Name: ________________

Student ID: ________________

Section: ________________

Instructor: Scott Glasgow

Math 314 (Calculus of Several Variables) **RED**

Exam 4 June 11,12, 2013

Instructions:

- For questions which require a written answer, show all your work. Full credit will be given only if the necessary work is shown, justifying your answer.
- Simplify your answers.
- Calculators are not allowed. Textbooks are not allowed. Notes are not allowed.
- Should you have need for more space than is allotted to answer a question, use the back of the page the problem is on and indicate this fact.
- Talking about the exam with other students before the graded exam is returned to you is a violation of the Honor Code.

Part I: Multiple Choice *Mark the correct answer on the bubble sheet provided*.

1. (5 points) Suppose that we want to compute the double integral

$$I := \iint_R f(x,y)dA \qquad (1)$$

given that i) $f(x,y) \geq 0$ for all $(x,y) \in R$ and that ii) R is a rectangular region. (Think of R as a rectangle in the $z=0$ plane.) The connection we may make to compute this integral is that $I =$

a) the volume of the solid that lies above the rectangle R and below the surface $z = f(x,y)$.
b) the average value of f over the region R
c) the average value of f over the region R divided by the area of R
d) the area of R
e) the area of the surface $z = f(x,y)$ over the region R times the average value of f over R
f) the area of the surface $z = f(x,y)$ over the region R

Solution: The correct answer is a). One tautological way to know this is that (1) is taken to be the definition of said volume in the book. Yet one can show this notion agrees other notions.

2. (5 points) Suppose that for $(x,y) \in R$ we have $f(x,y) \in [c,d]$, and let

$$A := \iint_R 1 dA \qquad (2)$$

Then the average value $\overline{f}$ of f over R satisfies

a) $\overline{f} \in [Ac, Ad]$
b) $\overline{f} \in [c/A, d/A]$
c) $\overline{f} \in [c/A, Ad]$
d) $\overline{f} \in [Ac, d/A]$
e) $\overline{f} = A$
f) $\overline{f} \in [c,d]$

Solution: The correct answer is f). Not sure what to say about that.

3. (5 points) Suppose f is continuous and nonnegative over rectangle $R=\{(x,y)|x\in[a,b],y\in[c,d]\}$, and define integral I as in (1). Then

a) I is the limit of a double Riemann sum
b) $I=\int_a^b dx\int_c^d dy\, f(x,y)$
c) I is the volume of a solid with base R
d) $I=\int_c^d dy\, \int_a^b dx\, f(x,y)$
e) all of the above
f) none of the above

Solution: The correct answer is e).

4. (5 points) Consider the iterated integral

$$I=\int_a^b dx\int_0^{g(x)} dy\, f(x,y), \tag{3}$$

where f and g are continuous. If g is strictly ***decreasing*** over $x\in[a,b]$, then which of the following iterated integrals (or sums of such) are numerically equivalent to I ? (Hint: Draw a picture of the graph of g . From that picture, note that the bounding curve $C=\{(x,g(x))|x\in[a,b]\}$ of the domain of integration can also be rewritten as $C=\{(g^{-1}(y),y)|y\in[g(b),g(a)]\}$.)

a) $\int_{g(b)}^{g(a)} dy\int_a^b dx\, f(x,y)$, b) $\int_{g(b)}^{g(a)} dy\int_{g^{-1}(y)}^b dx\, f(x,y)$, c) $\int_0^{g(a)} dy\int_a^b dx\, f(x,y)$,

d) $\int_0^{g(b)} dy\int_a^b dx\, f(x,y)$, e) $\int_0^{g(b)} dy\int_a^b dx\, f(x,y)+\int_{g(b)}^{g(a)} dy\int_{g^{-1}(y)}^b dx\, f(x,y)$,

f) $\int_0^{g(b)} dy\int_a^b dx\, f(x,y)+\int_{g(b)}^{g(a)} dy\int_a^{g^{-1}(y)} dx\, f(x,y)$.

Solution: The correct answer is f). Draw a picture.

5. (5 points) Given that f has continuous mixed partial derivatives, and $R=\{(x,y)|x\in[a,b],y\in[c,d]\}$, what is the value of the integral

$$I=\iint_R f_{xy}(x,y)dA? \qquad (4)$$

a) $f(a,d)+f(a,c)-f(b,d)-f(b,c)$, b) $f(a,d)-f(b,d)-f(b,c)+f(a,c)$,

c) $f(b,d)+f(b,c)-f(a,d)-f(a,c)$, d) $f(b,d)-f(b,c)-f(a,d)+f(a,c)$,

e) $f(b,d)+f(b,c)+f(a,d)+f(a,c)$, f) $f(b,d)-f(b,c)+f(a,d)-f(a,c)$.

<u>Solution</u>: By Fubini and the equality of mixed partials, and the fundamental theorem, we have

$$\begin{aligned}
I&=\iint_R f_{xy}(x,y)dA=\int_a^b dx\int_c^d dy\, f_{xy}(x,y)=\int_a^b dx\int_c^d dy\,\partial_y f_x(x,y)\\
&=\int_a^b dx\int_{y=c}^{y=d} df_x(x,y)=\int_a^b dx\,\left[f_x(x,y)\right]_{y=c}^{y=d}=\int_a^b dx\,\left(f_x(x,d)-f_x(x,c)\right)\\
&=\int_a^b dx\,\frac{d}{dx}\left(f(x,d)-f(x,c)\right)=\int_a^b d\left(f(x,d)-f(x,c)\right)\\
&=\left[f(x,d)-f(x,c)\right]_{x=a}^{x=b}=\left(f(b,d)-f(b,c)\right)-\left(f(a,d)-f(a,c)\right)\\
&=f(b,d)-f(b,c)-f(a,d)+f(a,c).
\end{aligned} \qquad (5)$$

Hence the correct answer is d).

6. (5 Points) Suppose a region D lies entirely within a rectangle $R=\{(x,y)|x\in[a,b],y\in[c,d]\}$, and suppose a function F is the same as another function f on D but is zero for all $(x,y)\in R-D$. ($R-D$ denotes the points in R not in D). Then we always have

a) $\iint_D F(x,y)dA=\iint_R f(x,y)dA$

b) $\iint_D F(x,y)dA\le\iint_R f(x,y)dA$

c) $\iint_D f(x,y)dA=\iint_R F(x,y)dA$

d) $\iint_D F(x,y)dA\ge\iint_R f(x,y)dA$

e) $\iint_D f(x,y)dA>\iint_R F(x,y)dA$

f) $\iint_D F(x,y)dA<\iint_R f(x,y)dA$

Solution: c) defines the object on the left there in terms of the object on the right.

Part II: *In the following problems, show all work, and simplify your results*.

7. (10 points) Given that D is the triangular region in the xy-plane with vertices $(0,0)$, $(1,0)$, and $(1,1)$, what is the value of the integral

$$I = \iint_D dA\ (x^2 + y^2)? \tag{6}$$

Solution: We have D is a Type 1 (and Type 2) region, one that can be written as

$$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\}, \tag{7}$$

whence, since $f(x,y) := x^2 + y^2$ is continuous there, by a certain theorem (related to Fubini's) we have

$$I \overset{\text{def}}{=} \iint_D dA\, f(x,y) \overset{\text{thm}}{=} \int_0^1 dx \int_0^x dy\ (x^2 + y^2). \tag{8}$$

So by two uses of the fundamental theorem, we have

$$\begin{aligned} \int_0^1 dx \int_0^x dy\ (x^2 + y^2) &= \int_0^1 dx \left[x^2 y + \frac{1}{3} y^3 \right]_{y=0}^{y=x} = \int_0^1 dx \left(x^2 x + \frac{1}{3} x^3 \right) = \int_0^1 dx \left(\frac{4}{3} x^3 \right) \\ &= \left[\frac{1}{3} x^4 \right]_{x=0}^{x=1} = \frac{1}{3} 1^4 = \frac{1}{3}. \end{aligned} \tag{9}$$

8. (15 points) A lamina occupies a region D of the xy-plane enclosed by the lines $x=0$, $y=2x$, and $y=2$. If the density of the lamina is

$$\rho(x,y)=30(x+y), \tag{10}$$

what is the moment of inertia of the lamina about the origin $(0,0)$? (Equivalently, what is the moment of inertia of the object when rotated about the z axis?)

<u>Solution</u>: By definition

$$\begin{aligned} I_{(0,0)} &:= \iint_R dA\, d^2\big((x,y),(0,0)\big)\rho(x,y) \\ &= \iint_R dA\, \left\|\langle x,y\rangle-\langle 0,0\rangle\right\|^2 \rho(x,y) = \iint_R dA\, \left\|\langle x,y\rangle\right\|^2 \rho(x,y) \\ &= \iint_R dA\, \left(x^2+y^2\right)\rho(x,y), \end{aligned} \tag{11}$$

whence, for the problem at hand

$$\begin{aligned} I_{(0,0)} &= \iint_R dA\, \left(x^2+y^2\right)\rho(x,y) = \iint_R dA\, \left(x^2+y^2\right)30(x+y) \\ &= 30\iint_R dA\, \left(x^3+x^2y+y^2x+y^3\right) \end{aligned} \tag{12}$$

Now we have

$$D=\left\{(x,y)\middle|\, 0\le x\le 1\,,\ 2x\le y\le 2\right\} \tag{13}$$

So by a certain theorem (related to Fubini's) we have

$$
\begin{aligned}
I_{(0,0)} &= 30\iint_R dA\ \left(x^3+x^2y+y^2x+y^3\right) = 30\int_0^1 dx\ \int_{2x}^2 dy\left(x^3+x^2y+xy^2+y^3\right) \\
&= 30\int_0^1 dx\ \left[x^3y+\frac{1}{2}x^2y^2+\frac{1}{3}xy^3+\frac{1}{4}y^4\right]_{y=2x}^{y=2} \\
&= 30\int_0^1 dx\ \left(2x^3+\frac{2^2}{2}x^2+\frac{2^3}{3}x+\frac{2^4}{4}-\left(x^3(2x)+\frac{1}{2}x^2(2x)^2+\frac{1}{3}x(2x)^3+\frac{1}{4}(2x)^4\right)\right) \\
&= 30\int_0^1 dx\ \left(2x^3+2x^2+\frac{2^3}{3}x+2^2-\left(2+2+\frac{2^3}{3}+2^2\right)x^4\right) \\
&= 30\left[\frac{2}{4}x^4+\frac{2}{3}x^3+\frac{2^2}{3}x^2+2^2x-\frac{1}{5}\left(2+2+\frac{2^3}{3}+2^2\right)x^5\right]_{x=0}^{x=1} \\
&= 30\left(\frac{2}{4}1^4+\frac{2}{3}1^3+\frac{2^2}{3}1^2+2^21-\frac{1}{5}\left(2+2+\frac{2^3}{3}+2^2\right)1^5\right) = 30\left(\frac{1}{2}+\frac{2}{3}+\frac{2^2}{3}+2^2-\frac{2^5}{3\cdot 5}\right) \\
&= 30\left(\frac{1}{2}+6-\frac{2^5}{3\cdot 5}\right) = 30\frac{\left(3\cdot 5\cdot 13-2^6\right)}{2\cdot 3\cdot 5} = 195-64 = 131.
\end{aligned} \tag{14}
$$

We can also write the lamina as

$$
D=\left\{(x,y)\middle|\,0\le y\le 2\ ,\ 0\le x\le y/2\right\} \tag{15}
$$

whence we could have also computed that

$$
\begin{aligned}
I_{(0,0)} &= 30\iint_R dA\ \left(x^3+x^2y+y^2x+y^3\right) = 30\int_0^2 dy\ \int_0^{y/2} dx\left(x^3+yx^2+y^2x+y^3\right) \\
&= 30\int_0^2 dy\ \left[\frac{1}{4}x^4+\frac{1}{3}yx^3+\frac{1}{2}y^2x^2+y^3x\right]_0^{y/2} \\
&= 30\int_0^1 2dy\ \left[\frac{1}{4}x^4+\frac{1}{3}2yx^3+2y^2x^2+2^3y^3x\right]_0^{y} \\
&= 2^2\cdot 3\cdot 5\int_0^1 dy\ \left(\frac{1}{4}+\frac{2}{3}+2+2^3\right)y^4 \\
&= 2^2\cdot 3\cdot 5\left[\left(\frac{1}{4}+\frac{2}{3}+2+2^3\right)\frac{y^5}{5}\right]_{y=0}^{y=1} \\
&= 3+2^3+2^3\cdot 5\cdot 3 = 3+2^3\cdot 2^4 = 3+128 = 131.
\end{aligned} \tag{16}
$$

9. (10 points) Let D be the region in the second quadrant of the xy-plane that lies between the circles of radius $r_0\ (>0)$ and $R_0\ (>r_0)$.

a) Give a description of D in polar coordinates.

Solution: The second quadrant has the (azimuthal) angle $\theta \in \left[\pi/2, \pi\right]$, whence

$$D = \left\{ (r,\theta) \middle| r_0 < r < R_0, \quad \pi/2 \le \theta \le \pi \right\}. \tag{17}$$

b) For density $\rho(x,y) = x^2 y$, find the mass M of the lamina defined by D. Evaluate the integral by using polar coordinates. (Note that if $R_0 = 2$ and $r_0 = 1$, the answer should be 31/15.)

Solution: By definition, and then by Fubini and (17), along with the Jacobian rule, we have

$$\begin{aligned} M &= \iint_D dA\, \rho(x,y) = \iint_D dA\, x^2 y \\ &= \iint_D r dr d\theta \left(r^2 \cos^2\theta\, r \sin\theta \right) \\ &= \int_{r_0}^{R_0} dr \int_{\pi/2}^{\pi} d\theta\, r^4 \cos^2\theta \sin\theta \\ &= \int_{r_0}^{R_0} dr\ r^4 \int_{\pi/2}^{\pi} d\theta\ \cos^2\theta \sin\theta \\ &= \int_{r_0}^{R_0} d\frac{r^5}{5} \int_{\pi/2}^{\pi} d - \frac{\cos^3\theta}{3} \\ &= \left(\frac{R_0^{\ 5} - r_0^{\ 5}}{5} \right) \left(\frac{-\cos^3\pi + \cos^3 \pi/2}{3} \right) \\ &= \left(\frac{R_0^{\ 5} - r_0^{\ 5}}{5} \right) \frac{-(-1)^3 + 0}{3} = \frac{R_0^{\ 5} - r_0^{\ 5}}{15}. \end{aligned} \tag{18}$$

where we also used the change of coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{19}$$

10. (10 points) Use cylindrical coordinates to find the volume of the solid E that lies above the generalized paraboloid $z = k_1 + 2\left(x^2 + y^2\right)^2$ and below the generalized paraboloid $z = k_2 - 2\left(x^2 + y^2\right)^2$, where $k_2 > k_1$. Hint: the answer is

$$V = \pi \frac{\left(k_2 - k_1\right)^{3/2}}{3}. \tag{20}$$

<u>Solution</u>: First we note that the paraboloids meet when

$$z_{bottom} := k_1 + 2\left(x^2 + y^2\right)^2 = k_2 - 2\left(x^2 + y^2\right)^2 =: z_{top}, \tag{21}$$

i.e., when

$$r^4 = \left(x^2 + y^2\right)^2 = \frac{k_2 - k_1}{4} =: r_0^{\,4} > 0, \tag{22}$$

whence we have that the region in question is

$$E = \left\{ (r, \theta, z) \middle| 0 \le r \le r_0 := \sqrt[4]{\frac{k_2 - k_1}{4}},\ \ 0 \le \theta < 2\pi,\ \ k_1 + 2r^4 \le z \le k_2 - 2r^4 \right\}. \tag{23}$$

Thus we have that the volume of E is given by

$$\begin{aligned} V_E &= \iiint_E dV = \iiint_E r dr d\theta dz = \int_0^{2\pi} d\theta \int_0^{r_0} dr\ r \int_{k_1+2r^4}^{k_2-2r^4} dz = \\ &= 2\pi \int_0^{r_0} dr\ r\left(k_2 - 2r^4 - \left(k_1 + 2r^4\right)\right) \\ &= 2\pi \int_0^{r_0} dr\ \left(\left(k_2 - k_1\right) r - 4r^5\right) \\ &= 2\pi \int_0^{r_0} d\left(\left(k_2 - k_1\right)\frac{r^2}{2} - \frac{4r^6}{6}\right) = \pi\left(\left(k_2 - k_1\right) r_0^{\,2} - \frac{4r_0^{\,6}}{3}\right) \\ &= \pi\left(4r_0^{\,4} r_0^{\,2} - \frac{4r_0^{\,6}}{3}\right) = \pi \frac{8}{3} r_0^{\,6} = \pi \frac{8}{3}\left(\frac{k_2 - k_1}{4}\right)^{3/2} = \pi \frac{\left(k_2 - k_1\right)^{3/2}}{3}. \end{aligned} \tag{24}$$

11. (15 points) For $\alpha > 0$, let B be the conical solid bounded below by the cone $z = \alpha\sqrt{x^2 + y^2}$, and bounded above by the sphere $x^2 + y^2 + z^2 = R_0^{\ 2}, R_0 > 0$.

a) Give a description of B in spherical coordinates and b) find the volume V of the solid. Hint: For $\alpha = 1$ and $R_0 = \sqrt{2}$ the volume is given by

$$V = \frac{4}{3}\pi\left(\sqrt{2} - 1\right). \tag{25}$$

Solution: First we note that the sphere and cone meet when

$$\begin{aligned} & R_0^{\ 2} = x^2 + y^2 + z^2, z = \alpha\sqrt{x^2 + y^2} \\ & \Leftrightarrow \\ & \tan\phi = \frac{\sqrt{x^2 + y^2}}{z} = \alpha^{-1} =: \tan\phi_0, \\ & \rho = \sqrt{x^2 + y^2 + z^2} = R_0, \end{aligned} \tag{26}$$

thus

$$B = \left\{ (\rho, \theta, \phi) \middle| 0 \le \rho \le R_0,\ \ 0 \le \theta < 2\pi,\ \ 0 \le \phi \le \tan^{-1}\left(\frac{1}{\alpha}\right) =: \phi_0 \right\} \tag{27}$$

answering a). For b) we have

$$\begin{aligned} V &= \iiint_B dV = \iiint_B \rho^2 \sin\phi \ d\rho d\theta d\phi \\ &= \int_0^{2\pi} d\theta \int_0^{\phi_0} d\phi \ \sin\phi \int_0^{R_0} d\rho \ \rho^2 \\ &= 2\pi \int_0^{\phi_0} d\left(-\cos\phi\right) \int_0^{R_0} d\,\frac{\rho^3}{3} \ = 2\pi\left(-\cos\phi_0 + \cos 0\right)\frac{R_0^{\ 3}}{3} \\ &= 2\pi\left(1 - \cos\phi_0\right)\frac{R_0^{\ 3}}{3}. \end{aligned} \tag{28}$$

Of course we have (draw a picture)

$$\cos\phi_0 = \cos\left(\tan^{-1}\left(\frac{1}{\alpha}\right)\right) = \frac{\alpha}{\sqrt{1 + \alpha^2}}, \tag{29}$$

whence

$$V = 2\pi\left(1-\cos\phi_0\right)\frac{R_0^{\ 3}}{3} = 2\pi\left(1-\frac{\alpha}{\sqrt{1+\alpha^2}}\right)\frac{R_0^{\ 3}}{3}. \tag{30}$$

12. (10 points) Consider the change of variable

$$\begin{gathered} x = st\cos\theta, \quad y = st\sin\theta, \quad z = \frac{t^2-s^2}{2} \\ \Leftrightarrow \\ \tan\theta = \frac{y}{x}, \quad s^2 = \sqrt{x^2+y^2+z^2}-z, \quad t^2 = \sqrt{x^2+y^2+z^2}+z. \end{gathered} \tag{31}$$

Here (x,y,z) denotes a point in 3-space labeled by Cartesian coordinates, while (s,t,θ) labels a point with "parabolic" coordinates. We take the convention that $s,t \geq 0$. In Cartesian coordinates, the volume element is given by

$$dV = dxdydz, \tag{32}$$

while in parabolic coordinates we have

$$dV = J(s,t,\theta)\,dsdtd\theta \tag{33}$$

for some function $J(s,t,\theta)$ to be determined—by you. Mark, get set, go, find $J(s,t,\theta)$.

Solution: From the first part of (31) we get

$$\begin{aligned} dx &= t\cos\theta ds + s\cos\theta dt - st\sin\theta d\theta \\ dy &= t\sin\theta ds + s\sin\theta dt + st\cos\theta d\theta \\ dz &= -sds + tdt + 0d\theta \end{aligned} \tag{34}$$

whence, since the product of differentials in (32) is actually a wedge product we have

$$\begin{aligned} J(s,t,\theta) &= \left|\det\begin{bmatrix} t\cos\theta & s\cos\theta & -st\sin\theta \\ t\sin\theta & s\sin\theta & st\cos\theta \\ -s & t & 0 \end{bmatrix}\right| \\ &= \left|-s\det\begin{bmatrix} s\cos\theta & -st\sin\theta \\ s\sin\theta & st\cos\theta \end{bmatrix} - t\det\begin{bmatrix} t\cos\theta & -st\sin\theta \\ t\sin\theta & st\cos\theta \end{bmatrix}\right| \\ &= \left|-s^3t\det\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} - st^3\det\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right| \\ &= |st|\left(s^2+t^2\right). \end{aligned} \tag{35}$$

With convention $s, t \geq 0$, this is

$$J\left(s,t,\theta\right) = st\left(s^2 + t^2\right). \tag{36}$$