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Section: _002____________

Instructor: Scott Glasgow

Math 314 (Calculus of Several Variables) **RED**

Exam 1 May 16-17, 2013

Instructions:

- For questions which require a written answer, show all your work. Full credit will be given only if the necessary work is shown, justifying your answer.
- Simplify your answers.
- Calculators are not allowed. Textbooks are not allowed. Notes are not allowed.
- Should you need more space than is allotted to answer a question, use the back of the page the problem is on and indicate this fact.
- Talking about the exam with other students before the graded exam is returned to you is a violation of the Honor Code.

Part I: Multiple Choice *Mark the correct answer on the bubble sheet provided*.

1. (5 points) If $\mathbf{r}(t) = \left\langle \frac{e^t - 1}{t}, \frac{t^2 - t}{t^2 + t}, t \log t \right\rangle$, then $\lim_{t \to 0^+} \mathbf{r}(t) = \lim_{t \downarrow 0} \mathbf{r}(t)$ is

a) $\langle 1, -1, 1 \rangle$ b) $\langle 1, 1, 1 \rangle$ c) $\langle 1, -1, -1 \rangle$

d) $\langle 1, -1, 0 \rangle$ e) $\langle 1, 1, -1 \rangle$ f) does not exist (including infinite)

<u>Solution:</u> We have the "forms"

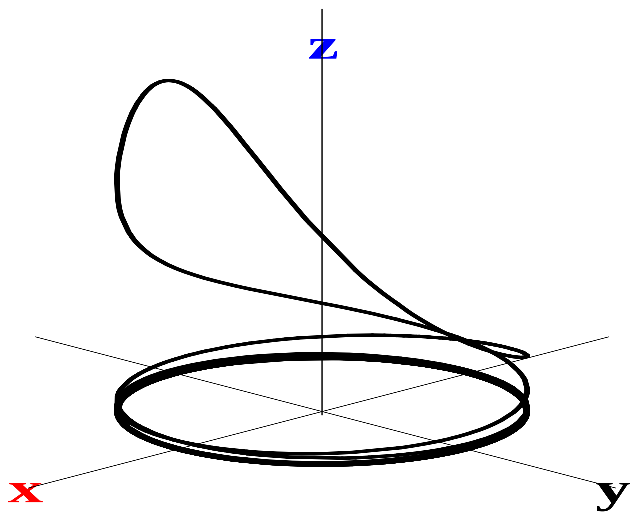
$$\lim_{t \downarrow 0} \frac{e^t - 1}{t} = \frac{0}{0}, \qquad \lim_{t \downarrow 0} \frac{t^2 - t}{t^2 + t} = \frac{0}{0}, \qquad \lim_{t \downarrow 0} t \log t = \lim_{t \downarrow 0} \frac{\log t}{t^{-1}} = \frac{-\infty}{+\infty}, \tag{1}$$

whence by L'Hôpital's rule (and simpler ideas)

$$\begin{aligned}
\lim_{t \downarrow 0} \mathbf{r}(t) &= \lim_{t \downarrow 0} \left\langle \frac{e^t - 1}{t}, \frac{t^2 - t}{t^2 + t}, \frac{\log t}{t^{-1}} \right\rangle = \left\langle \lim_{t \downarrow 0} \frac{e^t - 1}{t}, \lim_{t \downarrow 0} \frac{t^2 - t}{t^2 + t}, \lim_{t \downarrow 0} \frac{\log t}{t^{-1}} \right\rangle \\
&= \left\langle \lim_{t \downarrow 0} \frac{\frac{d}{dt}\left(e^t - 1\right)}{\frac{d}{dt} t}, \lim_{t \downarrow 0} \frac{\frac{d}{dt}\left(t^2 - t\right)}{\frac{d}{dt}\left(t^2 + t\right)}, \lim_{t \downarrow 0} \frac{\frac{d}{dt} \log t}{\frac{d}{dt} t^{-1}} \right\rangle \\
&= \left\langle \lim_{t \downarrow 0} \frac{e^t}{1}, \lim_{t \downarrow 0} \frac{2t - 1}{2t + 1}, \lim_{t \downarrow 0} \frac{t^{-1}}{-t^{-2}} \right\rangle = \left\langle e^0, \frac{2 \cdot 0 - 1}{2 \cdot 0 + 1}, \lim_{t \downarrow 0} -t \right\rangle \\
&= \left\langle 1, \frac{-1}{1}, -0 \right\rangle = \langle 1, -1, 0 \rangle.
\end{aligned} \tag{2}$$

So the answer is d).

2. (5 points) Consider the following graph of a space curve:



Which of the following vector-valued functions give rise to this graph?

a) $\left\langle \cos(t), \sin(t), t\left(1+t^2\right)^{-1} \right\rangle$ b) $\langle \cos t, \sin t, \cos 2t \rangle$ c) $\left\langle \cos(t), \sin(t), \left(1+t^2\right)^{-1} \right\rangle$

d) $\langle \cos t, \sin t, \tanh t \rangle$ e) $\langle \cos t, \sin t, \cos^2 t \rangle$ f) $\langle \cos t, \sin t, \cos 2t \rangle$

<u>**Solution**</u>: The correct answer is c):

$$\mathbf{r}(t) = \left\langle \cos(t), \sin(t), \frac{1}{1+t^2} \right\rangle; \tag{3}$$

it sports all the right symmetries and asymptotic values. For examples,

$$\begin{gathered} \mathbf{r}(t) = \left\langle \cos(t), \sin(t), \frac{1}{1+t^2} \right\rangle =: \langle X(t), Y(t), Z(t) \rangle \\ \Rightarrow \\ X(-t) = X(t), \quad Y(-t) = -Y(t), \quad Z(-t) = Z(t), \\ \lim_{t \to \pm\infty} \langle X(t), Y(t), Z(t) \rangle = \langle \text{d.n.e.}, \text{d.n.e.}, 0 \rangle, \end{gathered} \tag{4}$$

where, further, the "d.n.e. components" actually correspond to periodic motion, as in the graph above.

3. (5 points) The parametric equations for the tangent line of the space curve

$$\mathbf{r}(t) = \left\langle e^t, 2+\sin(t), 3+\log\left(t+1\right)\right\rangle \tag{5}$$

at the point $\langle 1,2,3\rangle$ can be written as

a) $x = 1+e^t, y = 2+\cos(t), z = 3+\frac{1}{t+1}$, b) $x = e^t, y = \cos(t), z = \frac{1}{t+1}$, c) $x = t, y = t, z = t$,

d) $x = 1-t, y = 2-t, z = 3+t$, e) $x = 1-t, y = 2-t, z = 3-t$, f) none of the above

Solution: The correct answer is e): the tangent line is always of the form

$$\mathbf{R}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0), \qquad t \in \mathbb{R}. \tag{6}$$

Since

$$\mathbf{r}(t_0) := \left\langle e^{t_0}, 2+\sin(t_0), 3+\log\left(t_0+1\right)\right\rangle = \langle 1,2,3\rangle \quad \Leftrightarrow \quad t_0 = 0, \tag{7}$$

we find that in (6) we must choose $t_0 = 0$, whence, since

$$\mathbf{r}'(t) = \left\langle e^t, \cos(t), \frac{1}{t+1}\right\rangle \tag{8}$$

the tangent line is given by

$$\mathbf{R}(t) = \mathbf{r}(0) + t\mathbf{r}'(0), \qquad t \in \mathbb{R} \tag{9}$$

where

$$\mathbf{r}'(0) = \left\langle e^0, \cos(0), \frac{1}{0+1}\right\rangle = \langle 1,1,1\rangle, \tag{10}$$

and, so,

$$\begin{aligned}\mathbf{R}(t) &= \mathbf{r}(0) + t\mathbf{r}'(0) = \langle 1,2,3\rangle + t\langle 1,1,1\rangle = \langle 1,2,3\rangle + \langle t,t,t\rangle \\ &= \langle 1+t, 2+t, 3+t\rangle, \qquad t \in \mathbb{R},\end{aligned} \tag{11}$$

Replacing t with $-t$ (which maps $\mathbb{R}$ to itself) we get the results in e).

4. (5 points) Let $b > a$. The arc length function $s(t)$ for a space curve $\mathbf{r}(t) : t \in [a,b]$ is

a) $s(t) = \int_a^t du \|\mathbf{r}(u)\|, \;\; t \in [a,b]$,

b) $s(t) = \int_a^t du \sqrt{\mathbf{r}'(u) \cdot \mathbf{r}'(u)}, \; t \in [a,b]$,

c) $s(t) = \int_a^t du \; \mathbf{r}'(u), \; t \in [a,b]$,

d) $s(t) = \int_a^t du \; \mathbf{r}(u), \;\; t \in [a,b]$,

e) $s(t) = \int_a^t du \|\mathbf{r}'(u)\|^2$,

f) f) none of the above.

Solution: By definitions we certainly have

$$s(t) := \lim_{\|\Delta\| \downarrow 0} \sum_i \|\Delta \mathbf{r}_i\| =: \int_a^t \|d\mathbf{r}(u)\|, \qquad t \in [a,b], \tag{12}$$

whence with the chain rule, etc., we certainly get

$$\begin{aligned} s(t) &= \int_a^t \left\| du \frac{d\mathbf{r}}{du}(u) \right\| = \int_a^t |du| \left\| \frac{d\mathbf{r}}{du}(u) \right\| = \int_a^t du \|\mathbf{r}'(u)\| \\ &= \int_a^t du \sqrt{\mathbf{r}'(u) \cdot \mathbf{r}'(u)}, \qquad t \in [a,b], \end{aligned} \tag{13}$$

and, so, the only correct answer is b).

5. (5 points) Given that the arc length function $s(t)$ of a space curve $\mathbf{r}(t) : t \in [a,b]$ satisfies

$$\frac{ds(t)}{dt} = \|\mathbf{r}'(t)\|, \tag{14}$$

and the unit tangent vector (function) is given by

$$\mathbf{T}(t) := \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \tag{15}$$

and that one straightforward representation of the curvature (function) is given by

$$\kappa(t(s)) = \left\| \frac{d\mathbf{T}(t(s))}{ds} \right\|, \tag{16}$$

another representation of curvature not referencing the arc length function is

a) $\dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$ b) $\dfrac{\|\mathbf{T}(t)\|}{\|\mathbf{r}(t)\|}$ c) $\dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}(t)\|}$

d) $\dfrac{\|\mathbf{T}(t)\|}{\|\mathbf{r}'(t)\|}$ e) $\|\mathbf{T}'(t)\|$ f) $\dfrac{1}{\|\mathbf{T}'(t)\|}$.

<u>Solution</u>: The solution is a): by the Chain Rule

$$\kappa(t) = \left\| \frac{d\mathbf{T}(t)}{ds} \right\| = \left\| \frac{d\mathbf{T}(t)}{dt} / \frac{ds}{dt} \right\| = \left\| \mathbf{T}'(t) / \|\mathbf{r}'(t)\| \right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}. \tag{17}$$

Part II: *In the following problems, show all work, and simplify your results*.

6. (25 points) Find the arc length function $s(t)$ for the space curve

$$\mathbf{r}(t) = \left\langle e^{2t}\sin(t), e^{2t}\cos(t), 7\right\rangle \quad t \in [0, 2\pi] \tag{18}$$

such that $s(0) = 0$. Then reparameterize the curve according to arc length s. To check your results (and which is worth some points), confirm that for your specific answer $\tilde{\mathbf{r}}(s) := \mathbf{r}(t(s))$ you do in fact get the necessary result that

$$\left\| \frac{d}{ds}\tilde{\mathbf{r}}(s) \right\| = \left\| \tilde{\mathbf{r}}'(s) \right\| \equiv 1, \tag{19}$$

the "$\equiv$"in (19) indicating "independent of s". In lieu of that specific computation, you can show that (19) holds generally, which is actually much easier.

<u>Solution</u>: We have

$$\mathbf{r}'(u) = \left\langle e^{2u}\left(2\sin(u) + \cos(u)\right), e^{2u}\left(2\cos(u) - \sin(u)\right), 0\right\rangle \tag{20}$$

thus by (12) we have

$$s(t) = \int_a^t du\sqrt{\mathbf{r}'(u)\cdot\mathbf{r}'(u)} = \int_0^t du\sqrt{\mathbf{r}'(u)\cdot\mathbf{r}'(u)} \tag{21}$$

where

$$\begin{aligned}
\mathbf{r}'(u)\cdot\mathbf{r}'(u) &= e^{4u}\left(2\sin(u) + \cos(u)\right)^2 + e^{4u}\left(2\cos(u) - \sin(u)\right)^2 \\
&= e^{4u}\begin{pmatrix} \left(4\sin^2(u) + 4\sin(u)\cos(u) + \cos^2(u)\right) + \\ \left(4\cos^2(u) - 4\cos(u)\sin(u) + \sin^2(u)\right) \end{pmatrix} \\
&= e^{4u}\left(5\sin^2(u) + 5\cos^2(u)\right) = 5e^{4u}
\end{aligned} \tag{22}$$

whence

$$s(t) = \int_0^t du\sqrt{\mathbf{r}'(u)\cdot\mathbf{r}'(u)} = \int_0^t du\sqrt{5e^{4u}} = \sqrt{5}\int_0^t du\ e^{2u} = \sqrt{5}\left[\frac{e^{2u}}{2}\right]_0^t = \sqrt{5}\,\frac{e^{2t}-1}{2}, \tag{23}$$

and we find that

$$t(s) = \frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right). \tag{24}$$

Thus we can write the curve as

$$\begin{aligned}\tilde{\mathbf{r}}(s) &:= \mathbf{r}(t(s)) = \left\langle e^{2\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right)}\sin(\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right)), e^{2\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right)}\cos(\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right)), 7\right\rangle \\ &= \left\langle \left(\frac{2s}{\sqrt{5}}+1\right)\sin(\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right)), \left(\frac{2s}{\sqrt{5}}+1\right)\cos(\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right)), 7\right\rangle.\end{aligned} \tag{25}$$

From (25) we find

$$\begin{aligned}\tilde{\mathbf{r}}'(s) &= \left\langle \begin{matrix} \frac{2}{\sqrt{5}}\sin(\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right)) + \left(\frac{2s}{\sqrt{5}}+1\right)\cos(\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right))\frac{1}{2}\frac{\frac{2}{\sqrt{5}}}{\frac{2s}{\sqrt{5}}+1}, \\ \frac{2}{\sqrt{5}}\cos(\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right)) - \left(\frac{2s}{\sqrt{5}}+1\right)\sin(\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right))\frac{1}{2}\frac{\frac{2}{\sqrt{5}}}{\frac{2s}{\sqrt{5}}+1}, 0 \end{matrix}\right\rangle \\ &= \frac{1}{\sqrt{5}}\left\langle 2\sin(\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right)) + \cos(\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right)), 2\cos(\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right)) - \sin(\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}+1\right)), 0\right\rangle \\ &=: \frac{1}{\sqrt{5}}\langle 2\sin(\theta)+\cos(\theta), 2\cos(\theta)-\sin(\theta), 0\rangle\end{aligned}$$

(26)

which gives

$$\begin{aligned}\|\tilde{\mathbf{r}}'(s)\| &= \left\|\frac{1}{\sqrt{5}}\langle 2\sin(\theta)+\cos(\theta), 2\cos(\theta)-\sin(\theta), 0\rangle\right\| \\ &= \frac{1}{\sqrt{5}}\sqrt{\left(2\sin(\theta)+\cos(\theta)\right)^2 + \left(2\cos(\theta)-\sin(\theta)\right)^2} \\ &= \frac{1}{\sqrt{5}}\sqrt{\left(4\sin^2(\theta)+4\sin(\theta)\cos(\theta)+\cos^2(\theta)\right)+\left(4\cos^2(\theta)-4\cos(\theta)\sin(\theta)+\sin^2(\theta)\right)} \\ &= \frac{1}{\sqrt{5}}\sqrt{5\sin^2(\theta)+5\cos^2(\theta)} = \frac{1}{\sqrt{5}}\sqrt{5} = 1,\end{aligned} \tag{27}$$

as advertized. Generally we simply note that

$$\begin{aligned}&\tilde{\mathbf{r}}(s) := \mathbf{r}(t(s)) \Rightarrow \\ &\tilde{\mathbf{r}}'(s) = \frac{d}{ds}\mathbf{r}(t(s)) = \frac{d}{dt(s)}\mathbf{r}(t(s))\frac{dt(s)}{ds} = \mathbf{r}'(t(s))\frac{dt(s)}{ds},\end{aligned} \tag{28}$$

and then get that, writing the arc length function as $S(t)$, and using (14),

$$\begin{aligned}&S(t(s)) = s \Rightarrow \\ &\|\mathbf{r}'(t(s))\|\frac{dt(s)}{ds} = S'(t(s))\frac{dt(s)}{ds} = \frac{d}{dt(s)}S(t(s))\frac{dt(s)}{ds} = \frac{d}{ds}S(t(s)) = 1 \\ &\Rightarrow \\ &\frac{dt(s)}{ds} = \frac{1}{\|\mathbf{r}'(t(s))\|},\end{aligned} \tag{29}$$

which in (28) gives

$$\tilde{\mathbf{r}}'(s) = \mathbf{r}'(t(s))\frac{dt(s)}{ds} = \mathbf{r}'(t(s))\frac{1}{\|\mathbf{r}'(t(s))\|} = \frac{\mathbf{r}'(t(s))}{\|\mathbf{r}'(t(s))\|} =: \mathbf{T}\big(t(s)\big) \tag{30}$$

and which readily then gives (19).

7. (15 points) Find the curvature of the curve in the previous problem.

Solution: From (17), which is

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \tag{31}$$

we decide to compute in turn

$$\mathbf{r}'(t) = \left\langle e^{2t}\left(2\sin(t)+\cos(t)\right), e^{2t}\left(2\cos(t)-\sin(t)\right), 0\right\rangle \tag{32}$$

then, as above,

$$\|\mathbf{r}'(t)\| = \sqrt{5}e^{2t} \tag{33}$$

then

$$\begin{aligned}\mathbf{T}(t) &:= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\left\langle e^{2t}\left(2\sin(t)+\cos(t)\right), e^{2t}\left(2\cos(t)-\sin(t)\right), 0\right\rangle}{\sqrt{5}e^{2t}} \\ &= \frac{1}{\sqrt{5}}\left\langle 2\sin(t)+\cos(t), 2\cos(t)-\sin(t), 0\right\rangle\end{aligned} \tag{34}$$

then

$$\begin{aligned}\mathbf{T}'(t) &= \frac{d}{dt}\frac{1}{\sqrt{5}}\left\langle 2\sin(t)+\cos(t), 2\cos(t)-\sin(t), 0\right\rangle \\ &= \frac{1}{\sqrt{5}}\left\langle 2\cos(t)-\sin(t), -2\sin(t)-\cos(t), 0\right\rangle\end{aligned} \tag{35}$$

then

$$
\begin{aligned}
\left\|\mathbf{T}'(t)\right\| &= \left\|\frac{1}{\sqrt{5}}\left\langle 2\cos(t)-\sin(t), -2\sin(t)-\cos(t), 0\right\rangle\right\| \\
&= \frac{1}{\sqrt{5}}\sqrt{\left(2\cos(t)-\sin(t)\right)^2+\left(-2\sin(t)-\cos(t)\right)^2} \\
&= \frac{1}{\sqrt{5}}\sqrt{\left(2\cos(t)-\sin(t)\right)^2+\left(2\sin(t)+\cos(t)\right)^2} \\
&= \frac{1}{\sqrt{5}}\sqrt{\left(4\cos^2(t)-4\cos(t)\sin(t)+\sin^2(t)\right)+\left(4\sin^2(t)+4\sin(t)\cos(t)+\cos^2(t)\right)} \\
&= \frac{1}{\sqrt{5}}\sqrt{5\cos^2(t)+5\sin^2(t)} = \frac{1}{\sqrt{5}}\sqrt{5} = 1,
\end{aligned}
\tag{36}
$$

whence

$$
\kappa(t) = \frac{\left\|\mathbf{T}'(t)\right\|}{\left\|\mathbf{r}'(t)\right\|} = \frac{1}{\sqrt{5}e^{2t}} = \frac{e^{-2t}}{\sqrt{5}}. \tag{37}
$$

8. (15 points) Find the tangential and normal components $a_T(t)$ and $a_N(t)$ of acceleration $\mathbf{a}(t)=\mathbf{r}''(t)$ for the space curve

$$\mathbf{r}(t)=\langle \sin(t),\cos(t),7\rangle. \tag{38}$$

How does your result make intuitive sense?

__Solution__: We can write

$$\mathbf{a}(t)=\mathbf{r}''(t)=a_T(t)\mathbf{T}(t)+a_N(t)\mathbf{N}(t) \tag{39}$$

where, since the indicated vectors are orthonormal,

$$\begin{aligned}
\mathbf{T}(t)\cdot\mathbf{r}''(t)&=\mathbf{T}(t)\cdot\left(a_T(t)\mathbf{T}(t)+a_N(t)\mathbf{N}(t)\right)\\
&=a_T(t)\mathbf{T}(t)\cdot\mathbf{T}(t)+a_N(t)\mathbf{T}(t)\cdot\mathbf{N}(t)\\
&=a_T(t)(1)+a_N(t)(0)=a_T(t),\\
\mathbf{N}(t)\cdot\mathbf{r}''(t)&=\mathbf{N}(t)\cdot\left(a_T(t)\mathbf{T}(t)+a_N(t)\mathbf{N}(t)\right)\\
&=a_T(t)\mathbf{N}(t)\cdot\mathbf{T}(t)+a_N(t)\mathbf{N}(t)\cdot\mathbf{N}(t)\\
&=a_T(t)(0)+a_N(t)(1)=a_N(t),
\end{aligned} \tag{40}$$

i.e., given the various definitions,

$$a_T(t)=\mathbf{T}(t)\cdot\mathbf{r}''(t),\qquad a_N(t)=\mathbf{N}(t)\cdot\mathbf{r}''(t)=\frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}\cdot\mathbf{r}''(t). \tag{41}$$

So since we have

$$\begin{aligned}
\mathbf{r}(t)&=\langle \sin(t),\cos(t),7\rangle\\
\mathbf{r}'(t)&=\langle \cos(t),-\sin(t),0\rangle,\\
\|\mathbf{r}'(t)\|&=\sqrt{(\cos(t))^2+(-\sin(t))^2}=1,\\
\mathbf{T}(t)&=\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}=\frac{\mathbf{r}'(t)}{1}=\mathbf{r}'(t)=\langle \cos(t),-\sin(t),0\rangle,\\
\mathbf{T}'(t)&=\langle -\sin(t),-\cos(t),0\rangle,\\
\|\mathbf{T}'(t)\|&=\sqrt{(-\sin(t))^2+(-\cos(t))^2}=1,\\
\mathbf{N}(t)&=\frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}=\frac{\mathbf{T}'(t)}{1}=\mathbf{T}'(t)=\langle -\sin(t),-\cos(t),0\rangle=-\mathbf{r}(t)+\langle 0,0,7\rangle,\\
\mathbf{r}''(t)&=\langle -\sin(t),-\cos(t),0\rangle=\mathbf{N}(t),
\end{aligned} \tag{42}$$

we get

$$
\begin{aligned}
a_T(t) &= \mathbf{T}(t)\cdot\mathbf{r}''(t) = \langle \cos(t), -\sin(t), 0\rangle \cdot \langle -\sin(t), -\cos(t), 0\rangle = -\cos(t)\sin(t) + \sin(t)\cos(t) \\
&= 0, \\
a_N(t) &= \mathbf{N}(t)\cdot\mathbf{r}''(t) = \mathbf{N}(t)\cdot\mathbf{N}(t) = 1.
\end{aligned} \tag{43}
$$

This makes sense since this is just motion of constant speed round a circle; in such case we already know the acceleration is constant and directed inwards. (The normal direction $\mathbf{N}(t)$ is clearly inwards: $\mathbf{N}(t) = -\mathbf{r}(t) + \langle 0, 0, 7\rangle$.)

9. (20 points) Show that the definitions

$$\mathbf{T}(t) := \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \qquad \mathbf{n}(t) := \mathbf{T}'(t) \tag{44}$$

(and differentiability) give

$$\mathbf{n}(t) = \frac{\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} - \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^3}\mathbf{r}'(t) \tag{45}$$

and then

$$\mathbf{T}(t)\cdot\mathbf{n}(t) = 0. \tag{46}$$

Assume of course that $\|\mathbf{r}'(t)\| \neq 0$, etc.

Solution: We note

$$\|\mathbf{r}'(t)\| := \left(\mathbf{r}'(t)\cdot\mathbf{r}'(t)\right)^{1/2} \tag{47}$$

whence

$$\begin{aligned}
\frac{d}{dt}\|\mathbf{r}'(t)\| &= \frac{d}{dt}\left(\mathbf{r}'(t)\cdot\mathbf{r}'(t)\right)^{1/2} = \frac{1}{2}\left(\mathbf{r}'(t)\cdot\mathbf{r}'(t)\right)^{-1/2}\left(\left(\frac{d}{dt}\mathbf{r}'(t)\right)\cdot\mathbf{r}'(t) + \mathbf{r}'(t)\cdot\frac{d}{dt}\mathbf{r}'(t)\right) \\
&= \frac{1}{2}\left(\mathbf{r}'(t)\cdot\mathbf{r}'(t)\right)^{-1/2}\left(\mathbf{r}''(t)\cdot\mathbf{r}'(t) + \mathbf{r}'(t)\cdot\mathbf{r}''(t)\right) \\
&= \frac{1}{2}\left(\mathbf{r}'(t)\cdot\mathbf{r}'(t)\right)^{-1/2}\left(2\mathbf{r}'(t)\cdot\mathbf{r}''(t)\right) = \left(\mathbf{r}'(t)\cdot\mathbf{r}'(t)\right)^{-1/2}\mathbf{r}'(t)\cdot\mathbf{r}''(t) \\
&= \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\left(\mathbf{r}'(t)\cdot\mathbf{r}'(t)\right)^{1/2}} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}.
\end{aligned} \tag{48}$$

So

$$\begin{aligned}\mathbf{n}(t) &:= \mathbf{T}'(t) := \frac{d}{dt}\mathbf{T}(t) = \frac{d}{dt}\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t)\|\frac{d}{dt}\mathbf{r}'(t) - \mathbf{r}'(t)\frac{d}{dt}\|\mathbf{r}'(t)\|}{\|\mathbf{r}'(t)\|^2} \\ &= \frac{\|\mathbf{r}'(t)\|\mathbf{r}''(t) - \mathbf{r}'(t)\frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}}{\|\mathbf{r}'(t)\|^2} = \frac{\|\mathbf{r}'(t)\|^2\mathbf{r}''(t) - (\mathbf{r}'(t)\cdot\mathbf{r}''(t))\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|^3} \\ &= \frac{\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} - \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^3}\mathbf{r}'(t),\end{aligned} \tag{49}$$

and, so,

$$\begin{aligned}\|\mathbf{r}'(t)\|^4\,\mathbf{T}(t)\cdot\mathbf{n}(t) &= \mathbf{r}'(t)\cdot\left(\|\mathbf{r}'(t)\|^2\mathbf{r}''(t) - (\mathbf{r}'(t)\cdot\mathbf{r}''(t))\mathbf{r}'(t)\right) \\ &= \|\mathbf{r}'(t)\|^2\mathbf{r}'(t)\cdot\mathbf{r}''(t) - (\mathbf{r}'(t)\cdot\mathbf{r}''(t))\mathbf{r}'(t)\cdot\mathbf{r}'(t) \\ &= \|\mathbf{r}'(t)\|^2\mathbf{r}'(t)\cdot\mathbf{r}''(t) - (\mathbf{r}'(t)\cdot\mathbf{r}''(t))\|\mathbf{r}'(t)\|^2 = 0,\end{aligned} \tag{50}$$

giving (46) as advertized.