Math 313 Midterm I KEY
Winter 2011
section 003
Instructor: Scott Glasgow

Write your name very clearly on this exam. In this booklet, write your mathematics clearly, legibly, in big fonts, and, most important, "have a point", i.e. make your work logically and even pedagogically acceptable. (Other human beings not already understanding 313 should be able to learn from your exam.) To avoid excessive erasing, first put your ideas together on scratch paper, then commit the logically acceptable fraction of your scratchings to this exam booklet. More is not necessarily better: say what you mean and mean what you say.

Honor Code: After I have learned of the contents of this exam by any means, I will not disclose to anyone any of these contents by any means until after the exam has closed. My signature below indicates I accept this obligation.

Signature:

(*Exams without this signature will not be graded.*)

1) True or False: the product $A^T A$ is always well-defined (for any size matrix A). Justify you answer.

5pts

Solution

True: the number of columns of the matrix on the left will always be the same as the number of rows of the matrix on the right, which is the only criterion that must be met to form the product.

2) Put the following matrix in reduced row-echelon form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 4 & -2 & 1 & 4 \end{bmatrix} \tag{1}$$

10pts

Solution

The row reduction may proceed as follows:

$$\begin{aligned} &\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 4 & -2 & 1 & 4 \end{bmatrix}^{\substack{R2-R1 \\ R3-4R1}} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & -6 & -3 & 0 \end{bmatrix}^{\substack{R2/(-2) \\ R3/(-3)}} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}^{R3-2R2} \\ &\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}^{R1-R3} \sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}^{R1-R2} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}. \end{aligned} \tag{2}$$

However the row reduction proceeds, the row echelon form is unique—the last matrix indicated in (2) is ***the*** answer.

3) Solve the following system of 2 equations and 2 unknowns by performing Gauss-Jordon elimination on the relevant augmented matrix.

$$\begin{aligned} x + y &= 3, \\ x + 2y &= 5. \end{aligned} \tag{3}$$

5pts

Solution

The augmented matrix and its row reduction appear below:

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \end{bmatrix} \overset{R_2-R_1}{\sim} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix} \overset{R_1-R_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}. \tag{4}$$

Thus the solution to (3) is $(x, y) = (1, 2)$.

4) Assuming A and B are invertible matrices of the same size, show that

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{5}$$

10 points

Solution

$B^{-1}A^{-1}$ is the inverse of AB[1] if and only if

$$(AB)\left(B^{-1}A^{-1}\right) = \left(B^{-1}A^{-1}\right)(AB) = I, \tag{6}$$

to whit we first note that, by the associative property of matrix multiplication,

$$\begin{gathered} (AB)\left(B^{-1}A^{-1}\right) = \left(A\left(BB^{-1}\right)\right)A^{-1} \\ \text{and} \\ \left(B^{-1}A^{-1}\right)(AB) = \left(B^{-1}\left(A^{-1}A\right)\right)B. \end{gathered} \tag{7}$$

Then, by the definition of the inverses, in particular that an inverse is both a right and a left inverse, we have, respectively, that

$$\begin{gathered} (AB)\left(B^{-1}A^{-1}\right) = \left(A\left(BB^{-1}\right)\right)A^{-1} = (AI)A^{-1} \\ \text{and} \\ \left(B^{-1}A^{-1}\right)(AB) = \left(B^{-1}\left(A^{-1}A\right)\right)B = \left(B^{-1}I\right)B. \end{gathered} \tag{8}$$

Using now the fact that the identity matrix is in fact the "multiplicative identity" *from either side* we get

[1] Here we used the definite article "the", as in "the inverse", since it can be shown that all such indicated inverses are the same.

$$\begin{gathered}(AB)\left(B^{-1}A^{-1}\right)=(AI)A^{-1}=AA^{-1}\\ \text{and}\\ \left(B^{-1}A^{-1}\right)(AB)=\left(B^{-1}I\right)B=B^{-1}B.\end{gathered}\tag{9}$$

Finally we use again the definition of the inverses. In particular, using that an inverse is both a right and a left inverse, we have, respectively, that

$$\begin{gathered}(AB)\left(B^{-1}A^{-1}\right)=AA^{-1}=I\\ \text{and}\\ \left(B^{-1}A^{-1}\right)(AB)=B^{-1}B=I,\end{gathered}\tag{10}$$

which is the required (6).

5) Show that if a matrix A is invertible, the system $A\mathbf{x}=\mathbf{b}$ has one and only one solution $\mathbf{x}$, namely $\mathbf{x}=A^{-1}\mathbf{b}$. Here, as one of you pointed out, it is important to note that there is really only one such inverse, i.e. A^{-1} denotes only one object. [Warning: there are two things to prove here, namely a) that ***if*** the system has a solution, then it can only be $\mathbf{x}=A^{-1}\mathbf{b}$, and that b) $\mathbf{x}=A^{-1}\mathbf{b}$ actually does solve the system. Here then you will have addressed the "one and only one" issues in reverse order: you may first show that a) there is at most one solution, and b) that there is in fact one solution (rather than none). In parts a) and b) you will use that A^{-1} is A's left and right inverse, ***respectively***.]

10 points

<u>Solution</u>

If the system has a solution $\mathbf{x}$, then, for any such $\mathbf{x}$, we may certainly write

$$A\mathbf{x}=\mathbf{b}\tag{11}$$

without implicitly lying, and then, by left application of A^{-1} to (11), as well as by the associative property of matrix multiplication, obtain that

$$\mathbf{x}=I\mathbf{x}=\left(A^{-1}A\right)\mathbf{x}=A^{-1}(A\mathbf{x})=A^{-1}\mathbf{b}\ .\tag{12}$$

In (12) we also used that A^{-1} is a left inverse of A, as well as the fact that the so-called identity matrix I is in fact a "multiplicative identity". Here then we have just showed that ***if*** (11) has a solution, it's got to be $\mathbf{x}=A^{-1}\mathbf{b}$. Thus we have showed that (11) has at most one solution. But our demonstration does not yet preclude there being no solution. To preclude that possibility, we confirm that, for the only promising candidate, namely $\mathbf{x} = A^{-1}\mathbf{b}$, we get

$$A\mathbf{x} = A\left(A^{-1}\mathbf{b}\right) = \left(AA^{-1}\right)\mathbf{b} = I\mathbf{b} = \mathbf{b}, \tag{13}$$

so that our candidate was successful. (Here we have used the associative property of matrix multiplication, the fact that A^{-1} is a right inverse of A, as well as the fact that the so-called identity matrix I is in fact a "multiplicative identity".) Thus we have showed that the system has one and only one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

6) Assume that both the matrix B and the matrix C are inverses of the matrix A. Show that B and C are just two aliases for the same matrix, i.e. show that in fact $B = C$.

10 points

<u>Solution</u>

The descriptions of B and C demand that

$$AB = BA = I = AC = CA. \tag{14}$$

Using the associative property of matrix multiplication in two different ways on the product BAC we get

$$\begin{gathered} BAC = B(AC) = BI = B \\ \text{and} \\ BAC = (BA)C = IC = C, \end{gathered} \tag{15}$$

so that indeed

$$\begin{gathered} B = BAC = C \\ \Rightarrow \\ B = C \end{gathered} \tag{16}$$

as claimed. Note that in (15) we also used that a) C is a right inverse of A, b) B is a left inverse of A, and that c) the identity matrix acts as both a right and left multiplicative identity. One can alternately approach this problem by considering the product CAB, but then by using that a) C is a left inverse of A, b) B is a right inverse of A, and again that c) the identity matrix acts as both a right and left multiplicative identity.

7) Find the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{17}$$

by row reducing $\left[A|I\right]$ to $\left[I|A^{-1}\right]$. Assume the parameters $a,b,c,$ and d do not take on any special values, nor have a special relationship among them—that is row reduce naively, without worrying about any divisions by hidden zeros.

10 points

<u>Solution</u>

The naïve row reduction mentioned may proceed as follows:

$$\begin{aligned}\left[A|I\right] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]^{aR2-cR1} &\sim \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right]^{(ad-bc)R1-bR2} \\ \sim \left[\begin{array}{cc|cc} a(ad-bc) & 0 & ad & -ab \\ 0 & ad-bc & -c & a \end{array}\right]^{R1/a} &\sim \left[\begin{array}{cc|cc} ad-bc & 0 & d & -b \\ 0 & ad-bc & -c & a \end{array}\right]^{R1/a} \\ \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ 0 & 1 & \end{array}\right]^{R1/a} &= \left[I|A^{-1}\right].\end{aligned} \tag{18}$$

8) For the given set of objects, together with the indicated notions of addition and scalar multiplication, determine whether each of the ten vector space axioms holds: real triples (x,y,z), where

$$(x,y,z)+(x',y',z') := (x+x', y+y', z+z'), \quad k(x,y,z) := (kx, y, z). \tag{19}$$

<u>15pts</u>

<u>Solution</u>

Closure axioms 1) and 6) hold because sums and products of real numbers give real numbers, and because on the right hand sides of equation (19) the objects are again triples of those real numbers. 2) through 5) will also hold, since they reference only vector addition, which in (19) is the standard notion (giving the 10 axioms as *theorems*). For axiom 7) we have

$$
\begin{aligned}
k\left(\left(x,y,z\right)+\left(x',y',z'\right)\right)=k\left(x+x',y+y',z+z'\right)&=\left(k\left(x+x'\right),y+y',z+z'\right)\\
&=\left(kx+kx',y+y',z+z'\right)=\left(kx,y,z\right)+\left(kx',y',z'\right)\\
&=\ k\left(x,y,z\right)+k\left(x',y',z'\right),
\end{aligned}
\tag{20}
$$

as required. But for axiom 8) we have the generally distinct results

$$
\begin{gathered}
\left(k+m\right)\left(x,y,z\right)=\left(\left(k+m\right)x,y,z\right)=\left(kx+mx,y,z\right),\ \text{and}\\
k\left(x,y,z\right)+m\left(x,y,z\right)=\left(kx,y,z\right)+\left(mx,y,z\right)=\left(kx+mx,y+y,z+z\right)=\left(kx+mx,2y,2z\right),
\end{gathered}
\tag{21}
$$

so that this axiom does not hold. Axioms 9) and 10) hold in an obvious way—essentially because the scalar multiplication is normal in the one slot it affects.

9) Prove that for any (real) vector space $\left(V,\mathbb{R},+,\bullet\right)$ (satisfying the ten axioms)—no matter how bizarre the addition $+$ and the scalar multiplication $\bullet$ are—we must have $k\bullet\mathbf{z}=\mathbf{z}$ for any scalar k ($\in\mathbb{R}$), where $\mathbf{z}$ is the additive identity. Be sure to list the axioms used in your proof. Feel free (but not compelled!) to use the fact that

$$
\begin{gathered}
\mathbf{w}+\mathbf{u}=\mathbf{u}\Rightarrow\mathbf{w}=\mathbf{z},\ \text{or}\\
\mathbf{u}+\mathbf{w}=\mathbf{u}\Rightarrow\mathbf{w}=\mathbf{z},
\end{gathered}
\tag{22}
$$

i.e. that if a vector $\mathbf{w}\in V$ acts like $\mathbf{z}\in V$ even for just one $\mathbf{u}\in V$, then it is $\mathbf{z}$. On the other hand, you may also do what you did in the relevant homework problem (which invents this fact for you).

<u>15pts</u>

<u>Solution</u>

By axiom 7)

$$
k\bullet\mathbf{z}+k\bullet\mathbf{z}=k\bullet\left(\mathbf{z}+\mathbf{z}\right),
\tag{23}
$$

and then by axiom 4) we get then

$$
k\bullet\mathbf{z}+k\bullet\mathbf{z}=k\bullet\left(\mathbf{z}+\mathbf{z}\right)=k\bullet\mathbf{z}.
\tag{24}
$$

But now this is (22) with $\mathbf{w}=k\bullet\mathbf{z}$ (and, less important, $\mathbf{u}=k\bullet\mathbf{z}$), so $\mathbf{w}=k\bullet\mathbf{z}=\mathbf{z}$ by the last parts of (22).

10) By use of the relevant theorem, determine whether the following is a subspace of M_{nn}: (M_{nn} is the vector space of $n\times n$ matrices with ordinary matrix addition and scalar multiplication.) the set of all $n\times n$ matrices A such that $AB=BA$ for

some fixed/specific/particular $n \times n$ matrix B. MAKE SURE AND REFERENCE AND **USE** THE THEOREM in determining your conclusion.

<u>15pts</u>

<u>Solution</u>

Since such A's certainly form a non-empty subset of M_{nn}, then, theorem, they form a subspace of M_{nn} iff for all scalars c and c' and for every A and A' in the set we have

$$\left(cA + c'A'\right)B = B\left(cA + c'A'\right). \tag{25}$$

Now (25) always holds and, so, the set is actually a subspace of M_{nn}: we have

$$\left(cA + c'A'\right)B = \left(c\left(AB\right) + c'\left(A'B\right)\right) = \left(c\left(BA\right) + c'\left(BA'\right)\right) = B\left(cA + c'A'\right) \tag{26}$$

by matrix algebra, together with A and A''s membership in the set (giving both $AB = BA$ and $A'B = BA'$).

11) By use of the relevant theorem, determine whether the following is (always) true or (sometimes) false: The intersection $W_1 \cap W_2$ of two subspaces W_1 and W_2 of a given vector space V is itself a subspace of V. MAKE SURE AND REFERENCE AND **USE** THE THEOREM in determining your conclusion.

<u>15pts</u>

<u>Solution</u>

Since subspaces are vector spaces and, so, by definition, non-empty, there is a chance that $W_1 \cap W_2$ is non-empty (so that we can apply the theorem mentioned in the previous problem). Indeed each of W_1 and W_2 contain the additive identity $\mathbf{z} \in V$, so that in fact $W_1 \cap W_2 \supset \{\mathbf{z}\} \neq \varnothing$. ($\varnothing$ is notation for the empty set.) So let c and c' be any two (real) scalars, and let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors in $W_1 \cap W_2$, and in particular consider whether the arbitrary linear combination $c\mathbf{u} + c'\mathbf{v}$ is in $W_1 \cap W_2$. Well, since $\mathbf{u}$ and $\mathbf{v}$ are both in subspace W_1, then, by closure of vector spaces under linear combinations, $c\mathbf{u} + c'\mathbf{v} \in W_1$, and then since $\mathbf{u}$ and $\mathbf{v}$ are both in subspace W_2, then, by closure of vector spaces under linear combinations, $c\mathbf{u} + c'\mathbf{v} \in W_2$. Consequently $c\mathbf{u} + c'\mathbf{v}$ is in $W_1 \cap W_2$: $W_1 \cap W_2$ is a non-empty subset of V closed under linear combination and, so, is a subspace of V.

12) Determine whether the following is (always) true or (sometimes) false: The set $S=\{\mathbf{u}-\mathbf{v},\mathbf{v}-\mathbf{w},\mathbf{w}-\mathbf{u}\}$ is linearly dependent, where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any three vectors. If it is true, prove it, otherwise give a counter example.

10pts

Solution

Consider the equation

$$k_1\cdot(\mathbf{u}-\mathbf{v})+k_2\cdot(\mathbf{v}-\mathbf{w})+k_3\cdot(\mathbf{w}-\mathbf{u})=\mathbf{z}. \tag{27}$$

where $\mathbf{z}$ denotes the additive identity in the relevant vector space. By the axioms of a vector space, the left-hand side of (27) can be rearranged until we get

$$(k_1-k_3)\cdot\mathbf{u}+(k_2-k_1)\cdot\mathbf{v}+(k_3-k_2)\cdot\mathbf{w}=\mathbf{z}. \tag{28}$$

Note that choosing $k_1=k_2=k_3=1\neq 0$, we get (by theorem) that (28) hence (27) holds. (Of course one could have just seen that (27) itself holds with this choice, by axiom 10, etc.) Thus, definition, $S=\{\mathbf{u}-\mathbf{v},\mathbf{v}-\mathbf{w},\mathbf{w}-\mathbf{u}\}$ is (always) linearly dependent.

13) For $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$, prove the triangle inequality $\|\mathbf{u}+\mathbf{v}\|\le\|\mathbf{u}\|+\|\mathbf{v}\|$. Note in your calculation the step at which you use the Cauchy-Schwarz inequality.

10 points

Solution

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})=\mathbf{u}\cdot\mathbf{u}+2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v}\\
&=\|\mathbf{u}\|^2+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2\\
&\le\|\mathbf{u}\|^2+2|\mathbf{u}\cdot\mathbf{v}|+\|\mathbf{v}\|^2\\
&\le\|\mathbf{u}\|^2+2\|\mathbf{u}\|\|\mathbf{v}\|+\|\mathbf{v}\|^2 \qquad \text{(Cauchy-Schwarz)}\\
&=\left(\|\mathbf{u}\|+\|\mathbf{v}\|\right)^2\\
&\Leftrightarrow\\
\|\mathbf{u}+\mathbf{v}\| &\le\|\mathbf{u}\|+\|\mathbf{v}\|.
\end{aligned} \tag{29}$$

14) Determine whether

$$\mathbf{x} \in Span\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \tag{30}$$

where

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}. \tag{31}$$

10 points

<u>Solution</u>

In problem 2) we showed that

$$\mathbf{x} = 2\mathbf{u} + 1\mathbf{v} - 2\mathbf{w} \in Span\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}, \tag{32}$$

that is

$$\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{33}$$

so in fact (30) holds.