mathbf{u}_n : a_1, \ldots, a_m, c_1, \ldots, c_n \in \mathbb{R} \text{ or } \mathbb{C}\}, \end{aligned} \tag{30}$$

i.e., so that $V \subset \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_n\}$) and also says b) for every list of scalars $a_1, \ldots, a_m, c_1, \ldots, c_n$, the right hand side of (29) is an element of V (so that $V \supset \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_n\}$). b) is easy: since V is a vector space, and since $\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_n$ are all elements of V, then, by closure of vector spaces under linear combination, every object of the form of the right hand side of (29) is also an element of V. a) is harder. It comes from (26), which, given the definition of kernel, says

$$\mathbf{v} - (a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m) \in \ker T. \tag{31}$$

Since $\ker T$ has basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, we have

$$\ker T = \operatorname{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\} := \{c_1\mathbf{u}_1 + \ldots + c_n\mathbf{u}_n : c_1, \ldots, c_n \in \mathbb{R} \text{ or } \mathbb{C}\}, \tag{32}$$

and (31) indicates then that for every $\mathbf{v} \in V$ there are scalars $c_1, \ldots, c_n$ such that

$$\mathbf{v} - (a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m) = c_1\mathbf{u}_1 + \ldots + c_n\mathbf{u}_n, \tag{33}$$

which is equivalent to the claim associated with (29) after using the vector space axioms to move the $\mathbf{v}$'s from one side of the $=$ to the other side: We have just shown (27).

We now want to show that $\{\mathbf{v}_1,\ldots,\mathbf{v}_m,\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ is linearly independent, i.e. show that

$$a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V \Rightarrow a_1=\ldots=a_m=c_1=\ldots=c_n=0. \tag{34}$$

So we start on the left hand side of (34) and try to find a path to the right hand side, given that a) $\{\mathbf{v}_1,\ldots,\mathbf{v}_m\}$ is linearly independent, b) $\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ is linearly independent and spans the kernel of T, and c) that $\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_m)\}$ is linearly independent and spans the image of T. Perhaps we will only use some of these facts at this late stage. Let's see.

Applying linear $T:V\to W$ to the left hand side of (34) we get

$$\begin{gathered} a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V \Rightarrow \\ a_1T(\mathbf{v}_1)+\ldots+a_mT(\mathbf{v}_m)+c_1T(\mathbf{u}_1)+\ldots+c_nT(\mathbf{u}_n)=T(\mathbf{z}_V)=\mathbf{z}_W \end{gathered} \tag{35}$$

Then using that each of the $\mathbf{u}$'s is in the kernel of T, that scalar multiples of the additive identity $\mathbf{z}_W$ give $\mathbf{z}_W$, that sums of $\mathbf{z}_W$ give $\mathbf{z}_W$, and that $\mathbf{z}_W$ "does nothing" to anything in W, we get (35) becomes

$$\begin{gathered} a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V \Rightarrow \\ a_1T(\mathbf{v}_1)+\ldots+a_mT(\mathbf{v}_m)=\mathbf{z}_W. \end{gathered} \tag{36}$$

But since $\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_m)\}$ is linearly independent, we have

$$\begin{gathered} a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V \Rightarrow \\ a_1T(\mathbf{v}_1)+\ldots+a_mT(\mathbf{v}_m)=\mathbf{z}_W \Rightarrow a_1=\ldots=a_m=0, \end{gathered} \tag{37}$$

i.e., in short,

$$a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V \Rightarrow a_1=\ldots=a_m=0. \tag{38}$$

But since a statement implies itself, we could also write this as

$$\begin{aligned} &a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V \Rightarrow \\ &a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V, a_1=\ldots=a_m=0 \Rightarrow \\ &c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V, a_1=\ldots=a_m=0, \end{aligned} \tag{39}$$

the last implication following by the fact that the zero multiple of anything in V is $\mathbf{z}_V$, that sums of $\mathbf{z}_V$ give $\mathbf{z}_V$, and/or that $\mathbf{z}_V$ does nothing to anything in V. But then since $\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ is independent, we then have

$$\begin{aligned} a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V &\Rightarrow c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V, a_1=\ldots=a_m=0 \\ &\Rightarrow a_1=\ldots=a_m=0=c_1=\ldots=c_m, \end{aligned} \tag{40}$$

i.e.,

$$a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m+c_1\mathbf{u}_1+\ldots+c_n\mathbf{u}_n=\mathbf{z}_V \Rightarrow a_1=\ldots=a_m=0=c_1=\ldots=c_m \tag{41}$$

which is the same as (34).

So now with $\{\mathbf{v}_1,\ldots,\mathbf{v}_m,\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ linearly independent (which implies all nonempty subsets are independent, including the important ones we've thought about recently) and with (27), we have

$$\begin{aligned} \dim V &= \dim \operatorname{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_m,\mathbf{u}_1,\ldots,\mathbf{u}_n\} = \left|\{\mathbf{v}_1,\ldots,\mathbf{v}_m,\mathbf{u}_1,\ldots,\mathbf{u}_n\}\right| = m+n \\ &= \left|\{\mathbf{v}_1,\ldots,\mathbf{v}_m\}\right| + \left|\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}\right| = \left|\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_m)\}\right| + \left|\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}\right| \\ &= \dim \operatorname{Span}\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_m)\} + \dim \operatorname{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_n\} \\ &= \dim \operatorname{Im} T + \dim \ker T, \end{aligned} \tag{42}$$

which is the desired (28), i.e. the dimension theorem. In (42) we also used $\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_m)\}$ is linearly independent.

8) Explain why it is that if A is a (real) $m \times n$ matrix with $n > m$ (≥ 1), then the kernel of A, i.e.

$$\ker A := \left\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0} \in \mathbb{R}^m\right\}, \tag{43}$$

can't just be the zero vector. (That is, show that $\ker A \neq \left\{\mathbf{0} \in \mathbb{R}^n\right\}$). Hint: use the dimension theorem. (Aren't you glad I reminded you of that?)

15pts

Solution

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, so that T is a linear map from a finite dimensional vector space to another one, and so that $\ker A = \ker T$. The dimension theorem (28) in this context says that

$$n = \dim V = \dim \operatorname{Im} T + \dim \ker T = \dim \operatorname{Im} T + \dim \ker A \le m + \dim \ker A. \qquad (44)$$

Here we also used that $\operatorname{Im} T$ is a subspace of $\mathbb{R}^m$, so that $\dim \operatorname{Im} T \le \dim \mathbb{R}^m = m$. So then with $n > m \Leftrightarrow n - m \ge 1$ [since these are (nonnegative) integers], from (44) we have

$$\dim \ker A \ge n - m \ge 1 \Rightarrow \dim \ker A \ge 1 \qquad (45)$$

and $\ker A$ can't be $\{\mathbf{0} \in \mathbb{R}^n\}$ (which we say has dimension 0).

9) If W is a subspace of a finite dimensional vector space V, show that if $\dim W = \dim V$, then in fact $W = V$, i.e. both $W \subset V$ and, more interesting/telling, $W \supset V$ also. [I'll do the first part for you: the definition of subspace is that it is (first and foremost) a (nonempty) subset, hence $W \subset V$. So now move on to show $W \supset V$.]

15pts

Solution

To show $W \supset V$, we must show that every vector $\mathbf{v}$ in V is also in W. Now W and V are both finite dimensional vector spaces (in fact W has the same dimension as V does, which is finite), and, so, both have bases provided $V \ne \{\mathbf{z}_V\}$; write $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ in the case of W ($\subset V$), $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in the case of V, with $n \in \mathbb{N} = \{1, 2, \ldots\}$. (If $V = \{\mathbf{z}_V\}$, so that $\dim V = 0$ and V does not have a basis, then, since $W \subset V$ and $W \ne \varnothing$, then $W = \{\mathbf{z}_V\} = V$ and we have our result.) Here we have used the same cardinality $n = \dim V = \dim W$. ($n \in \mathbb{N} = \{1, 2, \ldots\}$)So since

$$W = \operatorname{Span}\{\mathbf{w}_1, \ldots, \mathbf{w}_n\} := \{c_1\mathbf{w}_1 + \ldots + c_n\mathbf{w}_n : c_1, \ldots, c_n \text{ scalars}\}, \qquad (46)$$

it becomes our goal to show that for every $\mathbf{v} \in V$ there are scalars $c_1, \ldots, c_n$ such that

$$\mathbf{v} = c_1\mathbf{w}_1 + \ldots + c_n\mathbf{w}_n. \qquad (47)$$

Suppose this were not the case, i.e. that for at least one particular/special $\mathbf{v} = \mathbf{v}_{\text{special}} \in V$, we have $\mathbf{v}_{\text{special}}$ cannot be written as on the right hand side of (47), i.e. cannot be written as a linear combination of the $\mathbf{w}$'s. (Clearly $\mathbf{v}_{\text{special}} \ne \mathbf{z}_V = \mathbf{z}_W = \mathbf{z}$ since we could then use zero scalars.) Then since $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a linearly independent set of vectors in W hence in V, the set $\{\mathbf{w}_1,\ldots,\mathbf{w}_n,\mathbf{v}_{\text{special}}\}$ would be a linearly independent set of vectors in V. But this last occurrence is impossible: it is impossible to have a linearly independent set of vectors with cardinality $n+1$ in an n dimensional vector space. Thus every vector $\mathbf{v}\in V$ is of the form indicated in (47) and $W\supset V$. Combined with $W\subset V$ we have then $W=V$.

Here we used a couple of big guns/theorems. Alternatively we could a) describe why $\{\mathbf{w}_1,\ldots,\mathbf{w}_n,\mathbf{v}_{\text{special}}\}$ is linearly independent (suppose it isn't and get contradiction with $\mathbf{v}_{\text{special}}$ is not of the form (47) yet $\{\mathbf{w}_1,\ldots,\mathbf{w}_n\}$ is independent), and then b) describe why it is impossible to have a set of $n+1$ linearly independent vectors in an n dimensional space (suppose you could and use the basis with only n elements and find a nontrivial solution to the relevant "independence equation").

10) Let $\{\mathbf{b}_1,\ldots,\mathbf{b}_n\}$ be a set of $n\geq 1$ vectors in a real innerproduct space, and construct a real matrix $G=G_{n\times n}$ of inner products of these $\mathbf{b}$'s as follows:

$$G_{ij}:=\langle \mathbf{b}_i,\mathbf{b}_j\rangle. \tag{48}$$

Show that the set of vectors $\{\mathbf{b}_1,\ldots,\mathbf{b}_n\}$ is linearly independent if and only if $\operatorname{Rank}G=n$, i.e., using the dimension theorem, if and only if $\operatorname{nullity}(G):=\dim\ker G=0$ (i.e. if and only if $G\mathbf{x}=\mathbf{0}\Rightarrow\mathbf{x}=\mathbf{0}$). (I am making life easiest on you in writing the last statement.)

<u>15pts</u>

<u>Solution</u>

We have $\{\mathbf{b}_1,\ldots,\mathbf{b}_n\}$ is linearly independent if and only if

$$\sum_{j=1}^{n}x_j\mathbf{b}_j = x_1\mathbf{b}_1+\ldots+x_n\mathbf{b}_n=\mathbf{z}\Rightarrow x_1=\ldots=x_n=0. \tag{49}$$

Now the first statement here implies that for $i\in\{1,\ldots,n\}$

$$\left\langle \mathbf{b}_i,\sum_{j=1}^{n}x_j\mathbf{b}_j\right\rangle=\sum_{j=1}^{n}x_j\langle\mathbf{b}_i,\mathbf{b}_j\rangle=:\sum_{j=1}^{n}x_jG_{ij}=\sum_{j=1}^{n}G_{ij}x_j=(G\mathbf{x})_i=\langle\mathbf{b}_i,\mathbf{z}\rangle=0. \tag{50}$$

Here we used the linearity axiom(s) of the real inner product. So we have shown

$$i \in \{1,\ldots,n\}, \sum_{j=1}^{n} x_j \mathbf{b}_j = \mathbf{z} \Rightarrow i \in \{1,\ldots,n\}, \left\langle \mathbf{b}_i, \sum_{j=1}^{n} x_j \mathbf{b}_j \right\rangle = (G\mathbf{x})_i = 0. \tag{51}$$

Now start on the right hand side of (51). Using the linearity of the inner product we find this right hand side implies

$$0 = x_i \cdot 0 = x_i \left\langle \mathbf{b}_i, \sum_{j=1}^{n} x_j \mathbf{b}_j \right\rangle = \left\langle x_i \mathbf{b}_i, \sum_{j=1}^{n} x_j \mathbf{b}_j \right\rangle \tag{52}$$

for each i, and then

$$0 = \sum_{i=1}^{n} 0 = \sum_{i=1}^{n} \left\langle x_i \mathbf{b}_i, \sum_{j=1}^{n} x_j \mathbf{b}_j \right\rangle = \left\langle \sum_{i=1}^{n} x_i \mathbf{b}_i, \sum_{j=1}^{n} x_j \mathbf{b}_j \right\rangle = \left\langle \sum_{j=1}^{n} x_j \mathbf{b}_j, \sum_{j=1}^{n} x_j \mathbf{b}_j \right\rangle \tag{53}$$

using linearity of the inner product again, and then changing the name of the dummy variable of summation in the last step (to make it more obvious we are taking an inner product of a vector with itself). But since innerproduct $\langle , \rangle$ is positive, (53) implies

$$\sum_{j=1}^{n} x_j \mathbf{b}_j = \mathbf{z} \tag{54}$$

which is the (main part of) left hand side of (51). So we have just shown that if $\langle , \rangle$ really is an innerproduct, then

$$\sum_{j=1}^{n} x_j \mathbf{b}_j = \mathbf{z} \Leftrightarrow (G\mathbf{x})_i = 0 \text{ for } i \in \{1,\ldots,n\} \overset{\substack{\text{definition of zero} \\ \text{vector in } \mathbb{R}^n}}{\Leftrightarrow} G\mathbf{x} = \mathbf{0}. \tag{55}$$

Thus from (49) we have

$$(x_1 \mathbf{b}_1 + \ldots + x_n \mathbf{b}_n = \mathbf{z} \Rightarrow x_1 = \ldots = x_n = 0) \Leftrightarrow (G\mathbf{x} = \mathbf{0} \Rightarrow x_1 = \ldots = x_n = 0), \tag{56}$$

i.e.,

$$(x_1 \mathbf{b}_1 + \ldots + x_n \mathbf{b}_n = \mathbf{z} \Rightarrow x_1 = \ldots = x_n = 0) \Leftrightarrow (G\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}). \tag{57}$$

(57) says

$$\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \text{ is linearly independent iff } \ker G = \{\mathbf{0}\} \tag{58}$$

which was to be shown (in the last/easiest version of the problem statement).