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Section: _002____________

Instructor: Scott Glasgow

Math 314 (Calculus of Several Variables) **RED**

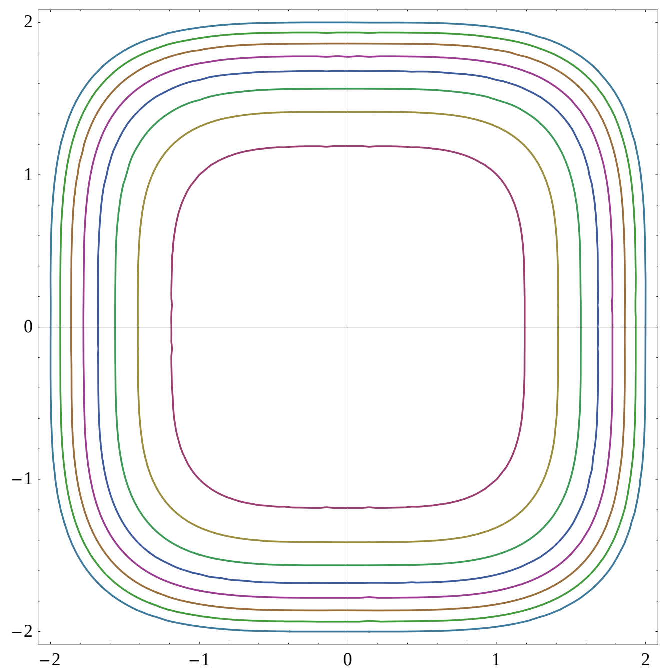
Exam 3 May 24-28, 2013

Instructions:

- For questions which require a written answer, show all your work. Full credit will be given only if the necessary work is shown, justifying your answer.
- Simplify your answers.
- Calculators are not allowed. Textbooks are not allowed. Notes are not allowed.
- Should you need more space than is allotted to answer a question, use the back of the page the problem is on and indicate this fact.
- Talking about the exam with other students before the graded exam is returned to you is a violation of the Honor Code.

Part I: Multiple Choice *Mark the correct answer on the bubble sheet provided*.

1. (5 points) Which function has the following level curves?



a) $f(x,y) = x^2 + y^2$ b) $f(x,y) = x^3 + y^3$ c) $f(x,y) = x^4 + y^4$

d) $f(x,y) = x + y$ e) $f(x,y) = xy$ f) $f(x,y) = x^2 y^2$

<u>Solution:</u> The answer is c), by elimination of the other impossibilities if nothing else: a) gives circles, b) allows $y \sim -x$ for large values of those variables, d) gives straight contours, and e) and f) both give hyperbolae.

2. (5 points) Here are three limits:

(i) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ (ii) $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ (iii) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Which of these limits exist?

a) none of them b) only (i) c) only (ii)
d) only (iii) e) only (i) and (ii) f) only (i) and (iii)
g) only (ii) and (iii) h) all of them

Solution: The correct answer is a):

$$\begin{aligned}\lim_{x\to 0}\frac{xy(x)^2}{x^2+y(x)^4} &=: \lim_{x\to 0}\frac{xz(x)}{x^2+z(x)^2} = \lim_{x\to 0}\frac{z(x)+xz'(x)}{2x+z(x)z'(x)} = \lim_{x\to 0}\frac{2z'(x)+xz''(x)}{2+z(x)z''(x)+z'(x)^2} \\ &= \frac{2z'(0)+0\cdot z''(0)}{2+z(0)z''(0)+z'(0)^2} = \frac{2z'(0)}{2+0\cdot z''(0)+z'(0)^2} = \frac{2z'(0)}{2+z'(0)^2}\end{aligned} \tag{1}$$

shows the value of the function depends on how the point $(0,0)$ is approached, hence that limit does not exist. Likewise for the second limit above:

$$\lim_{x\to 0}\frac{xy(x)^3}{x^2+y(x)^6} =: \lim_{x\to 0}\frac{xz(x)}{x^2+z(x)^2} = \cdots = \frac{2z'(0)}{2+z'(0)^2}. \tag{2}$$

Also the third does not exist:

$$\begin{aligned}\lim_{x\to 0}\frac{xy(x)}{x^2+y(x)^2} &= \lim_{x\to 0}\frac{y(x)+xy'(x)}{2x+2y(x)y'(x)} = \lim_{x\to 0}\frac{y'(x)+y'(x)+xy''(x)}{2+2y'(x)^2+2y(x)y''(x)} \\ &= \frac{2y'(0)+0\cdot y''(0)}{2+2y'(0)^2+2y(0)y''(0)} = \frac{2y'(0)}{2+2y'(0)^2+2\cdot 0\cdot y''(0)} = \frac{y'(0)}{1+y'(0)^2}.\end{aligned} \tag{3}$$

3. (5 points) Consider the fact that

$$\begin{gathered} z = f(x,y) \\ \Rightarrow \\ dz = f_x(x,y)dx + f_y(x,y)dy. \end{gathered} \tag{4}$$

This suggests that

a) $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ is the equation for the tangent plane of the surface $z = f(x, y)$ at the point (x_0, y_0, z_0),

b) $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ is the equation for the line normal to the tangent plane of the surface $z = f(x, y)$ at the point (x_0, y_0, z_0),

c) $z = f_x(x_0, y_0)x + f_y(x_0, y_0)y$ is the equation for the tangent plane of the surface $z = f(x, y)$ at the point (x_0, y_0, z_0),

d) $z = f_x(x_0, y_0)x + f_y(x_0, y_0)y$ is the equation for the line normal to the tangent plane of the surface $z = f(x, y)$ at the point (x_0, y_0, z_0),

e) $z + z_0 = f_x(x_0, y_0)(x + x_0) + f_y(x_0, y_0)(y + y_0)$ is the equation for the tangent plane of the surface $z = f(x, y)$ at the point (x_0, y_0, z_0),

f) none of the above.

<u>Solution</u>: The correct answer is a).

4. (5 points) Let $F(x,y,z)$ be differentiable at (x_0,y_0,z_0), and let $\nabla F(x_0,y_0,z_0) \neq \langle 0,0,0 \rangle$. Then the level surface given by

$$F(x,y,z) = F(x_0,y_0,z_0) \tag{5}$$

has a unique tangent plane. Its equation is

a) $F_x(x_0,y_0,z_0)x + F_y(x_0,y_0,z_0)y + F_z(x_0,y_0,z_0)z = 0$,
b) $F_z(x_0,y_0,z_0)z = F_x(x_0,y_0,z_0)x + F_y(x_0,y_0,z_0)y$,
c) $F_z(x_0,y_0,z_0)(z+z_0) = F_x(x_0,y_0,z_0)(x+x_0) + F_y(x_0,y_0,z_0)(y+y_0)$,
d) $F_z(x_0,y_0,z_0)(z-z_0) = F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0)$,
e) $F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$,
f) none of the above.

Solution: e)

5. (5 points) A function $z = f(x,y)$ is differentiable at a point (a,b) if only

a) (a,b) is in the domain of f ,
b) (a,b) is not on the boundary of the domain of f ,
c) the partial derivatives f_x and f_y exist on a small disk centered at (a,b) and are continuous at (a,b),
d) f is continuous at (a,b),
e) the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ exist,
f) none of the above.

Solution: c) is a theorem.

Part II: *In the following problems, show all work, and simplify your results*.

6. (15 points)

Theorem: For every $(x, y) \in \mathbb{R}^2$ we have

$$(x^2 + y^4)^5 \geq x^4 y^{12}. \tag{6}$$

Proof: We find that

$$(x^2 + y^4)^5 - x^4 y^{12} = x^{10} + 5x^8 y^4 + 10x^6 y^8 + 9x^4 y^{12} + 5x^2 y^{16} + y^{20} \tag{7}$$

which clearly can't be negative.

Use the **Theorem** above and the squeeze theorem, etc., to prove that the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^4} \tag{8}$$

exists, and use those theorems to determine the value of the limit.

<u>**Solution**</u>: We compute that

$$\begin{aligned} 0 &\leq \left(\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^4} \right)^4 = \lim_{(x,y)\to(0,0)} \left(\frac{xy^3}{x^2 + y^4} \right)^4 = \lim_{(x,y)\to(0,0)} \frac{x^4 y^{12}}{\left(x^2 + y^4\right)^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{x^4 y^{12}}{\left(x^2 + y^4\right)^5} \left(x^2 + y^4\right) = \lim_{(x,y)\to(0,0)} \frac{x^4 y^{12}}{\left(x^2 + y^4\right)^5} \lim_{(x,y)\to(0,0)} \left(x^2 + y^4\right) \\ &\leq \lim_{(x,y)\to(0,0)} 1 \lim_{(x,y)\to(0,0)} \left(x^2 + y^4\right) = \lim_{(x,y)\to(0,0)} \left(x^2 + y^4\right) = 0^2 + 0^4 = 0, \end{aligned} \tag{9}$$

whence

$$\left(\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^4} \right)^4 = 0 \Leftrightarrow \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^4} = 0. \tag{10}$$

7. (10 points) Write out the Chain Rule for the case in which $w = f(x, y, z, t)$ and each of x, y, z and t are (differentiable) functions of variable u and v. Specifically, determine the functions $A(u,v)$ and $B(u,v)$ in the differential

$$dw = A(u,v)\,du + B(u,v)\,dv. \tag{11}$$

Solution: We have

$$\begin{aligned} A(u,v) &= \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial u}, \\ B(u,v) &= \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial v}. \end{aligned} \tag{12}$$

8. (10 points) Make the definition

$$g(h) := f(x+ah, y+bh) \tag{13}$$

and compute $g'(0)$ in terms of any, all, or some of the following numbers:

$$a, b, f(x,y), f_x(x,y), f_y(x,y), f_{xx}(x,y), f_{yx}(x,y), f_{xy}(x,y), f_{yy}(x,y). \tag{14}$$

Solution: We have

$$\begin{aligned} g'(0) &:= \lim_{h\to 0}\frac{g(h)-g(0)}{h} = \lim_{h\to 0}\frac{f(x+ah,y+bh)-f(x,y)}{h} =: \left(D_{\langle a,b\rangle} f\right)(x,y) \overset{\text{thm}}{=} f_x(x,y)a + f_y(x,y)b \\ &= \left\langle f_x(x,y), f_y(x,y)\right\rangle \cdot \left\langle a,b\right\rangle =: \left(\nabla f\right)(x,y)\cdot\left\langle a,b\right\rangle, \end{aligned} \tag{15}$$

using a biggish theorem. Without that particular theorem we could write more fundamentally that

$$\begin{aligned} g(h) &:= f(x+ah, y+bh) \\ &\Rightarrow \\ g'(h) &= f_x(x+ah,y+bh)\frac{\partial\left(x+ah\right)}{\partial h} + f_y(x+ah,y+bh)\frac{\partial\left(y+bh\right)}{\partial h} \\ &= f_x(x+ah,y+bh)a + f_y(x+ah,y+bh)b, \\ &\Rightarrow \\ g'(0) &= f_x(x+a\cdot 0, y+b\cdot 0)a + f_y(x+a\cdot 0, y+b\cdot 0)b \\ &= f_x(x,y)a + f_y(x,y)b, \end{aligned} \tag{16}$$

using the basic theorem about partial derivatives and composed functions, i.e., using a rather general version of the chain rule.

9. (10 points) Assuming the function $F = F(x, y, z)$ has a nonvanishing gradient vector at a point (x_0, y_0, z_0) on the level surface

$$F(x, y, z) = k = F(x_0, y_0, z_0), \tag{17}$$

determine an equation giving all points (x, y, z) on the plane tangent to that level surface (dictated by (17)) at that point (x_0, y_0, z_0). Why is your equation "information free" when the gradient vector vanishes at (x_0, y_0, z_0)? In what way does the information free equation actually make sense?

<u>Solution</u>: From (17) we immediately have that such a tangent plane is given "in the small" by the differential

$$0 = dk = dF(x, y, z) = F_x(x, y, z)dx + F_y(x, y, z)dy + F_z(x, y, z)dz \tag{18}$$

at any given point (x, y, z) on the level surface, or by

$$0 = F_x(x_0, y_0, z_0)dx + F_y(x_0, y_0, z_0)dy + F_z(x_0, y_0, z_0)dz \tag{19}$$

at the specific point (x_0, y_0, z_0) in question. "In the large" we have (19) is

$$0 = F_x(x_0, y_0, z_0)\left(x - x_0\right) + F_y(x_0, y_0, z_0)\left(y - y_0\right) + F_z(x_0, y_0, z_0)\left(z - z_0\right) \tag{20}$$

which is the desired equation. If the gradient vector vanishes at the point in question, then the equation is $0 = 0$, which is satisfied by all points (x, y, z), and which makes sense because when the gradient vector vanishes, there is no unique tangent plane, all planes are tangent, and the union of their solution sets is all points in three space.

10. (10 points) Consider the plane dictated by the equation

$$0 = F_x(x_0, y_0, z_0)\left(x - x_0\right) + F_y(x_0, y_0, z_0)\left(y - y_0\right) + F_z(x_0, y_0, z_0)\left(z - z_0\right) \tag{21}$$

Find the line that goes through this plane orthogonally at the point (x_0, y_0, z_0). Use any valid format to communicate this line unambiguously. What assumption do you have to make so that your proposed equation(s) for a line actually give a line, not just a point?

Solution: A vector normal to the plane is

$$\mathbf{n}_0 := \nabla F(x_0, y_0, z_0) = \left\langle F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0)\right\rangle \tag{22}$$

whence, the vector equation for such a line is

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}_0 + t\mathbf{n}_0 = \left\langle x_0, y_0, z_0\right\rangle + t\left\langle F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0)\right\rangle \\ &= \left\langle x_0 + tF_x(x_0, y_0, z_0), y_0 + tF_y(x_0, y_0, z_0), z_0 + tF_z(x_0, y_0, z_0)\right\rangle \quad t \in \mathbb{R}. \end{aligned} \tag{23}$$

This “line” degenerates to a point iff $\nabla F(x_0, y_0, z_0) = \mathbf{0}$.

11. (10 points) By using a Lagrange Multiplier, calculate the maximum value of

$$g(a,b) := f_x(x,y)a + f_y(x,y)b \tag{24}$$

subject to the side condition that

$$a^2 + b^2 = 1. \tag{25}$$

Also compute the minimum value of $g(a,b)$ subject to (25). Assume that $\left\langle f_x(x,y), f_y(x,y) \right\rangle \neq \langle 0,0 \rangle$. What does your computation prove?

Solution: Define

$$h(a,b) := a^2 + b^2 = \left\| \langle a,b \rangle \right\|^2 \tag{26}$$

so that constraint (25) is the equation

$$h(a,b) = 1. \tag{27}$$

The constrained extrema happen exactly when

$$\left\{ \nabla g(a,b), \nabla h(a,b) \right\} \tag{28}$$

form a linearly dependent set. (This is equivalent to the actual Lagrange multiplier statement, but more flexible, as one might notice below.) This happens exactly when a certain obvious matrix formed from these vectors is singular, i.e., when the following equation holds:

$$\begin{aligned}
0 &= \det\begin{bmatrix} \nabla g(a,b)^T \\ \nabla h(a,b)^T \end{bmatrix} = \det\begin{bmatrix} g_a(a,b) & g_b(a,b) \\ h_a(a,b) & h_b(a,b) \end{bmatrix} = \det\begin{bmatrix} f_x(x,y) & f_y(x,y) \\ 2a & 2b \end{bmatrix} \\
&= 2\det\begin{bmatrix} f_x(x,y) & f_y(x,y) \\ a & b \end{bmatrix} = 2\left(f_x(x,y)b - f_y(x,y)a \right) \\
&\Leftrightarrow \\
&\langle a,b \rangle \in Span\left\{ \left\langle f_x(x,y), f_y(x,y) \right\rangle \right\} = Span\left\{ \nabla f(x,y) \right\} := \left\{ \lambda \nabla f(x,y) \middle| \lambda \in \mathbb{R} \right\} \\
&\Leftrightarrow \\
&\langle a,b \rangle = \lambda \nabla f(x,y),
\end{aligned} \tag{29}$$

the latter for some $\lambda \in \mathbb{R}$ to be determined by satisfying the constraint (25). Inserting (29)'s last result into (25) gives

$$1 = \left\|\left\langle a,b\right\rangle\right\|^2 = \left\|\lambda\nabla f(x,y)\right\|^2 = \lambda^2\left\|\nabla f(x,y)\right\|^2$$
$$\Leftrightarrow \tag{30}$$
$$\lambda = \pm\frac{1}{\left\|\nabla f(x,y)\right\|} =: \lambda_\pm .$$

For $\lambda_+ := 1/\left\|\nabla f(x,y)\right\|$ we get

$$\begin{aligned} g(a,b) &:= f_x(x,y)a + f_y(x,y)b = \nabla f(x,y)\cdot\left\langle a,b\right\rangle \\ &= \nabla f(x,y)\cdot\lambda_+\nabla f(x,y) \\ &= \lambda_+\left\|\nabla f(x,y)\right\|^2 = \frac{1}{\left\|\nabla f(x,y)\right\|}\left\|\nabla f(x,y)\right\|^2 = \left\|\nabla f(x,y)\right\|, \end{aligned} \tag{31}$$

while for $\lambda_- := -1/\left\|\nabla f(x,y)\right\|$ we clearly get

$$\begin{aligned} g(a,b) &:= \nabla f(x,y)\cdot\lambda_-\nabla f(x,y) \\ &= \lambda_-\left\|\nabla f(x,y)\right\|^2 = -\frac{1}{\left\|\nabla f(x,y)\right\|}\left\|\nabla f(x,y)\right\|^2 = -\left\|\nabla f(x,y)\right\|, \end{aligned} \tag{32}$$

the former then the maximum value, the latter the minimum value. Noting as in (15) that

$$\left(D_{\langle a,b\rangle} f\right)(x,y) \overset{\text{thm}}{=} f_x(x,y)a + f_y(x,y)b \tag{33}$$

we have just proved that the maximum value of the directional derivative at a point is the norm of the gradient vector of the function to be differentiated at said point, the minimum value the negation of that.

12. (10 points) Consider taking the "second directional derivative" as follows:

$$\left(D^2_{\langle a,b\rangle} f\right)(x,y) := g''(0), \tag{34}$$

where

$$g(h) := f(x+ah, y+bh). \tag{35}$$

Compute this second directional derivative in terms of any, all, or some of the following numbers:

$$a, b, f(x,y), f_x(x,y), f_y(x,y), f_{xx}(x,y), f_{yx}(x,y), f_{xy}(x,y), f_{yy}(x,y). \tag{36}$$

It turns out that you can write your final result (with the terms in (36)) as

$$\left(D^2_{\langle a,b\rangle} f\right)(x,y) = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \tag{37}$$

with matrix elements $M_{11}, M_{12} = M_{21}, M_{22}$ to be determined (by you). Noting that

$$\begin{aligned} \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} M_{11}a + M_{12}b \\ M_{21}a + M_{22}b \end{bmatrix} = a\left(M_{11}a + M_{12}b\right) + b\left(M_{21}a + M_{22}b\right) \\ &= M_{11}a^2 + M_{12}ab + M_{21}ba + M_{22}b^2, \end{aligned} \tag{38}$$

you see that I've given you the hint that your answer should look like

$$\begin{aligned} \left.\frac{\partial^2}{\partial h^2} f(x+ah, y+bh)\right|_{h=0} &= M_{11}a^2 + M_{12}ab + M_{21}ba + M_{22}b^2 \\ &= M_{11}a^2 + \left(M_{12} + M_{21}\right)ab + M_{22}b^2 \\ &= M_{11}a^2 + 2M_{12}ab + M_{22}b^2. \end{aligned} \tag{39}$$

If $g'(0) = 0$ yet $g''(0) > 0$, then g has a local min at $h = 0$. If this is true independent of vector $\langle a,b\rangle$ with norm 1, then that means that f has a local min at (x,y). It turns out that $g''(0)$ is positive independent of normalized $\langle a,b\rangle$ happens iff both of the eigenvalues of the symmetric matrix

$$M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \tag{40}$$

are positive. But the characteristic polynomial for matrix M is

$$\begin{aligned}
0 &= \det\left(\lambda I_{2x2} - M\right) = \det\begin{bmatrix} \lambda - M_{11} & -M_{12} \\ -M_{12} & \lambda - M_{22} \end{bmatrix} = \left(\lambda - M_{11}\right)\left(\lambda - M_{22}\right) - \left(-M_{12}\right)\left(-M_{12}\right) \\
&= \lambda^2 - \left(M_{11} + M_{22}\right)\lambda + \det M \\
&= \lambda^2 - \left(M_{11} + M_{22}\right)\lambda + M_{11}M_{22} - M_{12}{}^2.
\end{aligned} \tag{41}$$

And from this it's not hard to show that both eigenvalues are positive if and only if

$$\det M := M_{11}M_{22} - M_{12}{}^2 \tag{42}$$

is positive and so is $M_{11} + M_{22}$--in which case f has a local minimum at (x, y). But then the criteria for both eigenvalues to be positive (hence get a minimum for f) is also exactly

$$M_{11}M_{22} - M_{12}{}^2 > 0 \quad \text{and} \quad M_{11} \text{ or } M_{22} > 0. \tag{43}$$

To see (43) as equivalent to (42) positive and $M_{11} + M_{22}$ positive, note that if M_{11} and M_{22} are nonpostive, we can't get $M_{11} + M_{22}$ positive, and if only one of them is nonpositive, the product $M_{11}M_{22}$ will be nonpositive, and $M_{11}M_{22} - M_{12}{}^2$ can't be positive. So ultimately we get that (43) is a sufficient condition for f to have a local min at (x, y). Once you compute your answer to the first question, which is equivalent to finding the matrix elements $M_{11}, M_{12} = M_{21}, M_{22}$, tell me why it is that you could have expected that (43) is sufficient for f to have a minimum.

Solution: From (16) we already have

$$\begin{aligned}
g'(h) &= f_x(x+ah, y+bh)a + f_y(x+ah, y+bh)b \\
&\Rightarrow \\
g''(h) &= \left(f_{xx}(x+ah, y+bh)\frac{\partial\left(x+ah\right)}{\partial h} + f_{xy}(x+ah, y+bh)\frac{\partial\left(y+bh\right)}{\partial h} \right)a \\
&\quad + \left(f_{yx}(x+ah, y+bh)\frac{\partial\left(x+ah\right)}{\partial h} + f_{yy}(x+ah, y+bh)\frac{\partial\left(y+bh\right)}{\partial h} \right)b \\
&= \left(f_{xx}(x+ah, y+bh)a + f_{xy}(x+ah, y+bh)b \right)a \\
&\quad + \left(f_{yx}(x+ah, y+bh)a + f_{yy}(x+ah, y+bh)b \right)b \\
&\Rightarrow \\
g''(0) &= \left(f_{xx}(x,y)a + f_{xy}(x,y)b \right)a + \left(f_{yx}(x,y)a + f_{yy}(x,y)b \right)b \\
&= f_{xx}(x,y)a^2 + f_{xy}(x,y)ba + f_{yx}(x,y)ab + f_{yy}(x,y)b^2,
\end{aligned} \tag{44}$$

where we used the chain rule again, and whence we get

$$\begin{aligned}\left(D^2_{\langle a,b\rangle} f\right)(x,y) &:= f_{xx}(x,y)a^2 + f_{xy}(x,y)ba + f_{yx}(x,y)ab + f_{yy}(x,y)b^2 \\ &= f_{xx}(x,y)a^2 + \left(f_{xy}(x,y) + f_{yx}(x,y)\right)ab + f_{yy}(x,y)b^2 \\ &= f_{xx}(x,y)a^2 + 2 f_{xy}(x,y)ab + f_{yy}(x,y)b^2,\end{aligned} \tag{45}$$

and whence we can choose

$$M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix} = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{bmatrix}, \tag{46}$$

and so the sufficient condition (43) for a minimum is the usual one we've learned as a theorem. (And we've done a lot of the work here in proving that theorem.)