Name: _______________

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Section: _002___________

Instructor: Scott Glasgow

Math 314 (Calculus of Several Variables) **RED**

Exam 1 May 8-9, 2013

Instructions:

- For questions which require a written answer, show all your work. Full credit will be given only if the necessary work is shown, justifying your answer.
- Simplify your answers.
- Calculators are not allowed. Textbooks are not allowed. Notes are not allowed.
- Should you need more space than is allotted to answer a question, use the back of the page the problem is on and indicate this fact.
- Talking about the exam with other students before the graded exam is returned to you is a violation of the Honor Code.

Part I: Multiple Choice *Mark the correct answer on the bubble sheet provided*.

1. (5 points) Let $h>0$. The sphere with radius $r=5h$ and center $(4h,k,l)$ intersects the yz plane in the circle with
   a) center $(0,k,l)$ and radius $5h$, b) center $(0,k,l)$ and radius $4h$, c) center $(0,k,l)$ and radius $3h$, d) center $(0,k,l)$ and radius $9h$, e) center $(3h,k,l)$ and radius $9h$, f) center $(3h,k,l)$ and radius $9h$.

Solution: The yz plane is characterized by $x=0$, whence the equation

$$(x-4h)^2+(y-k)^2+(z-l)^2=r^2=(5h)^2 \tag{1}$$

for the indicated sphere becomes

$$\begin{gathered}(-4h)^2+(y-k)^2+(z-l)^2=r^2=(5h)^2\\ \Leftrightarrow\\ (y-k)^2+(z-l)^2=r^2-(4h)^2=(5h)^2-(4h)^2=9h^2=(3h)^2,\end{gathered} \tag{2}$$

which is a circle with center $(0,k,l)$ and radius $3h$. The correct answer is c).

2. (5 points) The unit vector in the direction of $\mathbf{v}=-4\langle a,b,c\rangle$ is

a) $\dfrac{\langle a,b,c\rangle}{\sqrt{a^2+b^2+c^2}}$, b) $\dfrac{-\langle a,b,c\rangle}{\sqrt{a^2+b^2+c^2}}$, c) $-\langle a,b,c\rangle$, d) $\langle a,b,c\rangle$, e) $-\dfrac{\langle a,b,c\rangle}{4}$, f) $\dfrac{\langle a,b,c\rangle}{4}$.

Assume $\langle a,b,c\rangle \neq \langle 0,0,0\rangle$.

Solution: The correct answer is b):

$$\begin{aligned}\frac{\mathbf{v}}{\|\mathbf{v}\|}&=\frac{-4\langle a,b,c\rangle}{\|-4\langle a,b,c\rangle\|}=\frac{-4\langle a,b,c\rangle}{|-4|\|\langle a,b,c\rangle\|}=\frac{-4\langle a,b,c\rangle}{4\|\langle a,b,c\rangle\|}=\frac{-\langle a,b,c\rangle}{\|\langle a,b,c\rangle\|}\\ &=\frac{-\langle a,b,c\rangle}{\sqrt{a^2+b^2+c^2}}\end{aligned} \tag{3}$$

3. (5 points) A vector equation for the line passing through the points $\mathbf{p}_0 = (a,b,c) = \langle a,b,c \rangle$ and $\mathbf{p}_1 = (\alpha,\beta,\gamma) = \langle \alpha,\beta,\gamma \rangle$ is

a) $\mathbf{r}(t) = \mathbf{p}_0 + t(\mathbf{p}_1 + \mathbf{p}_0)$, b) $\mathbf{r}(t) = (1+t)\mathbf{p}_0 + t\mathbf{p}_1$, c) $\mathbf{r}(t) = \mathbf{p}_0 + t\mathbf{p}_1$, d) $\mathbf{r}(t) = \mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0)$, e) $\mathbf{r}(t) = \mathbf{p}_1 + t\mathbf{p}_0$, f) $\mathbf{r}(t) = \mathbf{p}_0 + (1-t)\mathbf{p}_1$.

(Here we equate vectors from the origin to a point with that point.)

Solution: The correct answer is d): the vector equation is clearly

$$\mathbf{r}(t) = (1-t)\mathbf{p}_0 + t\mathbf{p}_1 = \mathbf{p}_0 - t\mathbf{p}_0 + t\mathbf{p}_1 = \mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0). \tag{4}$$

4. (5 points) Symmetric equations for the line passing through the points $\mathbf{p}_0 = (a,b,c) = \langle a,b,c \rangle$ and $\mathbf{p}_1 = (\alpha,\beta,\gamma) = \langle \alpha,\beta,\gamma \rangle$ are

(Assume the most general case.)

a) $\frac{x-a}{\alpha-a} = \frac{y-b}{\beta-b} = \frac{z-c}{\gamma-c}$, b) $\frac{x}{\alpha-a} = \frac{y}{\beta-b} = \frac{z}{\gamma-c}$, c) $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$, d) $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, e) $\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma}$, f) $\frac{x-a}{a} = \frac{y-b}{b} = \frac{z-c}{c}$.

Solution: The correct answer is a): from (4) we have generally that

$$
\begin{aligned}
\langle x,y,z\rangle &= \mathbf{r}(t)=\mathbf{p}_0+t\left(\mathbf{p}_1-\mathbf{p}_0\right)\\
&=\langle a,b,c\rangle+t\left(\langle \alpha,\beta,\gamma\rangle-\langle a,b,c\rangle\right)\\
&=\langle a,b,c\rangle+t\langle \alpha-a,\beta-b,\gamma-c\rangle\\
&=\langle a,b,c\rangle+\langle t(\alpha-a),t(\beta-b),t(\gamma-c)\rangle\\
&=\langle a+t(\alpha-a),b+t(\beta-b),c+t(\gamma-c)\rangle \qquad (5)\\
&\Leftrightarrow\\
x&=a+t(\alpha-a),\ \ y=b+t(\beta-b),\quad z=c+t(\gamma-c)\\
&\Leftrightarrow\\
\frac{x-a}{\alpha-a}&=\frac{y-b}{\beta-b}=\frac{z-c}{\gamma-c}=t.
\end{aligned}
$$

5. (5 points) Which of the following expressions gives the volume of the parallelepiped with sides $\mathbf{a}=\langle a_1,a_2,a_3\rangle$, $\mathbf{b}=\langle b_1,b_2,b_3\rangle$, and $\mathbf{c}=\langle c_1,c_2,c_3\rangle$? The choices are

$$
1)\ \left|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\right| \qquad 2)\ \left|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})\right| \qquad 3)\left|\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})\right| \qquad 4)\ \left|\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})\right| \qquad (6)
$$

a) 1) only, b) 2) only, c) 3) only, d) 4) only , e) 3) and 4) only f) all of the expressions in (6).

<u>Solution</u>: The correct answer is f).

Part II: *In the following problems, show all work, and simplify your results*.

6. (15 points) Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle \neq \langle 0,0,0 \rangle \neq \langle b_1, b_2, b_3 \rangle = \mathbf{b}$.

a) Find the vector projection $proj_{\mathbf{a}}\mathbf{b}$ of $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ onto $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$.
b) Find the scalar projection $comp_{\mathbf{a}}\mathbf{b}$ of $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ onto $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$.
c) Find the innerproduct

$$(\mathbf{b} - proj_{\mathbf{a}}\mathbf{b}) \cdot \mathbf{a} \tag{7}$$

Solution: Without memorizing anything except a principle/definition, the easiest way to solve this problem is to realize that $proj_{\mathbf{a}}\mathbf{b}$ is the multiple of vector $\mathbf{a}$ that makes (7) turn out to be zero. Thus $proj_{\mathbf{a}}\mathbf{b} = \lambda\mathbf{a}$ where the scalar λ is determined by

$$\begin{aligned} 0 &= (\mathbf{b} - proj_{\mathbf{a}}\mathbf{b}) \cdot \mathbf{a} = (\mathbf{b} - \lambda\mathbf{a}) \cdot \mathbf{a} \\ &= \mathbf{b} \cdot \mathbf{a} - \lambda\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} - \lambda\|\mathbf{a}\|^2 \\ &\Leftrightarrow \\ \lambda &= \frac{\mathbf{b} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}, \end{aligned} \tag{8}$$

and we get

$$proj_{\mathbf{a}}\mathbf{b} = \lambda\mathbf{a} = \frac{\mathbf{b} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{\mathbf{b} \cdot \mathbf{a}}{\|\mathbf{a}\|}\frac{\mathbf{a}}{\|\mathbf{a}\|} =: comp_{\mathbf{a}}\mathbf{b}\frac{\mathbf{a}}{\|\mathbf{a}\|}. \tag{9}$$

In (9) we indentify $comp_{\mathbf{a}}\mathbf{b} := \frac{\mathbf{b} \cdot \mathbf{a}}{\|\mathbf{a}\|}$ as the scalar we multiply a unit vector in the direction of $\mathbf{a}$ by in order to get $proj_{\mathbf{a}}\mathbf{b}$.

7. (15 points) Let points $\mathbf{p}_1 = (a_1, a_2, a_3) = \langle a_1, a_2, a_3 \rangle$, $\mathbf{p}_2 = (b_1, b_2, b_3) = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{p}_3 = (c_1, c_2, c_3) = \langle c_1, c_2, c_3 \rangle$ NOT be collinear. Thus there is a unique plane P containing all three points.

a) Find a nonzero vector $\mathbf{n}$ orthogonal to P in terms of these 3 points (written as vectors).
b) Find the area A of the triangle with the indicated 3 points as its vertices.

Solution: For a) we take the cross product of any two, non-collinear vectors in the plane. For example

$$\begin{aligned}\mathbf{n} &= (\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1) = \mathbf{p}_2 \times (\mathbf{p}_3 - \mathbf{p}_1) - \mathbf{p}_1 \times (\mathbf{p}_3 - \mathbf{p}_1) \\ &= \mathbf{p}_2 \times \mathbf{p}_3 - \mathbf{p}_2 \times \mathbf{p}_1 - \mathbf{p}_1 \times \mathbf{p}_3 + \cancel{\mathbf{p}_1 \times \mathbf{p}_1} \\ &= \mathbf{p}_2 \times \mathbf{p}_3 - \mathbf{p}_2 \times \mathbf{p}_1 - \mathbf{p}_1 \times \mathbf{p}_3 + \mathbf{0} \\ &= \mathbf{p}_2 \times \mathbf{p}_3 + \mathbf{p}_1 \times \mathbf{p}_2 + \mathbf{p}_3 \times \mathbf{p}_1 \\ &= \mathbf{p}_1 \times \mathbf{p}_2 + \mathbf{p}_2 \times \mathbf{p}_3 + \mathbf{p}_3 \times \mathbf{p}_1.\end{aligned} \tag{10}$$

Of course the negative of this expression is also such a vector, as is any non-zero multiple of it. (The advantage of the last form in (10) is that "it does not play favorites": it's an expression manifestly symmetrical under permutation of the vectors. This makes it worth writing down/computing.)

For b) we realize that the area A is just one half of the norm of the cross product of any two of the triangle's edges, whence

$$A = \frac{1}{2}\|\mathbf{n}\| = \frac{1}{2}\|(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1)\| = \frac{1}{2}\|\mathbf{p}_1 \times \mathbf{p}_2 + \mathbf{p}_2 \times \mathbf{p}_3 + \mathbf{p}_3 \times \mathbf{p}_1\|. \tag{11}$$

8. (20 points)
   a) Find an equation for the plane P that passes through the point $\mathbf{p}_0 = (x_0, y_0, z_0) = \langle x_0, y_0, z_0 \rangle$ and is perpendicular to $\mathbf{n} = \langle n_1, n_2, n_3 \rangle$.
   b) Let point $\mathbf{p}_1 = (x_1, y_1, z_1) = \langle x_1, y_1, z_1 \rangle$ NOT be in the plane P of part a). Find the point $\mathbf{p}_0{}'$ that IS in the plane P that is closest to $\mathbf{p}_1$.
   c) What is the distance from point $\mathbf{p}_1$ to plane P

<u>**Solution**</u>: For a) we note that a point $\mathbf{p} = (x, y, z) = \langle x, y, z \rangle$ is in P iff the vector from it to $\mathbf{p}_0 \in \mathrm{P}$ is orthogonal to $\mathbf{n}$, whence

$$(\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{n} = 0 \tag{12}$$

For b) we first note that any such special point $\mathbf{p}_0{}' \in \mathrm{P}$ satisfies (12), i.e.,

$$\left(\mathbf{p}_0{}' - \mathbf{p}_0\right) \cdot \mathbf{n} = 0. \tag{13}$$

Now the distance $d(\mathbf{p}_1, \mathbf{p}_0{}')$ from $\mathbf{p}_1 \notin \mathrm{P}$ to any $\mathbf{p}_0{}' \in \mathrm{P}$ is given by

$$d(\mathbf{p}_1, \mathbf{p}_0{}') = \left\| \mathbf{p}_1 - \mathbf{p}_0{}' \right\|. \tag{14}$$

This value will be least when the vector from $\mathbf{p}_0{}' \in \mathrm{P}$ to $\mathbf{p}_1 \notin \mathrm{P}$ is parallel to $\mathbf{n}$, i.e., when there's a scalar λ such that

$$\mathbf{p}_1 - \mathbf{p}_0{}' = \lambda \mathbf{n}. \tag{15}$$

Thus

$$\mathbf{p}_0{}' = -\lambda \mathbf{n} + \mathbf{p}_1 \tag{16}$$

and (13) gives

$$
\begin{aligned}
0 &= \left(\mathbf{p}_0{}' - \mathbf{p}_0\right) \cdot \mathbf{n} = \left(-\lambda \mathbf{n} + \mathbf{p}_1 - \mathbf{p}_0\right) \cdot \mathbf{n} \\
&= \left(-\lambda \mathbf{n} + \mathbf{p}_1 - \mathbf{p}_0\right) \cdot \mathbf{n} = -\lambda \mathbf{n} \cdot \mathbf{n} + \left(\mathbf{p}_1 - \mathbf{p}_0\right) \cdot \mathbf{n} \\
&\Leftrightarrow \\
\lambda &= \frac{\left(\mathbf{p}_1 - \mathbf{p}_0\right) \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} = \frac{\left(\mathbf{p}_1 - \mathbf{p}_0\right) \cdot \mathbf{n}}{\|\mathbf{n}\|^2}
\end{aligned} \tag{17}
$$

and the distance in (14) satisfies

$$
\begin{aligned}
d(\mathbf{p}_1, \mathbf{p}_0{}') &= \left\|\mathbf{p}_1 - \mathbf{p}_0{}'\right\| = \left\|\mathbf{p}_1 - \left(-\lambda \mathbf{n} + \mathbf{p}_1\right)\right\| \\
&= \left\|\mathbf{p}_1 + \lambda \mathbf{n} - \mathbf{p}_1\right\| = \left\|\lambda \mathbf{n}\right\| \\
&= |\lambda| \|\mathbf{n}\| \\
&= \left|\frac{\left(\mathbf{p}_1 - \mathbf{p}_0\right) \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right| \|\mathbf{n}\| = \frac{\left|\left(\mathbf{p}_1 - \mathbf{p}_0\right) \cdot \mathbf{n}\right|}{\|\mathbf{n}\|^2} \|\mathbf{n}\| \\
&= \frac{\left|\left(\mathbf{p}_1 - \mathbf{p}_0\right) \cdot \mathbf{n}\right|}{\|\mathbf{n}\|}.
\end{aligned} \tag{18}
$$

Of course this is the answer to c) by definition. The point $\mathbf{p}_0{}'$ in P closest to $\mathbf{p}_1$ is then

$$
\mathbf{p}_0{}' = -\lambda \mathbf{n} + \mathbf{p}_1 = -\frac{\left(\mathbf{p}_1 - \mathbf{p}_0\right) \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \mathbf{n} + \mathbf{p}_1 = -\frac{\mathbf{n} \cdot \left(\mathbf{p}_1 - \mathbf{p}_0\right)}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} + \mathbf{p}_1, \tag{19}
$$

We confirm that with the choice (19) we indeed have

$$
\begin{aligned}
\mathbf{n} \cdot \left(\mathbf{p}_0{}' - \mathbf{p}_0\right) &= \mathbf{n} \cdot \left(-\frac{\mathbf{n} \cdot \left(\mathbf{p}_1 - \mathbf{p}_0\right)}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} + \mathbf{p}_1 - \mathbf{p}_0\right) \\
&= -\frac{\mathbf{n} \cdot \left(\mathbf{p}_1 - \mathbf{p}_0\right)}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} \cdot \mathbf{n} + \mathbf{n} \cdot \left(\mathbf{p}_1 - \mathbf{p}_0\right) \\
&= -\mathbf{n} \cdot \left(\mathbf{p}_1 - \mathbf{p}_0\right) + \mathbf{n} \cdot \left(\mathbf{p}_1 - \mathbf{p}_0\right) = 0,
\end{aligned} \tag{20}
$$

i.e., the $\mathbf{p}_0{}'$ specified by (19) is in fact in the plane P.

9. (15 points)

a) Consider the equation $F(x,y)=0$. For a large class of functions F, this defines a (2-dimensional) surface in 3 space, or just a curve in 2 space. In 3 space, what type of surface is it?

b) What type of Quadric Surface is the graph of the equation $x^2+4y^2-z=0$?

Solution: a) this is a "cylinder"—the equation doesn't reference the variable z, whence for any triple (x,y,z) satisfying the equation, all other triples of the form $(x,y,z+k)$ also satisfy it. b) Solving for z we see that it's a paraboloid—an elliptic paraboloid.

10. (10 points) The Law of Cosines can be used to show that the angle θ between a vector $\mathbf{a} = \langle a_1, a_2 \rangle \neq \langle 0,0 \rangle$ and a vector $\mathbf{b} = \langle b_1, b_2 \rangle \neq \langle 0,0 \rangle$ satisfies

$$\begin{aligned} \cos\theta &= \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1a_1 + a_2a_2}\sqrt{b_1b_1 + b_2b_2}} \\ &= \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}}. \end{aligned} \tag{21}$$

But since $0 \le \cos^2\theta \le 1$, (21) would say that, for all pairs of numbers (a_1, a_2) and (b_1, b_2) (except those for which one or both pairs give the zero vector as above) it must be that

$$0 \le \frac{(a_1b_1 + a_2b_2)^2}{(a_1^2 + a_2^2)(b_1^2 + b_2^2)} \le 1. \tag{22}$$

The left-hand side of (22) seems clear. What about the right? Show that

$$(a_1b_1 + a_2b_2)^2 \le (a_1^2 + a_2^2)(b_1^2 + b_2^2), \tag{23}$$

or, equivalently, but simpler, show that, for all a_1, a_2, b_1, b_2

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2 \ge 0. \tag{24}$$

Hint: Expand out the expression $(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2$ and show that it is a perfect square.

Solution:

$$\begin{aligned} (a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2 &= a_1^2b_1^2 + a_2^2b_1^2 + a_1^2b_2^2 + a_2^2b_2^2 - (a_1^2b_1^2 + 2a_1b_1a_2b_2 + a_2^2b_2^2) \\ &= a_2^2b_1^2 + a_1^2b_2^2 - 2a_1b_1a_2b_2 \\ &= (a_2b_1 - a_1b_2)^2 \ge 0. \end{aligned} \tag{25}$$