1

Extra Credit Math Problem

Imagine terrorists set off a nuclear bomb in Omaha, Nebraska. The initial radiation level is 50 G per hour (1 Gray is 1 joule of energy absorbed per kilogram of tissue). A lethal dose is >5 G and an incapacitating dose is >1.5 G. You want to leave the city, but it takes **1.5** hours to get out of the city and you need to wait until radiation levels drop to where your total exposure will be less than 1.5 G before you can leave your home.

1. Estimate the amount of time that would elapse before it would first be safe to go outside (i.e. when the ambient radiation level first drops to a level where 1.5 hours of exposure at that level is less than incapacitation dose) by constructing a differential equation and solving it. Assume that 1 hour after the detonation the ambient radiation level is 30 G per hour.

Solution:

$$\frac{dy}{dt} = -ry$$

$$\frac{dy}{y} = -rdt$$

$$y = y_0 e^{-rt}$$

$$y(0) = 50$$

$$50 = y_0 e^0$$

$$y_0 = 50$$

$$y = 50e^{-rt}$$

$$30 = 50e^{-r*1}$$

$$\ln(\frac{50}{30}) = r = .511$$

$$y = 50e^{-\ln(\frac{50}{30})t}$$

$$1.5hr \times y\frac{G}{hr} = 1.5G$$

$$y = 1\frac{G}{hr}$$

$$\frac{1}{\ln(\frac{50}{30})}\ln(50) = 7.65hr$$

explain – recall that every formula should be part of a complete sentence.

(19 points)

2. The previous model assumes that ambient radiation exposurewill eventually gradually decrease to 0. In fact, we are constantly exposed to ambient radiation at a level of 2.4

mG per year from many sources. Construct a new differential equation that takes this into account and solve it (hint: it is of the same form as the equation in part 1 with a constant added).

Solution:

$$2.4\frac{mG}{yr} = 2.7\times10^{-7}\frac{G}{hr}$$

$$\frac{dy}{dt} = -ry + C$$

$$\frac{dy}{dt} + ry = C$$

$$\mu = e^{-\int -r dt}$$

$$\mu = e^{rt}$$

$$e^{rt}\frac{dy}{dt} + re^{rt}y = Ce^{rt}$$

$$\frac{d}{dt}(ye^{rt}) = \frac{d}{dt}(\frac{C}{r}e^{rt} + K)$$

$$y = \frac{C}{r} + Ke^{-rt}$$

ditto
(as in last problem)

....

$$y = 2.7\times10^{-7} + 50e^{-\ln(\frac{50}{30})t}$$

(19 points)

Determine whether the equation is exact. If it is exact, find the solution.

1) $(5x^4 + 8x - 16xy - 6x^2\cos y)\,dx + (7x^3 - 8x^2 y - 2x^3 \cos y)\,dy = 0$

Solution:

- definitions of M & N?

We see that $M_y(x,y) = -16x + 6x^2 \sin y$

and

$N_x(x,y) = -16x + 6x^2 \sin y$

so

$M_y(x,y) = N_x(x,y)$, so the equation is exact. (7 points)

Thus there is a $\Psi(x, y)$ such that

$$\Psi_x(x, y) = 5x^4 + 8x - 16xy - 6x^2 \cos y$$
$$\Psi_y(x, y) = 7x^3 - 8x^2 y - 2x^3 \cos y$$

Integrating the first of these equations, we obtain

$\Psi = \int M dx = \int 5x^4 + 8x - 16xy - 6x^2 \cos y$ which yields

$\Psi = x^4 + 4x^2 - 8x^2 y - 2x^3 \cos y + C(y)$ (10 points)

Q-font

Setting $\Psi_y = N$ gives

$$\Psi_y = -8x^2 - 2x^3 \sin y + C'(y) = 7x^3 - 8x^2 y - 2x^3 \cos y$$
$$C'(y) = 7x^3 - 8x^2 y - 2x^3 \cos y + 8x^2 + 2x^3 \sin y$$ (5 points)

- inconsistent with assumption that $C = C(y)$ (not $C = C(x,y)$)

Now we will integrate $C'(y)$ to find C(y).

$$\int C'(y) = \int (7x^3 - 8x^2 y - 2x^3 \cos y + 8x^2 + 2x^3 \sin y)dy$$
$$C(y) = \frac{7}{4} y^4 - 4x^2 y^2 + 2x\sin y + 8x^2 y - 2x^3 \cos y$$ (10 points)

To finalize the solution we now substitute C(y) into Ψ which gives

$$5x^4 + 4x^2 - 8x^2 y - 2x^3 \cos y + \frac{7}{4} y^4 - 4x^2 y^2 + 2x\sin y + 8x^2 y - 2x^3 \cos y = c$$

(5 points)

dubious because of "$C = C(y) = C(x,y)$ issue."

37 points

Alcohol poisoning occurs when someone consumes alcohol more quickly than their body can metabolize it. The body can metabolize one standard drink per hour, about 5 ounces of an alcoholic drink.

BAC stands for Blood Alcohol Content, and is the number of milligrams of alcohol per milliliter in your bloodstream. In New Jersey, the legal definition of drunkenness is a BAC of 0.08 mg/ml.

If you are a 160 lb. man who consumes1 drink in one hour, your BAC will be 0.02 mg per ml.

Alcohol absorption and elimination occur simultaneously and start with consumption. Absorption is the passage of alcohol into the blood. Elimination is the removal of alcohol from the body.

Elimination occurs at a constant rate for a given individual. The rate of decrease for a 160 lb. man is considered to be 15% of his BAC level per hour.

If Tom, a 160 lb. man, BAC level is at 0.18 mg/ml how long will it take for his BAC to decrease to 0.07 mg/ml so that he does not exceed the legal definition of drunkenness?

Solution:

Let B= BAC in mg/ml.
Let t= time in hours

The initial condition is B(0) = 0.18

Let $\frac{dB}{dt}$ = the rate of change of BAC levels, so

$$\frac{dB}{dt} = -.15\text{ B}$$

Solve for the Initial Value Problem

$$\frac{dB}{B} = -.15dt$$

Integrate both sides

$$\int \frac{dB}{B} = \int -0.15dt$$

Solve

$\ln|B|$ = -0.15t + C, where C is an arbitrary constant.

$e^\wedge (\ln |B|) = e^\wedge (-0.15t + C)$
$B = C_1 e^\wedge(-0.15t)$

Solve for C_1 at the initial condition, $B(0) = 0.18$

$0.18 = C_1 e^\wedge(-0.15(0))$
$0.18 = C_1$

better explanation

The equation is now

$B = 0.18e^\wedge(-0.15t)$

Solve for t when $B=0.07$

$$\frac{0.07}{0.18} = e^\wedge(-0.15t)$$

$$\ln\frac{0.07}{0.18} = \ln(e^\wedge(-0.15t))$$

$-0.9445 = -0.15t$

$t = 6.297$ hours

Answer: in 6.297 hours, Tom's BAC level will change from 0.18 to 0.07 mg/ml, so that he will no longer be legally drunk. But he still should not drive☺

37 points

-Ah! First problem that isn't just exponential decay.

Solve the initial value problem
$y' = -y/2t - t^2,\ y(1) = 2$

Solution:
Ch. 2.1. We first notice that this equation is not separable. We will then use an integrating factor to solve the equation. The integrating factor is found by getting the y terms on one side and everything else on the other

$y' + y/2t = -t^2$

Let $p(t) = 1/2t$, then the integrating factor is

$$\mu(t) = e^{\int p(t)dt} = e^{\int 1/2t dt} = e^{\ln 2t} = 2t$$

We then multiply through by the integrating factor getting

good explanation.

$2ty' + 2ty/2t = 2tt^2$ which simplifies to $2ty' + y = 2t^3$
or
$d/dt(2yt) = 2t^3 = d/dt(t^4/2 + c)$

Integrating both sides gives

$2yt = t^4/2 + c$ We then solve for y to get $y = t^3/4 + c/2t$

Using the initial condition $y(1) = 2$ we get

$2 = 1^3/4 + c/(2*1) \Rightarrow c = 7/2$

Which gives us

$y = t^3/4 + 7/4t$

Brandon Taylor

2

1 a) (5 pts.) Draw a slope field for $dp/dt = 530 - .5y$.

?

b) (5 pts.) Describe the behavior of the solution as $t \to \infty$.

p will go to 1060. – why?

sign error

c) (20 pts.) Find a general solution for $dp/dt = 530 - .5y$.

$$\frac{dp}{dt} = 530 - .5p \not\to \frac{dp}{.5\,dt} = p - 1060 \to \frac{dp}{p - 1060} = .5\,dt$$

$$\int \frac{dp}{p - 1060} = \int .5\,dt \to \ln|p - 1060| = .5t + C \to p - 1060 = \pm e^{.5t + C}$$

$$\to p = 1060 \pm e^{5t} \times e^{C} \to p = 1060 + ce^{5t}$$

Inconsistent (because of sign error).

2) (45points) One sunny afternoon you decide to go to the fair to take a break from your school work. You are having fun but before long the heat becomes sweltering. You look for a drinking fountain, but to no avail. You will be forced to pay for the overly priced fair drink or perish from heat exhaustion. You quickly decide that the largest drink is the best value even if it is 3 times the normal price. You are well aware however, that your supposedly 1500ml drink will only contain 1000ml of pop and 500ml worth of ice. As you buy your drink you glance at the mixers regulator. It's set at 50g of flavoring for every 1000ml of pop. *Perfect, just the way I like it.* But what bad luck you have, just as you are handed your drink, a young boy trying to pop balloons with a dart, throws one way out of control and it stabs right into the base of your cup. As you scramble to catch the stream of leaking fluid you realize several things. First you recognize that the same instant you bought the drink and the cup was punctured the ice began to melt at a rate of .2ml/s. You also know the exact volume your mouth holds and by timing how long it takes to fill from the leak, you determine it is leaking at a rate of .4ml/s. You continue to catch the slow trickle in your mouth as you realize a fundamental flaw you made in purchasing your drink. The day has not yet reached a maximum temperature and if you drink your drink too soon you will overheat again anyway. You need to save the drink for as long as possible to allow for the maximum effect of cooling. Meanwhile you are keeping cool with the small flow you are getting from the hole. You realize however that if you wait too long too much ice will melt and you will be left with a diluted drink that you would consider "nasty" and refuse to drink. This threshold of acceptability is at a concentration of 30g/1000ml. While poising the cup above your head with one hand you quickly begin to scribble some math you are now glad you learned from Professor Glasgow. You need to find an equation for the concentration (C) of flavoring in the drink at any given time, determine how long this equation will be valid and finally find at what instant you must chug the remaining pop to keep it from dropping below the acceptable level.

wow!

Can you finish your mad math scramble before this time passes?

We start by writing our differential equation for the change in amount of chemical with respect to time

$$\frac{dQ}{dt} = -.4\left(\frac{ml}{s}\right)\left(\frac{Q}{v}\right)$$

Because the volume changes over time, we need to find an equation for v. We again use a differential equation

$$\frac{dv}{dt} = -.4\left(\frac{ml}{s}\right) + .2\left(\frac{ml}{s}\right)$$

Integrating both sides we get

$$v = -.2\,t + k$$

Substitute initial data and solve for k

$$1000 = -.2(0) + k$$

$$k = 1000\ ml$$

$$v = -.2(t) + 1000$$

Substitute in for v

$$\frac{dQ}{dt} = -.4\left(\frac{Q}{1000 - .2\,t}\right)$$

Separate the variables

$$\frac{dQ}{Q} = -\frac{.4\,dt}{1000 - .2\,t}$$

To simplify the right side divide numerator and denominator by -.2

$$\frac{dQ}{Q} = \frac{2\,dt}{t - 5000}$$

Integrate both sides

$$\ln|Q| = 2\ln|t - 5000| + S_0$$

Exponentiate both sides

$$Q = S(t - 5000)^2$$

Using initial data solve for S

$$50 = S(0 - 5000)^2$$

$$S = \frac{50}{25000000}$$

$$S = \frac{1}{500000}$$

Substitute back into equation

$$Q = \frac{(t - 5000)^2}{500000}$$

This gives you the amount of flavor at any time however we want the concentration at any time. We get this by dividing by the volume

$$C = \frac{(t - 5000)^2}{500000(-.2\,t + 1000)}$$

We can simplify this equation if we do some rearranging

$$C = \frac{(t - 5000)^2}{-100000(t - 5000)}$$

$$C = \frac{t - 5000}{-100000}$$

$$\boxed{C = \frac{5000 - t}{100000}}$$

We now note that this equation will become invalid if either the ice runs out or the cup runs empty.

Because the ice is melting at a rate of .2ml/s we can easily solve for the time (t) when the ice is completely melted.

$$t = \frac{500}{.2}$$

$$t = 2500\ \text{s}$$

If we plug this time into our volume equation we get

$$v = -.2(2500) + 1000$$

$$v = 500\ \text{ml}$$

Because at this time there is still 500ml of drink the ice has obviously run out first and is therefore the limiting factor for the validity of this equation.

The equation is valid from $0 \le t \le 2500$s

We can now solve for t when the concentration hits 30g/L

$$\frac{30}{1000} = \frac{5000 - t}{100000}$$

$$3000 = 5000 - t$$

$$\boxed{t = 2000\ s}$$

whew!

3. (45 points) Solve the following differential equation:

$$\frac{t^2\,dy}{dt} = -\frac{1}{3}y + 5\,t^3 \qquad y(-3) = 1$$

Solution :

$$\frac{dy}{dt} + \frac{1}{3\,t^2}y = 5\,t$$

Find an integrating factor

$\int t^{-2}dt = -\frac{1}{t} + C \neq -t + C$

$$\mu = e^{\int \frac{1}{3\,t^2}dt} \cancel{=} e^{-\frac{1}{3}t}$$

Multiply through by the integrating factor

$$\int \frac{d}{dt}\left(e^{-\frac{1}{3}t}y\right) = \int 5\,te^{-\frac{1}{3}t}dt$$

$$e^{-\frac{1}{3}t}y = -15\,te^{-\frac{1}{3}t} - 45e^{-\frac{1}{3}t} + C$$

Solve for C

$$y = -15\,t - 45 + Ce^{\frac{1}{3}t}$$

plug in initial condition to solve for C

$$C = e^1$$

$$y = -15\,t - 45 + e^{\left(-\frac{1}{3}t + 1\right)}$$

4. For the following problem:

   a) Determine the value of b for which the equation is exact. **(10 Points)**

   b) Then solve the equation using that value of b. **(20 Points)**

$$(2\,xye^{2\,x} + ye^{2\,x} + 2\,x\sin(y))\,dx + \;(x\,e^{2\,x} + x^{b}\cos(y) + 3)\,dy = 0$$

Good - defined M & N

Solution:

First, check for exactness by letting

$M = 2\,xye^{2\,x} + ye^{2\,x} + 2\,x\sin(y)$ $\qquad$ $N = x\,e^{2\,x} + x^{b}\cos(y) + 3$

Then take the following partial derivatives

$M_y = 2\,xe^{2\,x} + e^{2\,x} + 2\,x\cos(y)$ $\qquad$ $N_x = 2\,xe^{2\,x} + e^{2\,x} + bx^{b-1}\cos(y)$

In order to be exact,

$$M_y = N_x$$

$$2\,xe^{2\,x} + e^{2\,x} + 2\,x\cos(y) = 2\,xe^{2\,x} + e^{2\,x} + bx^{b-1}\cos(y)$$

a) $\boxed{b = 2}$

Good explanation

Now we substitute 2 for b in the original equation:

$$(2\,xye^{2\,x} + ye^{2\,x} + 2\,x\sin(y))\,dx + (x\,e^{2\,x} + x^{2}\cos(y) + 3)\,dy = 0$$

And then solve by first integrating M with respect to x:

$$\psi_x = 2\,xye^{2\,x} + ye^{2\,x} + 2\,x\sin(y)$$

$$\psi = \int 2\,xye^{2\,x}\,dx + \int ye^{2\,x}\,dx + \int 2\,x\sin(y)\,dx$$

In order to integrate, either recognize that the first two terms are the derivative of $xye^{2\,x}$ or integrate the first term by parts, shown below:

$u = xy \qquad dv = 2e^{2\,x}dx$

$du = ydx \qquad v = e^{2\,x}$

$$\int 2\,xye^{2\,x}\,dx = xye^{2\,x} - \int ye^{2\,x}\,dx$$

$$\psi = xye^{2\,x} - \cancel{\int ye^{2\,x}\,dx} + \cancel{\int ye^{2\,x}\,dx} + \int 2\,x\sin(y)\,dx$$

$$\psi = xye^{2\,x} + x^{2}\sin(y) + C(y)$$

Then integrate N in terms of y:

$$\psi_y = x\,e^{2x} + x^2\cos(y) + 3$$

$$\psi = \int x\,e^{2x}\,dx + \int x^2\cos(y)\,dx + \int 3\,dx$$

$$\psi = x\,y e^{2x} + x^2\sin(y) + 3\,y + \mathcal{C}(x)$$

When we compare our two functions, we find that:

$$C(y) = 3\,y + K$$
$$\mathcal{C}(x) = K$$

So our solution to the differential equation is:

b) $\boxed{\psi = x\,y e^{2x} + x^2\sin(y) + 3\,y + K = 0}$

Correct, but non standard writing - dangerous! Normally write ψ = constant rather than $\psi(k) = 0$.

3

(40 pts)
Problem: Solve the initial value problem.

$$y' = \frac{6x^6 - 3x^3 + x}{10xy^4} \qquad y(0) = 2$$

Solution

First we can simplify the expression by dividing out an x and we get

$$y' = \frac{6x^5 - 3x^2 + 1}{10y^4}$$

(and differentials)

Then we set y'= dy/dx and separate the variables and we get

$$(10y^4)dy = (6x^5 - 3x^2 + 1)dx$$

Good explanation

Then we integrate both sides and obtain

$$2y^5 = x^6 - x^3 + x + c$$

To find the general solution of the problem we solve for y and obtain

$$y = \left(\frac{x^6 - x^3 + x + c}{2}\right)^{\frac{1}{5}}$$

We then apply the initial conditions and solve for the integration constant (c)

$$c = 64$$

Thus the solution to the initial value problem is

$$y = \frac{x^6 - x^3 + x + 64}{2}$$

Math 334-4

(50 pts)
Fui Vakapuna is handed the ball on the 40 yard line and runs at a speed of 5 yards/sec. As he is handed the ball, a defender grabs hold of him and is dragged for the extent of Vakapuna's run. As he is being dragged, he slows Fui by 1/9 of a yard per second for every yard that they travel. a.) Find an equation to describe Fui's travel. b.) Does Fui make it at least 40 yards for the touch down? c.) If so, how long does it take him to reach the endzone? If not, how fast does he need to run in order to score?

<u>Solution:</u>

Since Fui has not traveled any yards when he is handed the ball, $Y_0 = 0$.

$dy/dt = 5 - (1/9)y$ ~~This is not the relevant equation~~
$dy/dt + (1/9)y = 5$

O.K.! Good!

$\mu = e^{\int(1/9)dt} = e^{(1/9)t}$

$d/dt(\mu y) = 5\, e^{(1/9)t}$ * ⇒ ~~velocity v = v(y), y the distance traveled. So~~

$\int d/dt(\mu y) = \int 5\, e^{(1/9)t}$

$ye^{(1/9)t} = 45\, e^{(1/9)t} + c$ ~~v(y)~~

$y = 45 + ce^{(-1/9)t}$

$y(0) = 45 + c\,(1)$ explanation – full sentences.

$c = -45$

a) $y(t) = 45(1 - e^{(-1/9)t})$

$\lim_{t\to\infty}$ of $y(t) = 45\,(1 - 0) = 45$ yards.

b) Yes, Fui scores a Touchdown

$40 = 45(1 - e^{(-1/9)t})$
$40/45 = 1 - e^{(-1/9)t}$
$40/45 - 1 = -e^{(-1/9)t}$
$1 - 40/45 = e^{(-1/9)t}$
$\ln(1 - 40/45) = -t/9$
$t = -9\ln(1 - 40/45) = 19.78$ sec

(30pts)

Solve the initial value problem: $ty' - y/2 = t^2$ $y(0) = 0$

Solution:

Solve by integration factor.

Integration factor: $e^{\int -1/2t\,dt} = e^{-(\ln t)/2} = t^{-1/2}$

Multiply:

$$t^{-1/2}y' - t^{-1/2}/2y = t^{5/2}$$

Integrate:

$$\int t^{-1/2}y' - (t^{-1/2}/2)y\,dy = \int t^{5/2}dt$$

$$\longrightarrow (t^{-1/2}y) = 2/7(t^{7/2}) + C$$

Solve for y:

$$y = (2/7)(t^4) + Ct^{1/2}$$

Apply initial data to solve for C:

y(0) = 0

$$0 = (2/7)(0^4) + C(0^{1/2})$$

C = 0

Substitute into problem:

$$y = 2/7(t^4)$$

use complete sentences

(30pts)

Consider the initial value problem (IVP)

$x(x+2)e^x\sin(y) + 1/x + x^2e^x\cos(y)dy/dx = 0, \quad x > 0$ (1)

$y(1) = \pi/2$ (2)

a) Explain why the equation (1) is neither linear nor separable.
   a. sin(y), cos(y), and cos(y) dy/dx are all non-linear terms. Separation is also impossible.

b) Show that equation (1) is exact
   a. $M(x, y) = x(x+2)e^x\sin(y) + 1/x \quad N(x,y) = x^2e^x\cos(y)$
      $M_y = x(x+2)\, e^x\cos(y) \quad N_y = x(x+2)\, e^x\cos(y)$
      - Therefore, the condition is satisfied.

c) Describe the steps (at least three) to obtain the solution for the IVP (1)-(2). **Don't perform the steps described**.

1) Define $\Psi(x, y)$ such that $\Psi_y(x,y) = N(x,y)$ or $\Psi_x(x,y) = M(x,y)$ and integrate one of them.
2) Take the derivative of the result of step (1) with respect to the other variable.
   $\Psi(x, y) = \int N(x,y)dy + C(x)$
   $\Psi_x(x,y) = [\int N(x,y)dy]_x + C'(x)$ and set the result equal to M(x,y) or N(x,y).
   In this case, $= [\int N(x,y)dy]_x + C'(x) = M(x,y)$
3) Integrate the "derivative of the constant"
   $C(x) = \int C'(x)dx = \int\{M(x,y) = [\int N(x,y)dy]_x\}dx$
4) Plug this result into the expression obtained in step (1)
   $\Psi(x, y) = \int N(x,y)dy + C(x)$

   Then the solution is given by: $\Psi(x, y) = d$ where d is a constant.

(4)

1) Given the differential equation:

$(x+2)\sin y\, dx + x\cos y\, dy = 0$

a) Prove the differential equation is not exact.

Solution:

~~M_x~~ $\psi_x = (x+2)\sin y$ ~~N_y~~ $\psi_y = x\cos y$

~~M_{xy}~~ $\psi_{xy} = (x+2)\cos y$ ~~N_{yx}~~ $\psi_{yx} = \cos y$

$M_{xy} \neq N_{yx}$ - thus the equation is not exact

- definition of M, N & ψ confused

(5 points)

b) Find the integration factor, μ, such that μ is a function of x that makes the differential equation exact.

again, confusion on ψ, M, & N

Solution:

$\mu(M_x) + \mu(N_y) = 0$ <- this will be exact if $(\mu M)_y = (\mu N)_x$

taking the partial derivatives we get:

inconsistent with running (nonstandard) notation/ assumption.

$(\mu M)_y = \mu M_y + M\mu_y$ and $(\mu N)_x = \mu N_x + \mu_x N$

by moving equations and combining like terms we come to:

$M\mu_y - \mu_x N + \mu(M_y - N_x) = 0$

if μ is a function of only x then

$(\mu M)_y = \mu M_y$ so the above equation becomes:

$(\mu N)_x = \mu N_x + N\, d\mu/dx$

?

if $(\mu M)_y = (\mu N)_x$ as is necessary to be exact and solving for $d\mu/dx$ and we get:

$d\mu/dx = (\mu(M_y - N_x)) / N$

plugging in values we know from the differential equation above we can solve for μ:

$(x+2)\cos y - \cos y$ $x\cos y + \cos y$ $x+1$?

$$d\mu/dx = \frac{?}{x\cos y}\,\mu = \frac{?}{x\cos y}\,\mu = \frac{?}{x}\,\mu$$

now we solve the differential equation:

$$\frac{d\mu}{\mu} = \frac{x+1}{x}\,dx = \left(1 + \frac{1}{x}\right) dx$$

integrating both sides:

$$\int d\mu/\mu = \int dx + \int dx/x$$

and solving for μ:

$$\mathrm{Ln}|\mu| = x + \mathrm{Ln}|x|$$

$\mu = xe^x$ makes the function exact and can check by:

$\mu\, M_x = (x^2e^x + 2xe^x)\sin y$ $\qquad$ $\mu\, N_y = x^2e^x\cos y$

$\mu\, M_{xy} = (x^2e^x + 2xe^x)\cos y$ $\qquad$ $\mu N_{yx} = 2xe^x\cos y + x^2e^x\cos y$

$\mu\, M_{xy} = \mu N_{yx}$ is now exact – not this eq. ~~whi~~ whose exactness is in question.

(20 points)

c) Solve the differential equation using μ to make it exact.

awkward

Solution:

$\mu\, M_x = (x^2e^x + 2xe^x)\sin y$ $\qquad$ $\mu\, N_y = x^2e^x\cos y$

$M = \int (x^2e^x + 2xe^x)\sin y\, dx$ $\qquad$ $N = \int x^2e^x\cos y\, dy$

$= \sin y\left(\int x^2e^x\, dx + \int 2xe^x\, dx\right)$ $\qquad$ $= x^2e^x \int \cos y\, dy$

$= \sin y\left(x^2e^x - 2\int xe^x\, dx\right)$ $\qquad$ $= x^2e^x\sin y + g(x)$

$= \sin y(x^2e^x - 2(xe^x - \int e^x\, dx))$

$= \sin y(x^2e^x - 2xe^x - 2e^x) + g(y)$

$M = N$ $= \psi$?

$\sin y(x^2e^x - 2xe^x - 2e^x) + g(y) = x^2e^x\sin y + g(x)$

in an attempt to solve for g(y) and g(x) get rid of like terms

$$\sin y(-2xe^x - 2e^x) + g(y) = g(x)$$

plug g(x) back into the equation

$$\sin y(x^2e^x - 2xe^x - 2e^x) + g(y) = x^2e^x\sin y + (\sin y(-2xe^x - 2e^x) + g(y))$$

getting rid of like terms again leaves us with

$x^2e^x\sin y = C$ <- this is the solution of the differential equation
(15 points)

2) Solve the initial value problems.

a)

$$y' = 6(t^2)\cos(5t) - 5\sin(5t) - 6ty$$
$$y(0) = 2$$

<u>**Solution:**</u> $y(t) = \cos(5t) + 1.$?

6ty can be added to both sides to yield

$$y' + 6ty = 6(t^2)\cos(5t) - 5\sin(5t) \qquad \text{(I)}$$

Eq. (I) is now of the form

$$y' - p(t)y = g(t)$$

Thus, p(t)=6ty and the integrating factor, $\mu(t) = e^{\int 6t\,dt} = e^{3t^2}$. Multiplying all terms by $\mu(t)$, we obtain

$$e^{3t^2}y' + 6te^{3t^2}y = 6te^{3t^2}\cos(5t) - 5e^{3t^2}\sin(5t). \qquad \text{(II)}$$

The left side of Eq. (II) can be seen as the derivative of a product. The right side can also been seen as the derivative of another product. Doing the integrations we obtain

$$\int[e^{3t^2}y' + 6te^{3t^2}y]dt = \int[6te^{3t^2}\cos(5t) - 5e^{3t^2}\sin(5t)]dt$$

$$e^{3t^2}y = e^{3t^2}\cos(5t) + c.$$

Dividing by e^(3t^2), gives us y(t) by itself.

$$y(t) = \cos(5t) + c$$

Using the initial condition, y(0)=2, we obtain

$$y(0) = \cos(5*0) + c$$

$$c = 1$$

Thus, the solution to the initial value problem is

$$y(t) = \cos(5t) + 1.$$

(15 points)

b)

$$y' = (10 - 2x^3)y \qquad \text{(III)}$$
$$y(0) = 3$$

<u>Solution:</u> $y(x) = 3e^{10x - \frac{x^4}{2}}$?

Because $y' = \frac{dy}{dx}$, Eq. (III) becomes a separable function.

$$\frac{dy}{dx} = (10 - 2x^3)y$$

Now the equation can be rewritten as

$$\frac{dy}{y} = (10 - 2x^3)dx$$

Integrating both sides with respect to y and x, respectively, yields

$$\ln y = 10x - \frac{2}{4}x^4 + C$$

Thus,

$$y = Ae^{10x - \frac{x^4}{2}}$$

Using the initial condition y(0)=3, we can solve for A.

$$y(0) = Ae^{10\cdot 0 - \frac{0^4}{2}} = 3$$
$$3 = A$$

Thus, the final solution for Eq. (III) is

$$y(x) = 3e^{10x - \frac{x^4}{2}}$$

(15 points)

c. What is the behavior of the solution to Eq. (III) as $x \to \infty$?

<u>Solution:</u>
The end behavior can be studied by looking at

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} 3e^{10x - \frac{x^4}{2}} = 0$$

Thus, the solution approaches zero as $x \to \infty$.
(5 points)

3). Consider a lake with a constant volume V that contains, at time t, and amount of pollution called $Q(t)$ that is evenly distributed throughout the lake with a concentration $c(t)$, where

$$c(t) = \frac{Q(t)}{V}.$$

$r \geq ?$

Assume that the water enters with a concentration of the pollutant κ at rate r, and the water leaves the pond at the same rate. Pollutants are also added at a constant rate ρ. Answer the following:

a) If at time $t = 0$ the concentration of pollutant is c_0, find an expression for the concentration $c(t)$ at any time.

Solution:

a) $C_{in} = \kappa r + \rho$ $\qquad$ $C_{out} = -r\frac{Q(t)}{V}$ $\qquad$ $\frac{dQ}{dt} = C_{in} + C_{out}$

$$\frac{dQ}{dt} = \kappa r + \rho - r\frac{Q(t)}{V}$$

$$\frac{dQ}{dt} + \frac{r}{V}Q(t) = \kappa r + \rho$$

The integrating factor is $e^{\frac{r}{V}}$.

Multiplying both sides by the integrating factor yields:

$$e^{\frac{rt}{V}}\frac{dQ}{dt} + \frac{r}{V}e^{\frac{rt}{V}}Q(t) = (\kappa r + \rho)e^{\frac{r}{V}t}$$

Integrating both sides yields: $Q(t)e^{\frac{rt}{V}} = \frac{V}{r}(\kappa r + \rho)e^{\frac{rt}{V}} + C$

Divide and simplify to get $Q(t) = V\kappa + \frac{V\rho}{r} + Ce^{\frac{-rt}{V}}$

Substitute into equation to get:

$$c(t) = \frac{Q(t)}{V} = \frac{V\kappa + \frac{V\rho}{r} + Ce^{\frac{-rt}{V}}}{V} = \kappa + \frac{\rho}{r} + \frac{1}{v}Ce^{\frac{-rt}{V}}$$

Solve for C at initial time gives: $c(0) = \kappa + \frac{\rho}{r} + \frac{C}{V}$

$$c_0 = \kappa + \frac{\rho}{r} + \frac{C}{V}$$

$$C = c_0 - \kappa - \frac{\rho}{r}V$$

Then substituting that back into the equation for $c(t)$ gives the final answer of:

$$c(t) = \kappa + \frac{\rho}{r} + (c_0 - \kappa - \frac{\rho}{r})e^{\frac{-rt}{V}}$$

(20 points)

b) What is the limiting concentration as $t \to \infty$?

Solution:

As $t \to \infty$ $c(t) \to \kappa + \frac{\rho}{r}$

(5 points)

c) If pollutants suddenly stop entering the lake $(\kappa = 0, \rho = 0)$ then:

i. Determine how long it would take for the concentration of the pollutants to be reduced to half its original amount.
ii. Determine how long it would take for the concentration of the pollutants to be reduced to one quarter of its original amount.

Solution:

At the point when $(\kappa = 0, \rho = 0)$ the equation for the concentration at any time becomes $c(t) = c_0 e^{\frac{-rt}{V}}$

i) $\frac{1}{2} = e^{\frac{-rt}{V}}$ taking the natural log of both sides yields $\ln\frac{1}{2} = -\frac{rt}{V}$

multiplying both sides by -1 gives us $\ln 2 = \frac{rt}{V}$ then by solving for the time we get $t = \frac{V \ln 2}{r}$

(5 points)

ii) The same as part i) only this time instead of ½ we want ¼ of the pollutant. The only part that changes is what you take the natural log of, so by quick substitution we see that the answer is $t = \frac{V \ln 4}{r}$

(5 points)

d) If the lake's volume is 6 million gallons and the flow rate is 2176 gallons/year, how long, in years, would it take for the pollutants to drop to 15% of their original value (assuming that there are no pollutants entering the lake).

Solution:

d)Substituting the values directly into the equation to find the amount of pollutants at any time gives us the equation $.15 = e^{\frac{-(2176\, gallons/year)t}{6\times10^{6}\, gallons}}$

Solving for the time gives us that $t = 5231\, years$ wow!

(10 points)

4) The given equation can be used to measure population growth over time. Both r and P are positive constants.

$$\frac{dy}{dt} = ryLn\left(\frac{3P}{y}\right)$$

a). Using the given equation determined where the critical points will exist and whether each is asymmetrically stable or unstable.

Solution:

$$\frac{dy}{dt} = ryLn\left(\frac{3P}{y}\right)$$

Set the rate equal to zero and solve for each y individually.

$$0 = ryLn\left(\frac{3P}{y}\right)$$

$$0 = Ln\left(\frac{3P}{y_2}\right)$$

$$e^0 = \frac{3P}{y_2} \qquad\qquad y_1 = 0$$

$$y_2 = 3P$$

Critical Points will exist at y=0 and y=3P

stability of $y=0$ determined "nonstandardly" – $y<0$ not allowed, only $y>0$.

Stability can be determined by calculating the rate around the given critical points.

$y = 4P$

$$\frac{dy}{dt} = r(4P)Ln\left(\frac{3P}{(4P)}\right)$$

$$= r(4P)Ln\left(\frac{3}{4}\right)$$

$$= 4Ln\left(\frac{3}{4}\right)\Pr$$

$$= -1.15073\Pr$$

$y = 2P$

$$\frac{dy}{dt} = r(2P)Ln\left(\frac{3P}{(2P)}\right)$$

$$= r(2P)Ln\left(\frac{3}{2}\right)$$

$$= 2Ln\left(\frac{3}{2}\right)\Pr$$

$$= 0.81093\Pr$$

Since both r and P are known positive constants, they will not change the signs of the determined slopes. From the information found y=0 will be unstable and y=3P will be asymptotically stable.
(15 points)

? referenced this formula?

b) Using the derivation of "Gompertz initial equation," $y = Ke^{Ln\left(\frac{Y_0}{K}\right)e^{-rt}}$, find the time necessary for y(t)= .45K when K=55,300,000kg, r =76% and $\frac{y_0}{K} = 0.24$.

Solution:

$$y = Ke^{\left[Ln\left(\frac{y_0}{K}\right)e^{-rt}\right]}$$

$$Ln\left(\frac{y}{K}\right) = Ln\left(\frac{y_0}{K}\right)e^{-rt}$$

$$e^{rt} = Ln\left(\frac{y_0}{K}\right) \Big/ Ln\left(\frac{y}{K}\right)$$

$$t = \frac{Ln\left[\ln\left(\frac{y_0}{K}\right) \Big/ Ln\left(\frac{y}{K}\right)\right]}{r}$$

$$= \frac{Ln\left[Ln(.24) \Big/ Ln\left(\frac{.45K}{K}\right)\right]}{.76}$$

$$t = .76\frac{1}{year}$$

It will take $t = .76\frac{1}{year}$ to reach the given value for $y(t) = .45K$

(15 points)

5

1. Newton's law of cooling states that the temperature of an object v(t) satisfies

? missing something?

the differential equation where T is the constant surrounding temperature and k is a positive constant. (30pts)

$$dv/dt = -k(v - T),$$

?

a) Suppose v(0) = \`v Find the temperature of the object at anytime.(20pts)

b) If T = 70 degrees and k = .05/min. Find the time when v(t) is one half the original temperature.(10pts)

Answer)

a) Solve the equation using separation.

dv/(v – T) = -kdt, to, ln(v – T) = -kt + c, to, v – T = e^(-kt + c), to,

v = Ce^(-kt) + T

Use v(0) = \`v to find C

\`v = Ce^(-k*0) + T, to, \`v = C + T, to, \`v – T = C

Plug in C

v(t) = (\`v – T)e^(-kt) + T.

b) make \`v = 1, v(t) = ½, T = 70, and k = .05

½ = (1 – 70)e^(-.05t) + 70

Solve for t

(1/2) – 70 = 139/2 = (69)e^(-.05t), to, 139/138 = e^(-.05t), to, ln(139/138) = -.05t

t = (ln(139/138))/-.05 minutes.

better flow/ sentences

Population = y? $y \geq 0$?

2. (40pts)

The growth rate of the European swallow population in the village of Heathfield, England, is given by the equation $\frac{dy}{dt} = y^3 - 6y^2 - 72y$.

a) If $\frac{dy}{dt} = f(y)$, sketch the graph of f(y) versus y.(10pts)

b) Draw the phase line(10pts)

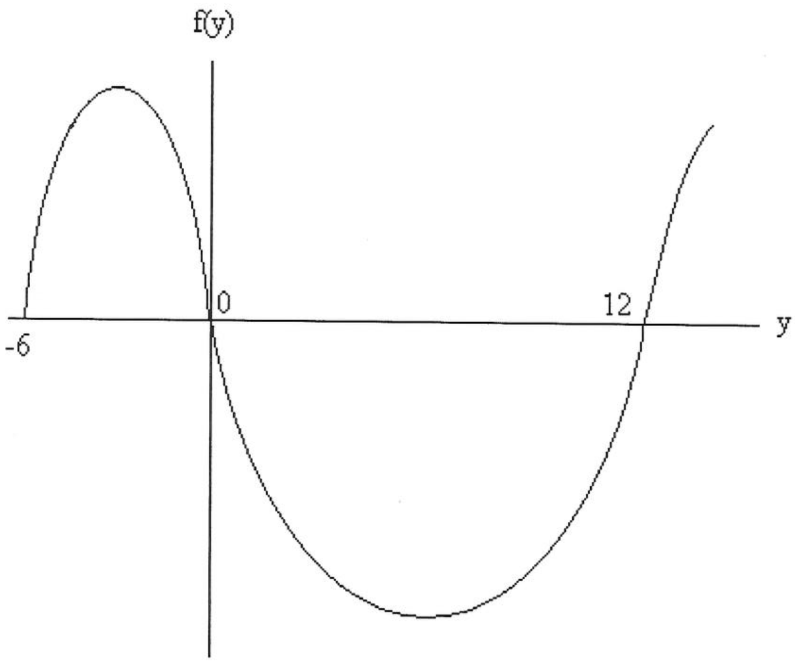
c) Determine the critical points and classify them as stable or unstable(10pts)

d) Sketch several graphs of solutions in the ty-plane on the same set of axis.(10pts)

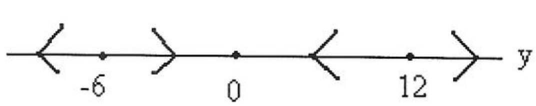
Solution:

$$\frac{dy}{dt} = y^3 - 6y^2 - 72y = y(y^2 - 6y - 72) = y(y+6)(y-12)$$

a)

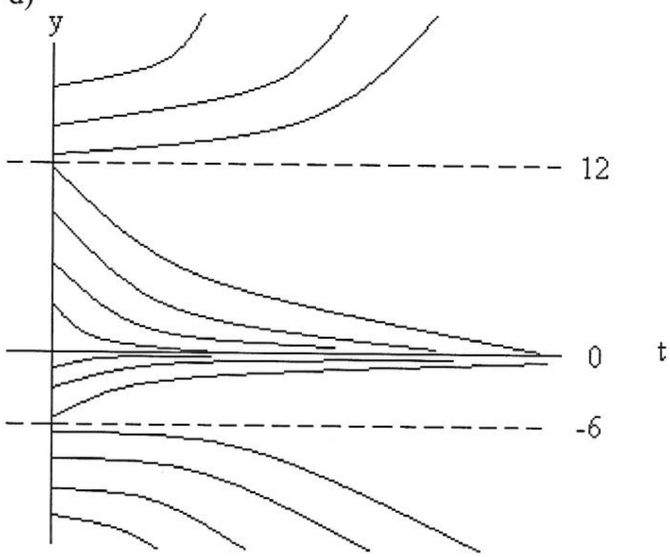


b)



c) y = -6 - unstable – nonphysical/irrelevant?
y = 0 - stable
y = 12 - unstable

d)



3. Determine whether the following differential equations are exact. If so, solve the differential equation. (40pts)

a) lnydx + (x+2y²)/y dy = 0 (15pts)

$M_y = 1/y$ $N_x = 1/y$ exact

Complete sentences

$\int(\ln y)dx = x\ln y + C$

$d(x\ln y + C)/dy = x/y + dC/dy$

$x/y + dC/dy = (x + 2y^2)/y$
$x/y + dC/dy = x/y + 2y$
$dC/dy = 2y$
$\int dC = \int 2y dy$
$C = y^2 + D$

$x\ln y + y^2 = c$

b) $(4x^2 - 3xy + 7)dx + [(-3/2)x^2 + 3y^2 + 2]dy = 0$(15pts)

M=?, N=?

$M_y = -3x$ $N_x = -3x$ exact

$\int(4x^2 - 3xy + 7)dx = (4/3)x^3 - (3/2)x^2y + 7x + C$

$d((4/3)x^3 - (3/2)x^2y + 7x + C)/dy = (-3/2)x^2 + dC/dy$

$(-3/2)x^2 + dC/dy = (-3/2)x^2 + 3y^2 + 2$
$dC/dy = 3y^2 + 2$
$\int dC = \int (3y^2 + 2)dy$
$C = y^3 + 2y + D$

!!

$(4/3)x^3 - (3/2)x^2y + 7x + y^3 + 2y = c$

c) $e^{x+2y}dx + e^{x+2y}dy = 0$ (10pts)

$M_y = 2e^{x+2y}$ $N_x = e^{x+2y}$ not exact

4. Section 2.2 and 2.3 (40pts)

Let P(t) be a model of the population growth of lemmings in South Africa. Assume that P(t) satisfies the logistic growth equation

$$\frac{dP}{dt} = .5\,P\,(1 - \frac{P}{1500}),\ y(0) = 35.$$

1. Is the differential equation separable? (5pts)
2. Is the equation linear?(5pts)
3. Solve the differential equation(25pts)
4. What is the long term behavior of the graph of P(t)?(5pts)

<u>Solution</u>

?

1. This equation is autonomous, that is to say that there is not variable t present and, therefore this equation is separable because all autonomous equations are separable.
2. No, the equation has P^2 which is not linear.

3. For the equation

$$\frac{dP}{dt} = .5\,P\,(1 - \frac{P}{1500}),\ y(0) = 35$$

$$= \frac{dP}{P(1-\frac{P}{1500})} = .5dt$$

$$= \int \frac{dP}{P(1-\frac{P}{1500})} = \int .5dt$$

Using partial fractions on the left hand side of the equations yields

$$\int \frac{dP}{P(1-\frac{P}{1500})} = \int (\frac{1}{P} + \frac{(1/1500)}{(1-(P/1500))})\,dP$$

Integrating with respect to P on the left and t on the right yields

$$\ln|P| - \ln|1-(P/1500)| = .5t + c$$

$$= \frac{P}{1-(P/1500)} = Ce^{.5t}$$, c being a constant,

Then solving for P,

$$P = \frac{1500Ce^{.5t}}{1500 + Ce^{.5t}}.$$

Now we apply the initial conditions y(0) = 35 and solve for C

$$C = \frac{(35)(1500)}{(1500-35)} = \frac{52500}{1465} \approx 35.836177.$$

Now substituting the derived value of C in to the equations found for P(t) yields

$$P(t) = \frac{52500}{35 + (1465)e^{-.5t}}.$$

4. Looking at the equation found in part 3 we see that

$$\lim_{t \to +\infty} P(t) = 1500.$$

6

1) A 1000 gallon holding tank that catches runoff from some chemical process initially has 800 gallons of water with 2 ounces of pollution dissolved in it. Polluted water flows into the tank at a rate of 3 gal/hr and contains 5 ounces/gal of pollution in it. A well mixed solution leaves the tank at 3 gal/hr as well. When the amount of pollution in the holding tank reaches 500 ounces the inflow of polluted water is cut off and fresh water will enter the tank at a decreased rate of 2 gallons while the outflow is increased to 4 gal/hr. Determine the amount of pollution in the tank at any time *t*.

Solution

Because this problem has two situations it will require two Initial Value Problems. The rate of change of chemical in the tank, *dQ/dt,* is equal to the rate at which chemical is flowing in minus the rate at which it is flowing out.

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

First Situation

The rate at which the chemical enters the tank is $\frac{3\,\cancel{gal}}{hour}\cdot\frac{5\,ounces}{\cancel{gal}} = \frac{15\,ounces}{hour}$

10

The rate at which the chemical leaves the tank is $\frac{3\cancel{gal}}{hour}\cdot\frac{Q(t)\,ounces}{800\,\cancel{gal}} = \frac{3\,Q(t)\,ounces}{800\,hour}$

The initial value is $Q(0)=2$ ounces

Therefore the first IVP for this problem is $\frac{dQ}{dt} = 15 - \frac{3}{800}Q(t) \quad Q(0) = 2 \quad 0 \le t \le t_{max}$

Where t_{max} is the time where the holding tank reaches 500 ounces of pollutant

This is a fairly simple linear differential equation.

$$\frac{dQ}{dt} + \frac{3}{800}Q(t) = 15 \text{ the integrating factor is therefore } \mu(t) = e^{\int \frac{3}{800}dt} = e^{\frac{3t}{800}}$$

Using $\mu(t)$ we obtain

$$e^{\frac{3}{800}}\cdot\frac{dQ}{dt} + e^{\frac{3t}{800}}\cdot\frac{3}{800}\cdot Q = e^{\frac{3t}{800}}\cdot 15 \quad or \quad e^{\frac{3t}{800}}\cdot Q = \int (e^{\frac{3t}{800}}\cdot 15)dt \quad = \quad \frac{15(800)}{3}\cdot e^{\frac{3t}{800}} + C$$

Using the initial value find the value for C and then solve for Q.

$$Q(t) = 4000 - 3998e^{-\frac{3t}{800}} \quad Q(0) = 2 \quad 0 \le t \le t_{max}$$

worth 45 points

Using this equation find the time (t) where Q=500

$$500 = 4000 - 3998e^{-\frac{3t}{800}} \Rightarrow \quad 3500 = 3998e^{-\frac{3t}{800}} \Rightarrow \quad \frac{3500}{3998} = e^{-\frac{3t}{800}} \Rightarrow \ln\left(\frac{3500}{3998}\right) = -\frac{3t}{800}$$

$$\Rightarrow \quad -\frac{800}{3}\ln\left(\frac{3500}{3998}\right) = t$$

Therefore $t_{max} = 35.475$

$$Q(t) = 4000 - 3998e^{-\frac{3t}{800}} \quad Q(0) = 2 \quad 0 \le t \le 35.475$$

Second Situation

After time has reached t_{max} there is a change in the system and therefore a new differential equation is needed to express the amount of chemical with respect to time. To solve the second situation we use the same procedure in the first situation to start out. The rate of change of chemical in the tank, *dQ/dt,* is equal to the rate at which chemical is flowing in minus the rate at which it is flowing out.

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

Fresh water is flowing into the tank and so the concentration of pollution in the incoming water is zero. However, don't "start over" at $t = 0$. Start this equation at t_{max}, the time at which the new process starts.

$$\frac{dQ}{dt} = -4\left(\frac{Q_2(t)}{800 - 2(t - t_{\max})}\right) \qquad Q_2(t_{\max}) = 500 \qquad t_{\max} \le t \le t_{empty}$$

$$800 - 2(t_{empty}) + 35.475 = 0 \quad \Rightarrow \quad t_{empty} = 435.475$$

During this time frame we are losing two gallons of water every hour of the process, the "-2" is in there to account for that. However, we can't just use t as we did in the first situation. When this new process starts up there needs to be 800 gallons of water in the tank and if we just use t there we won't have the required 800 gallons that we need in the equation. To make sure that the proper volume exist the difference in times is needed. In this way once we are one hour into the new process (*i.e* $t - t_{max} = 1$) we will have 798 gallons in the tank as required.

After inserting t_{max} and t_{empty} and simplifying the equation the IVP is

$$\frac{dQ}{dt} = -\frac{2Q_2(t)}{435.475 - t} \qquad Q_2(35.475) = 500 \qquad 35.475 \le t \le 435.475$$

This is a linear and separable equation

$$\frac{dQ}{Q} = -\frac{2}{435.475 - t}dt \quad \Rightarrow \quad \ln(Q) = 2\ln(435.475 - t) + c \quad \Rightarrow \quad Q = k(435.475 - t)^2$$

Using the initial value that $Q_2(35.475)=500$

$$500 = k(435.475 - 35.475)^2 \quad \Rightarrow \quad \frac{500}{(400)^2} = k = \frac{1}{320}$$

Therefore the equation for the amount of chemical in the tank at any time for the second situation is

$$Q_2(t) = \frac{(435.475 - t)^2}{320} \qquad Q_2(35.475) = 500 \qquad 35.475 \le t \le 435.475$$

The final answer is then,

$$\begin{cases} Q_1(t) = 4000 - 3998e^{-\frac{3t}{800}} & Q(0) = 2 & 0 \le t \le 35.475 \\ Q_2(t) = \dfrac{(435.475 - t)^2}{320} & Q_2(35.475) = 500 & 35.475 \le t \le 435.475 \end{cases}$$

write as

$$Q(t) = \begin{cases} 4000 \ldots, & 0 \le t \le 35.\ldots \\ \dfrac{(\quad)^2}{320}, & 35.\ldots \le t \le 435.\ldots \end{cases}$$

2) Question:

Use the existence and uniqueness theorem to find an interval in which the initial value problem

$$y' = \frac{3y}{x}, \quad \text{and} \quad y(x_o) = y_o$$

has a unique solution.

Solution:

In this example, *F(x,y) = 3y/x, and* $\frac{\partial F}{\partial y}(x,y) = \frac{3}{x}$. Both of these functions are defined for all $x_o \neq 0$, so the theorem tells us that for each $x_o \neq 0$ there must exist a unique solution defined in an open interval containing x_o.

— not motivated by question — need longer question.

By separating variables and integrating, we derive solutions to the equation

$$\frac{dy}{dx} = \frac{3y}{x}$$

$$\frac{dy}{3y} = \frac{dx}{x}$$

$$\frac{\ln|y|}{3} = \ln x + C$$

$$y = e^{3\ln(x)+C}$$

$$y = Cx^3$$

Notice that all of these solutions pass through the point (0,0), and that none of them pass through any point $(0,y_0)$ with $y_0 \neq 0$. So the initial value problem $y' = {}^{3y}/_{x}, y(0) = 0$, has infinitely many solutions, but the initial value problem

$y' = {}^{3y}/_{x},\ \ y(0) = y_o, y_o \neq 0$, has no solutions. Cool.

But for each (x_o,y_o) with $x_o \neq 0$, there is a unique ~~quadratic~~ cubic $y = Cx^3$ whose curve passes through the point (x_o,y_o). So the initial value problem $y' = \frac{3y}{x}$, and $y(x_o) = y_o$, $x_o \neq 0$, has a unique solution defined on some interval centered at the point x_o.

There is a unique solution only on the interval $x_0 - \lambda < x < x_0 + \lambda$, where $\lambda = |x_0|$.

In conclusion, the initial value problem has

In some meaningful sense the solution persists longer — solution is a cubic! 35 points

- A unique solution in an open interval containing x_0 if $x_0 \neq 0$;
- No solution if $x_0 = 0$ and $y_o \neq 0$.
- Infinitely many solutions if $(x_0, y_o) = (0,0)$.

Great!

3) Solve the initial value problem

$t\frac{dy}{dx} + 3y = 3$

$t\,dx + 3y\,dy = 3\,dx$

$y(2) = 11$

Write equation in the standard form

$$y' + \left(\frac{3}{t}\right)y = \frac{3}{t}$$

To solve the equation we need to find the integrating factor $\mu(t)$.

$$\mu = e^{\int \frac{3}{t}} \quad \mu = e^{3\ln(t)} \quad \mu = t^3$$

Then multiply the equation by $\mu = t^3$

$$t^3 y' + 3t^2 y + 3t^2$$

Then integrate both sides

$$\int (t^3 y) = \int 3t^2$$

Therefore

$$t^3 y = t^3 + c$$

Here c is some arbitrary constant and thus

$$y = t^3 + \frac{c}{t^3}$$

Using the initial values we find that

c=24

The solution to the initial value problem is

$$y = t^3 + \frac{24}{t^3}$$

35 ~~40~~ points

4) Solve the differential equation:

$$\frac{\partial y}{\partial x} = -\frac{(y\sin x + x^2)}{(2y - \cos x)}$$

Answer:

not ∂'s, just d's.

The equation simplifies to:

$(y\sin x + x^2)\partial x + (2y - \cos x)\partial y = 0$

$M_y(x, y) = \sin x$ M=? N=?

$N_x(x, y) = \sin x$

This equation is exact.

$\psi_x(x, y) = y\sin x + x^2$

$\psi_y(x, y) = 2y - \cos x$

Integrate the first equation to get this:

$\psi(x, y) = -y\cos x + \frac{x^3}{3} + h(y)$

Take the derivative of this with respect to y and set it equal to $\psi_y(x, y)$ above.

$\psi_y(x, y) = -\cos x + h'(y) = 2y - \cos x$

$h'(y) = 2y$

$h(y) = y^2$

$\psi(x, y) = -y\cos x + \frac{x^3}{3} + y^2$

better flow/sentences.

Therefore:

$c = -y\cos x + \frac{x^3}{3} + y^2$

35 points

7

From section 1.2

Suppose a swimming pool with a capacity of 50,000 Liters is half filled with pure water. At time $t = 0$, a valve is opened and the pool begins filling at 100 Liters per minute with chlorinated water containing 2 mg/L chlorine. When the pool is full, an overflow valve is opened, allowing excess water to drain. Determine the time required (in minutes) for the Chlorine concentration to reach 1.75 mg/L. Assume Chorine occupies zero volume, and is perfectly mixed. In the interest of simplicity, any irrational numbers should be left in exact form, ie sqrt(3).

Solution:

Great.

1. It should be immediately recognized that the concentration of Chlorine when the pool is initially full will be half the concentration of the incoming mixture, therefore making irrelevant (though simple to calculate) any information about the time period while the pool is filling. A derivation and inclusion of this function would turn out a piecewise function y(t). This is unnecessary.
2. At a rate of 100 L/min, the pool will be full in 250 minutes
3. Since the initial quantity of Chlorine is zero, and initial quantity of water is one-half the final quantity of mixture, the **useful initial value will be y(250) = 1 mg/L**
4. The total quantity of Chlorine can be represented by the differential equation, dy/dt = rate in – rate out
5. $\frac{dy}{dt} = \frac{2mg/L}{100L/\min} - \frac{100 * y}{50000L}$
6. $\frac{dy}{dt} = 200 - \frac{y}{500}$
   a. dy/dt has units of mg/min
   b. y(t) has units of mg
7. Solving the differential equation by method of an integrating factor yields: $y(t) = 100000 + ce^{-\frac{t}{500}}$
8. Using initial values of y(250) = 50,000 yields $c = -50000\sqrt{e}$
9. This gives us our final equation of $y(t) = 100000 - 50000\sqrt{e} * e^{-t/500}$
10. In order to find the time required for the Chlorine concentration to reach 1.75mg/L unfortunately requires a calculator, unless it is expressed symbolically:
    a. Setting y(t)= 1.75 * 50,000 = 87500 and solving for t yields:
    b. t = 250 * (4 * Ln (2) + 1) = 943.147 minutes

$\sqrt{e}\, e^{-t/500}$
$= e^{\frac{1}{2} - \frac{t}{500}}$

From section 2.2

Solve the initial value problem:

$$y' = xy^2 + 2y^2, \quad y(0) = 1$$

<u>Solution</u>

Solve as a separable equation

$$\frac{dy}{dx} = xy^2 + 2y^2$$

Factor the y^2 from the two terms in the right side of the equation

$$\frac{dy}{dx} = y^2(x-2)$$

Next, isolate the y and x terms on different sides of the equation

$$\frac{dy}{y^2} = (x-2)dx$$

Integrate each side

$$\int \frac{dy}{y^2} = \int (x-2)dx$$

$$-\frac{1}{y} = \frac{x^2}{2} - 2x + c$$

Solve the equation for y

$$y = \frac{-1}{\frac{x^2}{2} - 2x + c}$$

Solve the function 7 for our initial value,

$$1 = \frac{-1}{\frac{0^2}{2} - 2 \cdot 0 + c}$$

$$1 = \frac{-1}{c}$$

$$c = -1$$

And so our final equation is

$$y = \frac{-1}{\frac{x^2}{2} - 2x - 1}, \quad x \neq 2 \pm \sqrt{6}$$ actually $x \in (2-\sqrt{6}, 2+\sqrt{6})$

From section 2.3

Consider a tank that holds 50 gallons of fresh water. (No chemical at t=0) The capacity of the tank is 100 gallons. Water is poured in at a rate of 5 gal/minute. The well mixed concentration flows out at a rate of 1 gallon/minute. A toxic chemical is poured in at a rate G(t) where G(t) is defined as G(t)=10+(4/5)t grams/gallon. Find the equation for the concentration of the chemical at the time the tank overflows.

of concentration

Solution:

(dq/dt)=rate in-rate out
Rate in=5gal/minute * (10 +(4/5)t)gram/gallon
Rate out=1 gal/minute * Q(t)/(50+5t-1t)

Find an equation for the rate of change of the chemical

$$\frac{dq}{dt} = 50 + 4t - \frac{Q(t)}{50 + 5t - 1t}$$

To solve for the concentration Q(t), use an integrating factor.

$$e^{\int \frac{1}{(50+4t)}} = (50+4t)^{.25} \quad \text{or} \qquad \text{E^ INTEGRAL(1/(50+4t))=(50+4t)^(1/4)}$$

Multiply both sides of the equation by the integrating factor.

$$\frac{Q(t)*(50+4t)^{.25}}{50+4t} + \frac{dq}{dt}(50+4t)^{.25} = (50+4t)^{.25}$$

Integrate both sides

$$\int([Q(t)*(50+4t)^{.25}]') = (\frac{1}{9})*(50+4t)^{2.25} + C$$

$$Q(t) = \frac{1}{9}(50+4t)^2 + \frac{C}{(50+4t)^{.25}}$$

Solve for C by using the fact that Q(0)=0 (Given Condition)

C=-738.7

$$Q(t) = \frac{1}{9}(50+4t)^2 - \frac{738.7}{(50+4t)^{.25}}$$

The tank overflows at t=12.5, because 100-50=4t.

$$Q(12.5) = \frac{1}{9}(50+4*12.5)^2 - \frac{738.7}{(50+4*12.5)^{.25}}$$

The question does not ask for the solution, just the equation.

" — not well-defined question, i.e. it is ambiguous

"Determine $\phi(t)$ for an arbitrary value of n for the following equation. Let $\phi_0(t) = 0$."

$$y' = 2(y+1)$$

Solution

/ removes ambiguity — should be stated in question

We will use the method of successive approximations to solve the given initial value problem.

Setting $\phi_0(t) = 0$.

$\phi_1(t) = \int_0^t 2(\phi_0(s)+1)ds = \int_0^t 2(1)ds = 2t.$

We inserted the value of $\phi_0(t)$ to approximate $\phi_1(t)$ in following the method of successive approximations. We will continue this procedure until a pattern is developed to which we can formulate a general solution:

$$\phi_2(t) = \int_0^t 2(\phi_1(s)+1)ds = \int_0^t 2(2s+1)ds = \frac{2*2t^2}{2} + 2t.$$

$$\phi_3(t) = \int_0^t 2(\phi_2(s)+1)ds = \int_0^t 2((\frac{2*2s^2}{2} + 2s)+1)ds = \frac{2*2*2t^3}{3*2} + \frac{2*2t^2}{2} + 2t.$$

$$\phi_4(t) = \int_0^t 2(\phi_3(s)+1)ds = \int_0^t 2((\frac{2*2*2s^3}{3*2} + \frac{2*2s^2}{2} + 2s)+1)ds = \frac{2*2*2*2t^4}{4*3*2} + \frac{2*2*2t^3}{3*2} + \frac{2*2t^2}{2} + 2t$$

The numbers have been intentionally left un-multiplied to make it obvious what the problem will look like as a partial sum of an infinite series. From the pattern set it becomes clear that,

$\phi(t) = \sum_{k=1}^{\infty} \frac{2^k}{k!} t^k$.

this is not "$\phi(t)$ for an arbitrary value of n", but rather

$\lim_{n\to\infty} \phi_n(t)$, $\phi_n(t)$ defined in your solution (unfortunately not defined in your question).

16

Problem # 1

Sea World has had a mechanical problem with the systems that control the amount of chemicals in the water. Their 5 million gallon tank has a filtration system that cleans the water and adds chemicals; it operates at 10,000 gal/hr. However, the system in charge of the chemicals is adding 0.05 grams per gallon too much of a chemical which acts as a toxin in higher concentrations.

Great problem statement.

a) (25 pts.) Sea World has asked you to write a differential equation to model the situation, and then solve it to determine the amount of this chemical in the tank at any given time in terms of t and a single unknown constant. — which? physical interpretation?

Solution:

$$\frac{dq}{dt} = [chemical\_flow\_in] - [chemical\_flow\_out]$$

$$[chemical\_flow\_in] = 10{,}000 gal/hr(.05gm/gal)$$

$$[chemical\_flow\_out] = 10{,}000 gal/hr(\frac{q}{5x10^6 gal})$$

$$\frac{dq}{dt} = 10{,}000 gal/hr(.05gm/gal) - 10{,}000(\frac{q}{5x10^6 gal})$$

$$\frac{dq}{dt} = 10{,}000(.\frac{.05q}{5x10^6}) \text{ ?}$$

$$\frac{dq}{dt} = \frac{-10{,}000}{5x10^6}(q - .05(5x10^6))$$

$$\frac{\frac{dq}{dt}}{(q-.05(5x10^6))} = \frac{-10{,}000}{5x10^6}$$

$$\ln|q - .05(5x10^6)| = \frac{1}{500}t + A$$

$$q = e^{\frac{1}{500}t + A} + .05(5x10^6)$$

$$q = Ce^{\frac{1}{500}t} + .05(5x10^6)$$

Better, line-by-line dialogue needed here.

b) (10 pts.) Sea World needs to know the order of this equation and also if it is linear, explain these below.

Solution

The equation is first order, and this is found by looking at the highest derivative that appears in the equation, and since there isn't any multiplication of the variable this equation will be linear.

Problem # 2

Once the tank is fixed, Sea World has needs to calculate the number of bacteria in the tanks. They have found that the population can be approximated with the following differential equation for all times when t > 0, and have asked you to solve it.

$$2t^2y' + 6t^2y = 4t^3e^{-3t}$$

Hint: The method of integrating factors may be useful.

(35 pts.) Solve.

Solution:

$y' + 3y = 2te^{-3t}$	Simplify
$\mu = e^{\int 3dt}$	Find the value of the
$\mu = e^{3t}$	integrating factor ?
$(e^{3t})y' + 3(e^{3t})y = 2te^{-3t}(e^{3t})$	Plug in IF.
$(e^{3t})y' + 3(e^{3t})y = 2t$	Simplify
$\frac{d}{dt}(e^{3t}y) = 2t$	Reduce using product rule
$e^{3t}y = \int 2t\,dt$	Integrate
$e^{3t}y = t^2 + C$	
$y = t^2e + Ce^{-3t}$	Answer

Again, complete sentences. See book for examples of how to integrate eq.s & dialogue.

Problem # 3

Sea World has decided to use their tank for holding Whales. In order for the whales to live, the tank must be stocked with Plankton. Given the chemical makeup of the pool, and the exact type of plankton, Sea World has already established a differential equation to model the tank in terms of p (the current population in billions of plankton):

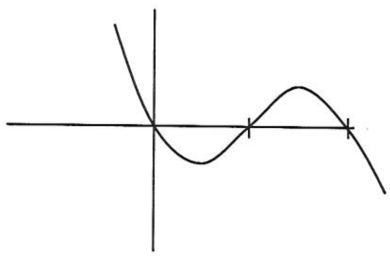
$$\frac{dp}{dt} = -5p(\frac{21-10p+p^2}{21})$$

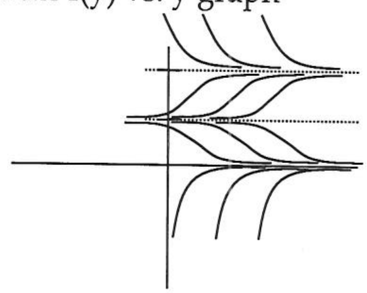
no antecedent

In order to know how much Plankton to add initially, Sea World has asked you to create both a phase graph, and a graph of f(y) vs. y for this equation. For future reference when selecting whales, they would like you to determine the carrying capacity of the tank, and the threshold value. Assumed in your analysis of these values is of course a description of each as asymptotically stable, unstable, or semistable. Remember to provide all answers in billions of plankton.

a) (10 pts.) Create the phase graph, and the f(y) vs. y graph

Solution:





b) (10 pts.) Give numerical values for each critical point, and classify each.

Solution:

zeros of the equation are 0, 3, and 7.
0 is asymptotically stable
3 is unstable
7 is asymptotically stable

how do we know these. (sure, sure follow from graph).

c) (5 pts.) What is the carrying capacity of the system

Solution:

7 (billion)

d) (5 pts.) What is the threshold value for the system

Solution:

3 (billion)

Problem # 4

Sea world has been using chlorine in their tanks to control algae outbreaks for some time. Recently they've been experiencing erratic algae control problems. They started monitoring algae level and have discovered what some would say are some very strange results. It turns out that some types of algae are not only resistant to chlorine but in many cases actually consume it and one such type of algae has found it's way into Sea World tanks. After careful monitoring and measuring the people in charge of tank maintenance have determined that the algae growth satisfies the following differential equation.

$$(2ca - a/c)dc + (c^2 - \ln(c))da = 0$$

a) (10 pts.) Show that the given differential equation is exact

Solution:

$$\frac{d}{da}(2ca - a/c) = 2c - 1/c$$
$$\frac{d}{dc}(c^2 - \ln(c)) = 2c - 1/c$$

Results are the same thus it is exact.

– no antecedent.

b) (20 pts.) Solve for $\psi(a,c)$

Solution:

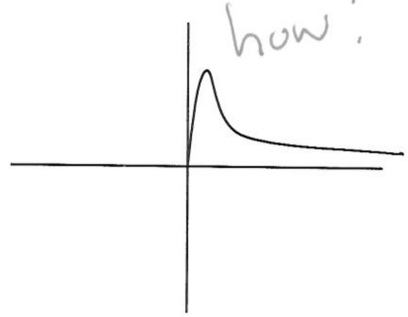
$$\int (2ca - a/c)dc = ac^2 - a\ln(c) + C_0(a) = \psi(a,c)$$
$$\int (c^2 - \ln(c))da = ac^2 - a\ln(c) + C_1(c) = \psi(a,c)$$
$$ac^2 - a\ln(c) + C_0(a) = ac^2 - a\ln(c) + C_1(c)$$
$$C_0 = C_1 \Rightarrow C_0 + C_1 = \kappa$$
$$\Psi(a,c) = ac^2 - a\ln(c) + \kappa$$

c) (20 pts.) Graph algae as a function of chlorine and give a recommendation on whether or not chlorine should be used in algae control at Sea World in the future

Solution:

how?

Chlorine appears to only have a significant impact when concentrations become extremely large. At these concentrations the chlorine is likely interfering with other life in the tank. Furthermore it appears that the algae will never be totally eradicated no matter how much chlorine is dumped in. An alternate algae control strategy is therefore recommended.



(26)

Problem 1 (45 points): A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 $\frac{mg}{cm^3}$ of the drug enters the patient's bloodstream at a rate of $100\frac{cm^3}{hr}$. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of $0.4hr^{-1}$.

a) Assuming that the drug is always uniformly distributed throughout the bloodstream write a differential equation for the amount of drug that is present in the bloodstream at any time.

Answer:

$$\text{Flow in} = 5\frac{mg}{cm^3}\cdot 100\frac{cm^3}{hr} = 500\frac{mg}{hr}$$

$$\text{Flow out} = 0.4hr^{-1}\cdot y(t)$$

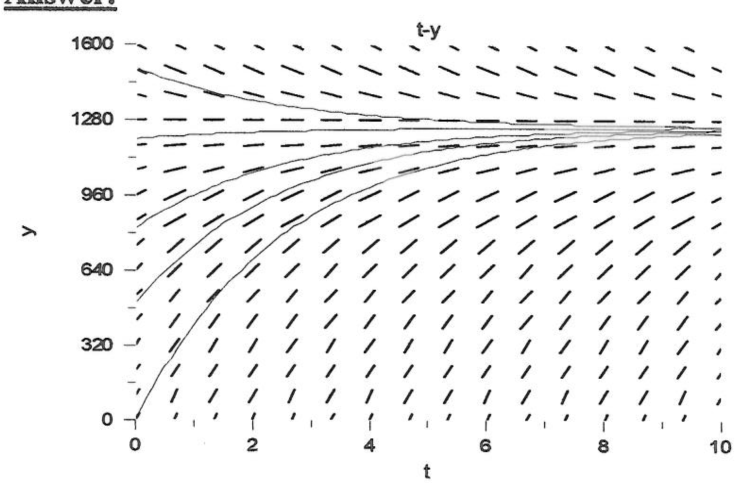
y(t) = amount of drug in bloodstream measured in mg

$$\frac{dy}{dt} = \text{Flow in - Flow out}$$

$$\frac{dy}{dt} = 500\frac{mg}{hr} - 0.4hr^{-1}\cdot y(t)$$

b) Draw the directional field for the differential equation in part (a). How much drug is present in the bloodstream after a long time?

Answer:



$$\frac{dy}{dt} = 0$$

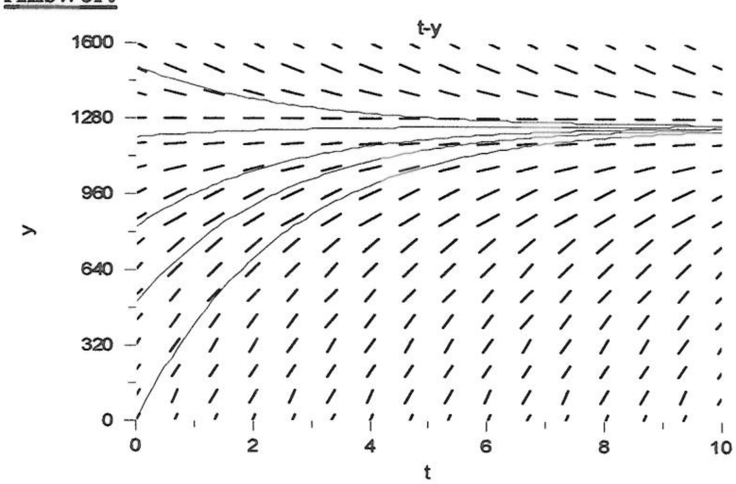
$$0 = 500\frac{mg}{hr} - 0.4hr^{-1}\cdot y(t)$$

$$0.4hr^{-1}\cdot y(t) = 500\frac{mg}{hr}$$

$$y(t) = \frac{500\frac{mg}{hr}}{0.4hr^{-1}} = 1250mg$$

c) Solve the differential equation found in part (a) and draw several integral curves of the equation (including the equilibrium point found in part (b)).

Answer:



$$\frac{dy}{dt} = 500\frac{mg}{hr} - 0.4hr^{-1} \cdot y(t)$$

$$\frac{dy}{dt} = .4hr^{-1} \cdot (1250mg - y(t))$$

$$\frac{dy}{(1250mg - y(t))} = .4hr^{-1}dt$$

$$\int \frac{dy}{(1250mg - y(t))} = \int .4hr^{-1}dt$$

$$\ln(1250mg - y(t)) = .4hr^{-1} \cdot t + C$$

$$1250mg - y(t) = C_1 e^{.4hr^{-1} \cdot t}$$

$$y(t) = 1250mg - C_1 e^{.4hr^{-1} \cdot t}$$

d) The equation found in part (a) can be described as an Ordinary Linear 1st Order Differential equation. Please describe the reason for each part of the classification below:

Ordinary:

Answer: There is only one variable (2 or more variables would change the classification to partial)

Linear:

Answer: Because all derivatives are in the form

$f^n(x) = [f(x) + f^1(x) + ... + f^{n-1}(x)]$?

1st Order:

Answer: Because the highest derivative is a first derivative

Problem 2 (30 points): Solve the initial value problem:

$$\frac{dy}{dt} = 1 + t^2 + y^2 + t^2 y^2 \qquad y(0) = 1$$

Solution:

This is a separable equation and we get

$$1 + t^2 + y^2 + t^2 y^2 = \left(1 + t^2\right)\left(1 + y^2\right)$$?

Because there are no roots to the equation $1 + y^2 = 0$ we can proceed with the separation of the two variables and integration. We have

$$\frac{dy}{1 + y^2} = \left(1 + t^2\right)dt$$

which gives

$$\int \frac{dy}{1 + y^2} = \int \left(1 + t^2\right)dt$$

Since

$$\int \frac{dy}{1 + y^2} = \tan^{-1}(y)$$

and

$$\int \left(1 + t^2\right)dt = t + \frac{t^3}{3}$$

we get

$$\tan^{-1}(y) = t + \frac{t^3}{3} + C$$

The initial condition $y(0) = 1$ gives

$$C = \tan^{-1}(1) = \frac{\pi}{4}$$

So the solution to the initial value problem is

$$y = \tan\left(t + \frac{t^3}{3} + \frac{\pi}{4}\right)$$

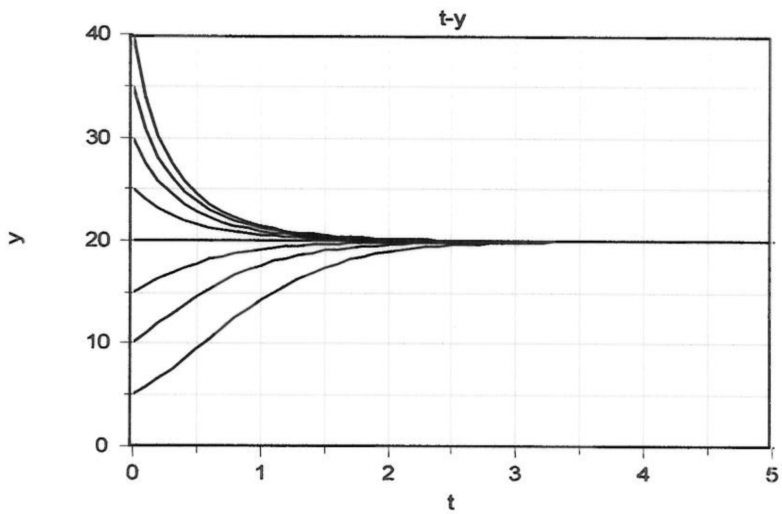
Problem 3 (45 Points): Assuming that a given population has no threshold limit, its population might be modeled by the equation: $f(y) = \frac{dy}{dt} = ry(1 - y/K)$ Where r is the rate of population growth and K is the limiting population factor. For $r = 2$ and $K = 20$ do the following:

A) Determine the critical points of $f(y)$ and classify each as stable, semi-stable, or unstable.
B) Sketch several graphs of solutions in the t-y plane
C) Integrate the differential equation (you don't need to solve for y explicitly).

Solution:

A) The critical points are the zeros of $f(y)$ which are at 0 and 20.
The $y = 0$ equilibrium is unstable and the $y = 20$ equilibrium is stable.

B) Several Graphs of the solution are shown below:



C) Separating the equation we obtain: $\frac{dy}{y(1 - y/20)} = 2dt$

Partial fractions can be used to "solve" the left side: $\frac{dy}{y(1-y/20)} = \frac{A}{y} + \frac{B}{1 - y/20}$

Where solving for A and B we obtain:

$$1 = \left(1 - \frac{y}{20}\right)A + By = A - \frac{Ay}{20} + By = A + (B - \frac{A}{20})y$$

Since the co-efficient on y is zero on the left side: $1 = A + (0)y$.
Therefore $A = 1$ and $B = 1/20$.

The resulting integration is therefore:

$$\ln(y) - \frac{\ln(1 - y/20)}{20} = 2t + c \Rightarrow \ln\left(\frac{y}{(1 - y/20)^{1/20}}\right) = 2t + c \Rightarrow \frac{y}{(1 - y/20)^{1/20}} = ce^{2t}$$

Problem 4 (30): Determine whether the equation is exact or not. If it is exact, find the solution.

$(3x^2-2xy + 2)\, dx + (6y^2 + x^2 + 3)dy = 0$

no antecedents

Solution:

1. First check to see if $M_y = N_x$ by taking the partial derivative.

$\frac{\partial}{\partial y}(3x^2-2xy + 2) = -2x \qquad \frac{\partial}{\partial x}(6y^2 + x^2 + 3) = -2x$

They are both equal so the equation is exact.

integration?

2. Now take the partial ~~derivative~~ with respect to x of the first half.

$\int (3x^2-2xy + 2)\, dx = x^3 - yx^2 + 2x + h(y)$

3. Take the partial derivative of your answer with respect to y and set it equal to the other half of the original equation and solve for $\frac{dy}{dh}$.

$\frac{\partial}{\partial y}\, x^3 - yx^2 + 2x + h(y) = -x^2 + \frac{dy}{dh} = 6y^2 + x^2 + 3$

$\frac{dy}{dh} = 6y^2 + 3$

4. Take the integral of your answer with respect to y.

$\int 6y^2 + 3\, dy = 2y^3 + 3y + C$

5. Combine this with the answer to the integral of the first half of the original equation and solve for C.

$C = x^3 - yx^2 + 2x + 2y^3 + 3y$

36

(20 points)

1) Solve the following first order linear differential equations
(Method of Integrating Factors)

a.) Find the particular solution of the following Initial Value Problem:

$$y' + 4y = 10 \qquad y(0) = 50$$

b.) Find the general solution of the following Differential Equation:

$$y' + \tan(t)y = \cos^2(t)$$

Solution:

a.)

i.) Find the integrating factor: $\mu(t) = e^{\int 4dt} = e^{4t}$

ii.) Multiply both sides: $e^{4t}(y') + e^{4t}(4y) = e^{4t}(10)$

iii.) Find relationship: $\frac{d}{dt}\left(ye^{4t}\right) = 10e^{4t}$

iv.) Antidifferentiate: $ye^{4t} = \int 10e^{4t}dt = 10\cdot\left(e^{4t}/4\right) + c$

v.) Solve for y: $y(t) = \frac{10}{4} + ce^{-4t}$

vi.) Use initial value data to solve for c:

$$y(0) = 50 = \frac{10}{4} + ce^{-4(0)} = \left(\frac{10}{4} + c\right) \qquad c = 50 - \frac{10}{4}$$

vii.) Solution: $y(t) = \frac{10}{4} + \left(50 - \frac{10}{4}\right)e^{-4t}$

b.)

i.) Find the integrating factor: $\mu(t) = e^{\int \tan(t)dt} = e^{-\ln(\cos(t))} = e^{\ln(\sec(t))} = \sec(t)$

ii.) Multiply both sides: $\sec(t)y' + \sec(t)\tan(t)y = \sec(t)\cos^2(t)$

iii.) Find relationship: $\frac{d}{dt}(\sec(t)y) = \sec(t)\cos^2(t)$

iv.) Antidifferentiate: $\sec(t)y = \int \sec(t)\cos^2(t)dt = \int \cos(t)dt = \sin(t) + k$

v.) Solve for y: $y(t) = \frac{\sin(t) + k}{\sec(t)} = \sin(t)\cos(t) + k\cos(t)$

(35 points)

2)

a.) Consider the following general first order differential equation,

$$\frac{dy}{dx} = f(x,y)$$

By substituting $M(x,y) = -f(x,y)$ and $N(x,y) = 1$, and if M is a function of only x and N is a function of y only, show that the equation is separable.

b.) Show that the given equation is homogeneous and then solve the differential equation.

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$$

Hint: If the right side of the equation can be expresses as a ratio of (y/x) then it is a homogeneous equation. When solving the differential equation use the substitution y=vx.

Solution:

a.) The above equation can be rewritten in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

By substituting M(x,y) = -f(x,y) and N(x,y) = 1.

Also, if M is a function of x and N is a function of y the equation can be written as,

$$M(x) + N(y)\frac{dy}{dx} = 0$$

To show that this is separable it can ge written in the differential form

$$M(x)dx + N(y)dy = 0$$

b.) Divide the numerator on the right side of the equation by x^2 and thus we can see that it is a homogeneous equation.

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \frac{y^2}{x^2}$$

By substituting y = vx we get

$$v + x\frac{dv}{dx} = 1 + v + v^2$$

$$xdv = (1 + v^2)dx$$

$$\frac{dv}{(1+v^2)} = \frac{1}{x}dx$$

Integrating

$$\int \frac{dv}{(1+v^2} = \int \frac{1}{x}dx$$

Leads to

$$\arctan v = \ln|x| + c$$

Substituting back gives us the answer

$$\arctan(\frac{y}{x}) = \ln|x| + c$$

(60 points)
3) A state originally has a population of 1,000,000 people. The immigration rate into the stats is 10,000 people per year and the same amount of the people emigrate from the state each year. Of the immigrants into the state 20% are illegal. It is assumed that the emigrants leave the state in the same proportion of legal to illegal citizens as are currently in the state. Illegal immigrants account for 2% of the state population. After 2 years the government stops all illegal immigration, and the immigration and emigration rates drop to 5,000 people per year.

What is the percentage of the population who illegally immigrated?

Solution:

$$\frac{dI}{dt} = .20(10{,}000) - \frac{I(t)}{1{,}000{,}000} * 10{,}000$$

$$\frac{dI}{dt} = 10{,}000(.20 - \frac{I(t)}{1{,}000{,}000})$$

$$\frac{\frac{dI}{dt}}{.20 - \frac{I(t)}{1{,}000{,}000}} = 10{,}000$$

$$\frac{d}{dt}(\ln(.20 - \frac{I(t)}{1{,}000{,}000})) = \frac{d}{dt}(10{,}000t + C)$$

$$.20 - \frac{I(t)}{1{,}000{,}000} = Ce^{10{,}000t}$$

$$I(t) = 1{,}000{,}000(.20 - Ce^{10{,}000t})$$

Now solve for C:

$$I(0) = .02(1{,}000{,}000)$$

$$.02(1{,}000{,}000) = 1{,}000{,}000(.20 - Ce^{0})$$

$$.02 = .20 - C$$

Now find the percentage of the population after 2 years:

$$I(2) = 1{,}000{,}000(.20 - .18e^{10{,}000(2)})$$

(35 points)

4) Determine if the following equations are exact, if so solve them.

a) $(3ye^{x} + x - 1)dx + (3e^{x} - y^{2} + 4)dy = 0$

b) $(\frac{y^{2}}{2x} - 4x)dx + (y\ln x^{2} - 3y^{2} + 2)dy = 0$

Solution:

Using Theorem 2.6.1 define M(x,y) and N(x,y) then proceed to see if $M_y(x,y) = N_x(x,y)$ if this is true then the equation is exact.

a.)

$$M(x,y) = (3ye^x + x - 1)$$
$$N(x,y) = (3e^x - y^2 + 4)$$
$$M_y = 3e^x$$
$$N_x = 3e^x$$
$$M_y = N_x$$

This equation is exact. Solving by integrating M and N

$$\int M = \int (3ye^x + x - 1)dx$$
$$\int (3ye^x + x - 1)dx = 3ye^x + \frac{x^2}{2} - x + C(y)$$
$$\int N = \int (3e^x - y^2 + 4)dy$$
$$\int (3e^x - y^2 + 4)dy = 3ye^x - \frac{y3}{3} + 4y + C(x)$$

Now these two equations must be combined to form the final equation.

$$f(x,y) = 3ye^x - \frac{y3}{3} + 4y + \frac{x^2}{2} - x + c$$

this is not the solution. solution is

$$\boxed{3ye^x - \frac{y^3}{3} + 4y + \frac{x^2}{2} - x = c}$$

b.)

$$M(x,y) = (\frac{y^2}{2x} - 4x)$$
$$N(x,y) = (y\ln x^2 - 3y^2 + 2)$$
$$M_y = \frac{y}{x}$$
$$N_x = \frac{y}{2x}$$
$$M_y \neq N_x$$

This is not an exact equation.

Question 1 –

4b

Say Mr. Red goes hiking and wants to drop rocks off of a bridge that he found. If Gravity pulls at a constant 9.00 m/s and Since the rock will follow F=ma (also known as F= m(v') then assuming a drag coefficient of .4. What equation will allow him to find out the equilibrium speed of the rock before it hits the ground assuming the rock falls into a semi-bottomless pit? The rock he throws weighs about 13.5 kg. You may want to draw a free body diagram to account for all of the forces.)

Solution 1 –

Answer) the equation is F=ma which is equal to

m(v') =mg-((drag)v)

or

m(dv/dt) = mg-((drag)v)

after diving by m or both sides gives:
dv/dt= g – ((drag(v)/m

of 7

b) If v = 45 m/s then what will the rate of change be?

Answer)
dv/dt= g – ((drag(v) / m   putting in values gives;
dv/dt= 9.00 m/s^2 - ((.4 * 45 m/s)/13.5 kg

dv/dt = 9.00 – 1.3333333   =   7 2/3

c) when will dv/dt = 0 ?

Answer) this happens when the drag times velocity divided by mass equals 9.00.

v= 9.00 m/s^2 * (.4 / 13.5) = 303.75 m/s

d) draw the directional field and the equilibrium speed as a function of time:

the equilibrium speed is 303.75 m/s because when dv/dt is zero gives a horizontal slope.

```
V       I\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\
        I````````````````````````````````````````````````````````````````
303.75  I----------------------------------------------------------------
        I
        I////////////////////////////////////////////////////////////////
        ----------time axis---------------------------------------
```

Problem 2 –

Find the solution of the given initial value problem:

$$y' = \frac{x^4 + e^{-7x}}{3 - y} \quad , \quad y(0) = 3$$

Solution 2 –

This is a separable differential equation so we begin by separating. Once we do this we have:

$$(3 - y)dy = (x^4 + e^{-7x})dx$$

Once we have separated we integrate both sides.

$$\int 3 - y dy = \int x^4 + e^{-7x} dx$$

And obtain:

$$3y - \frac{1}{2}y^2 = \frac{1}{5}x^5 - \frac{1}{7}e^{-7x} + c$$

Next we plug in the initial values to obtain a value for c.

$$3(3) - \frac{1}{2}(3)^2 = \frac{1}{5}(0)^5 - \frac{1}{7}e^{-7(0)} + c$$

$$9 - \frac{9}{2} = -\frac{1}{7} + c$$

$$c = \frac{9}{2} + \frac{1}{7} = \frac{65}{14}$$

This gives us our final equation and solution:

$$3y - \frac{1}{2}y^2 = \frac{1}{5}x^5 - \frac{1}{7}e^{-7x} + \frac{65}{14}$$

Problem 3 –

Given the following equation....

$$\left[x\cdot\left(y^2 + b\cdot x\cdot y\right)\right]\cdot dx + \left[x^2\cdot(y + x)\right]\cdot dy = 0$$

a)Find the value of q for which the given equation is exact.

b)Solve the equation using the value of q.

Solution 4 –

a) First, in order for this equation to be exact the following must be true.

$$M_y(x,y) = N_x(x,y)$$

So for our equation we have

$$M(x,y) = \left[x\cdot\left(y^2 + b\cdot x\cdot y\right)\right] \quad \text{and} \quad N(x,y) = \left[x^2\cdot(y + x)\right]$$

So solving for $M_y(x,y)$ and $N_x(x,y)$ we get

$$M_y(x,y) = 2\cdot x\cdot y + b\cdot x^2 \quad \text{and} \quad N_x(x,y) = 2\cdot x\cdot y + 3\cdot x^2$$

In order for the equation to be exact again

$$M_y(x,y) = N_x(x,y)$$

So for our values

$$2\cdot x\cdot y + b\cdot x^2 = 2\cdot x\cdot y + 3\cdot x^2$$

$\boxed{\text{Solving for the value of b we find that b = 3.}}$

b) We now can plug our b value into our original equation.

$$\left[x\cdot\left(y^2 + 3\cdot x\cdot y\right)\right]\cdot dx + \left[x^2\cdot(y + x)\right]\cdot dy = 0$$

The problem is now solvable by integrating each half of the equation.

$$\int x\cdot\left(y^2 + 3\cdot x\cdot y\right)dx + \int x^2\cdot(y + x)\,dy = 0$$

$$\left(\frac{x^2\cdot y^2}{2} + x^3\cdot y\right) + \left(x^3\cdot y + \frac{x^2\cdot y^2}{2}\right) = C$$

Where C is a constant.

Which can simplify to

$$\boxed{x^3\cdot y + \frac{x^2\cdot y^2}{2} = C}$$

Problem 4 –
In problem 4 let $\phi_0(t) = 0$ and use the method of successive approximations to solve the given initial value problem. (a) Determine $\phi_1, \ldots, \phi_3$. (b) Write whether the iterates appear to be converging (plots are helpful but not required).

4. $y' = t^2 - y^2, \quad y(0) = 0$

Solution 4 –

(a)

Note first that if $y = \phi(t)$, then the corresponding integral equation is

$$\phi(t) = \int_0^t [s^2 - \phi(s)^2]ds$$

If the initial approximation is $\phi_0(0) = 0$, it follows that

$$\phi_1(t) = \int_0^t [s^2 - \phi_0(s)^2]ds = \int_0^t [s^2 - 0^2]ds = \int_0^t s^2 ds = \frac{t^3}{3}$$

Similarly,

$$\phi_2(t) = \int_0^t [s^2 - \phi_1(s)^2]ds = \int_0^t [s^2 - \left(\frac{s^3}{3}\right)^2]ds = \int_0^t [s^2 - \left(\frac{s^6}{3\cdot 3}\right)]ds = \frac{t^3}{3} - \frac{t^7}{3\cdot 3\cdot 7}$$

And

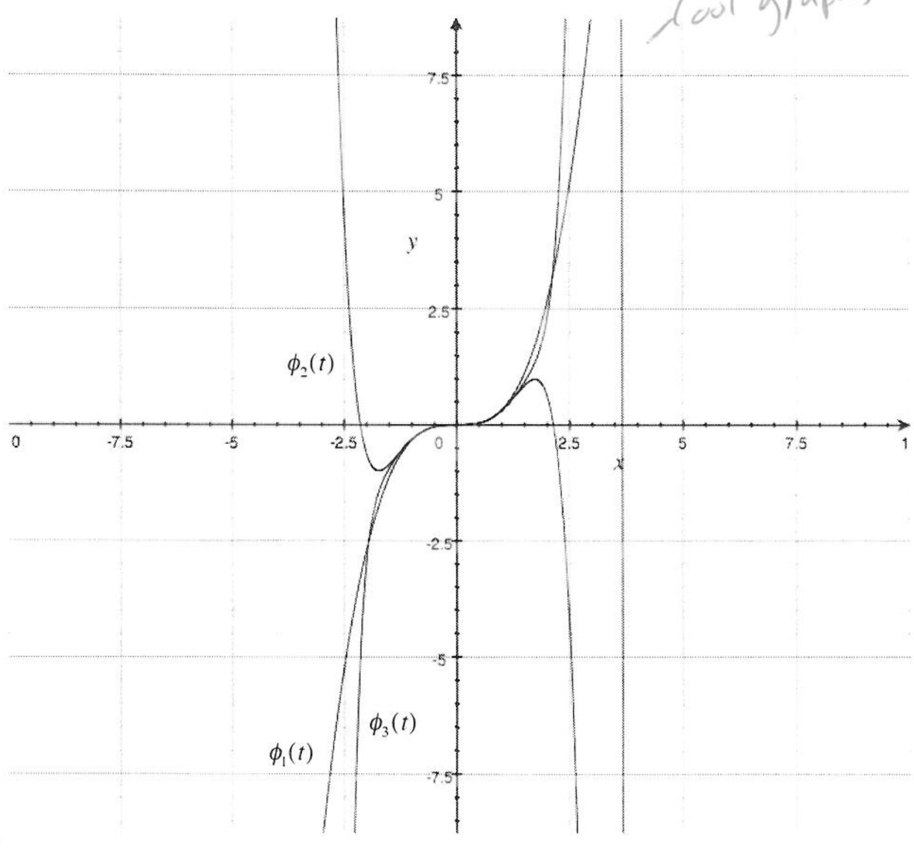
$$\phi_3(t) = \int_0^t [s^2 - \phi_2(s)^2]ds = \int_0^t [s^2 - \left(\frac{s^3}{3} - \frac{s^7}{3\cdot 3\cdot 7}\right)^2]ds = \int_0^t [s^2 - \left(\frac{s^6}{3\cdot 3} - \frac{2s^{10}}{3\cdot 3\cdot 3\cdot 7} + \frac{s^{14}}{3\cdot 3\cdot 3\cdot 3\cdot 7\cdot 7}\right)]ds$$

$$= \frac{s^3}{3} - \frac{s^7}{3\cdot 3\cdot 7} + \frac{2s^{11}}{3\cdot 3\cdot 3\cdot 7\cdot 11} - \frac{s^{14}}{3\cdot 3\cdot 3\cdot 3\cdot 7\cdot 7\cdot 14} = \frac{s^3}{3} - \frac{s^7}{63} + \frac{2s^{11}}{2079} - \frac{s^{14}}{55566}$$

(b)

Cool graphs

Plotting the three functions we see, yes, they appar to be converging (see graph):



56

(50) points

Solve the following differential equation including initial conditions.

$$x^3 y' + 4x^2 y = e^x$$

Given y(1) =1.

Solution:

$$y' + \frac{4}{x} y = \frac{e^x}{x^3}$$

$$u(x) = e^{\int \frac{4}{x}} = e^{3\ln x} = x^4$$

$$\int (x^4 y' + 4x^3 y) = \int xe^x$$

$$x^4 y = xe^x - e^x + c$$

$$y = \frac{e^x}{x^3} - \frac{e^x}{x^4} + \frac{c}{x^4}$$

Insert y(0)=0

$$1 = \frac{e^1}{1} - \frac{e^1}{1} + \frac{c}{1}$$

$$c = 1$$

Therefore:

$$\boxed{y = \frac{xe^x - e^x + 1}{x^4}}$$

(45) points

not a sentence

If the natural population of a fish farm is determined by the autonomous differential equation: (P represents hundreds of fish).

$$\frac{d}{dt}\mathrm{P}(t) = a\,P\,(b-P)$$

where a = 1 and b = 4.

a) If fish are harvested at a rate of h fish per week, what is the differential equation describing the fish population?

$$\frac{d}{dt}\mathrm{P}(t) = a\,P\,(b-P) - h$$

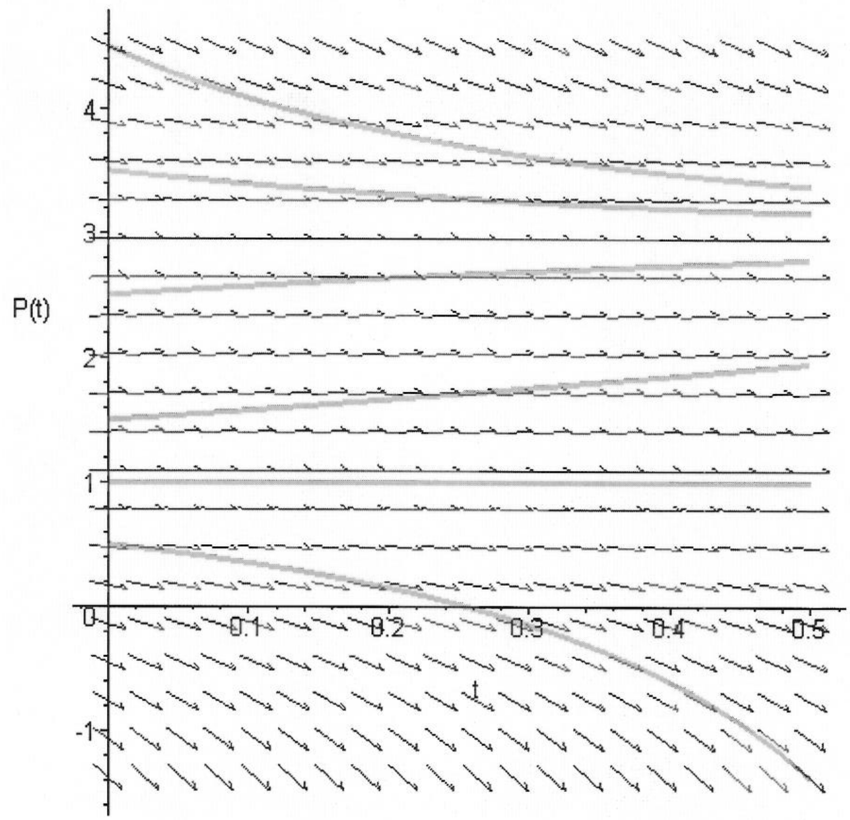
same "p"?

b) Draw possible solution curves for the differential equation indicating interesting initial values of p where h = 300.

equilibria? why?

There are horizontal asymptotes at P = 3 and P = 1. If P(0) is less than 1 then the population goes extinct. Otherwise the population converges on 3.

why? why?



c) If the initial fish population is 5 fish describe what will happen to the fish population if h = 3.

If the initial population of fish is 5 hundred and h = 3 then the population will converge down to 3 hundred.

d) Find a value for h where the fish population will go extinct if the initial population is 250 fish.

$$P = 2.5$$

$$2.5\,(4 - 2.5) - h < 0$$

$$3.75 - h < 0$$

$$h > 3.75$$

So if 375 fish are harvested each period then the initial population must be above 250 fish if you still want to have fish in the end.

(45) points

A 2000 gallon tank initially contains 400 gallons of water with 4 lbs of salt dissolved in it. Water with a salt concentration of $\frac{1}{2}(1+\cos(t))$ lb/gal is flowing into the tank at a rate of 4 gal/hr. A well-stirred mixture is draining from the tank at a rate of 2 gal/hr. (1)Set up the

initial value problem that describes this flow process and (2)Determine how much salt is

in the tank when it overflows.

$$Q'(t) = (4)\left[\frac{1}{2}(1+\cos(t))\right] - (2)\left(\frac{Q(t)}{400+2t}\right)$$

$$Q'(t) + \left(\frac{Q(t)}{200+t}\right) = 2(1+\cos(t))$$

$$u(t) = \exp^{\int \frac{1}{200+t}} = e^{\ln|200+t|} = (200+t) \qquad g(t) = 2(1+\cos(t))$$

$$Q(t) = \frac{1}{u(t)}\left[\int u(t)g(t)dt + c\right]$$

$$Q(t) = \frac{1}{(200+t)}\left[\int_0^t (200+s)(2(1+\cos(s)))ds + c\right]$$

$$Q(t) = \frac{2}{(200+t)}\left[200t + \frac{t^2}{2} + 200\sin(t) + \cos(t) + t\sin(t) - (1) + c\right]$$

Combine the -1 and c to just c.

more dialogue.

$$Q(t) = \frac{400t + t^2 + 400\sin(t) + 2\cos(t) + 2t\sin(t) + c}{(200+t)}$$

$$Q(0) = 4$$

$$4 = \frac{2+c}{200} \qquad c = 798$$

(1)The solution for the initial value problem

$$Q(t) = \frac{400t + t^2 + 400\sin(t) + 2\cos(t) + 2t\sin(t) + 798}{(200+t)}$$

(2)The tank overflows ~~and~~ at time $t = 800$?

$Q(800) =$ **962.2 lbs**

6b

1. Determine the order of the following differential equations and if they are linear or not.

1. $$\frac{t^4 d^4 y}{dt^4} + \frac{3td^3 y}{dt^3} + 5y = \cos(t)$$

2. $$\frac{d^3 y}{dt^3} + \cos(y) = \cos(t)$$

3. $$U_{xx} + 3\,U_{yy} + 2\,U_{zz} = 0$$

4. $$UU_{xxx} + 3\,U_{yy} + U_x + U_y = 0$$

Solution

1. 4th order linear
2. 3rd order nonlinear
3. 2nd order linear
4. 3rd order nonlinear

2. Find an integrating factor and solve for the equation:

$$x^2 \cdot \sin(y)\,dx + \sin(y)\cos(y)\,dy = 0$$

Solution ?

$$u := \frac{1}{\sin(y)}$$

$$x^2 dx + \cos(y)\,dy = 0$$

$0 = 0$ **exact** ?

$$\int x^2\,dx \qquad ? \qquad \int \cos(y)\,dy$$

$$\frac{x^3}{3} + c(y) \qquad ? \qquad \sin(y) + c(x)$$

$$\frac{x^3}{3} + \sin(y) + k$$

what is going on here?

3. Weeds are growing in a garden at a rate proportional to the current number of weeds in the garden. The owner of the garden is particularly lazy & only pulls 3 weeds a day. At time t=0 there are 20 weeds in the garden & at time t=1.753 weeks there are 38 weeds in the garden.

a) What is the value of the growth constant, r ?

b) How many weeds will be in the garden at t=3 weeks?

c) At what rate will the lazy gardener need to pull weeds in order to remove all the weeds in 1 week?

Solution

a) w = # of weeds

$$\frac{dw}{dt} = rw - 21 \qquad \left(\frac{21\,\text{weeds}}{\text{week}}\right)$$

$$\frac{dw}{dt} - rw = -21 \qquad \mu(t) = e^{-rt}$$

$$e^{-rt}w = \int -21e^{-rt}\,dt \rightarrow \frac{-21}{-r}e^{-rt} + c$$

$$\frac{21}{r} + ce^{rt} = w(t)$$

$$w(0) = 20 = \frac{21}{r} + c \qquad w(1.753) = 38 = \frac{21}{r} + ce^{1.753r}$$

$$20 - \frac{21}{r} = c \qquad \frac{\left(38 - \frac{21}{r}\right)}{e^{1.753r}}$$

much more dialogue, see my key, for example, cryptic

Setting the two c equations equal and solving for r gives:
r = 1.2 /week

b) With the growth constant r, we can now solve for c:

$$20 - \frac{21}{r} = c \rightarrow 20 - \frac{21}{1.2} = 2.5$$

cryptic

$$w(3) = 17.5 + 2.5e^{1.2(3)} = 108.996 \quad \textbf{weeds}$$

c) $\frac{dw}{dt} = 1.2w - r$ $\left(\frac{r}{1.2} + ce^{1.2t}\right) = w(t)$

$w(0) = \frac{r}{1.2} + c = 20$ $w(1) = \frac{r}{1.2} + ce^{1.2} = 0$

$\left(20 - \frac{r}{1.2}\right) = c$ $\frac{\left(\frac{-r}{1.2}\right)}{e^{1.2}} = c$

cryptic

Setting both c equations equal to one another and solving for r:

$$20 - \frac{r}{1.2} = \frac{\left(\frac{-r}{1.2}\right)}{e^{1.2}}$$

$$20e^{1.2} - e^{1.2}\frac{r}{1.2} = \frac{-r}{1.2}$$

$$24e^{1.2} - e^{1.2}r = -r$$

$$\left(1 - e^{1.2}\right)r = -24e^{1.2}$$

$$r := \frac{-24e^{1.2}}{1 - e^{1.2}} = 34.344 \frac{\text{weeds}}{\text{week}}$$

for limited values of t

4) Verify that both $y_1t = 1\text{-}t$ and $y_2t = -t^2 / 4$ are solutions of the initial value problem

$$y' = \frac{-t + \left(t^2 + 4y\right)^{\frac{1}{2}}}{2} \qquad y(2) = -1$$

o.k.

And determine where these solutions are valid.

solution

To test whether $y_1 t$ is a valid solution of the initial value problem, just plug it into the equation

$$\frac{-t + \left[t^2 + 4(1-t)\right]^{\frac{1}{2}}}{2} = \frac{-t + \left(t^2 + 4 - 4t\right)^{\frac{1}{2}}}{2} = \frac{-t + [(t-2)(t-2)]^{\frac{1}{2}}}{2} =$$

$$\frac{-t + t - 2}{2} = \frac{-2}{2} = -1$$

only when $t \geq 2$

for $y_2 t$:

$$\frac{-t + \left[t^2 + 4\left(\frac{-t^2}{4}\right)\right]^{\frac{1}{2}}}{2} = \frac{-t + \left(t^2 - t^2\right)^{\frac{1}{2}}}{2} = \frac{-t}{2}$$

for all t

Then just plug in your initial data y(2) = -1 $\rightarrow \quad \frac{-(2)}{2} = -1$

For $y_1 t$ the solution is only valid on the interval $t \geq 2$ because if $t < 2$ then the initial value solution y(2) = -1 is not satisfied. what?

For $y_2 t$ the solution is valid for any value of t because you will always get a solution that has the same slope as the initial value solution. what?

(40 *points*) 1 : $\dfrac{dy}{dx} = \dfrac{(x+4)^2}{3+y^2}$

a) "solve the given function."

b) determine where the solution is defined.

Solution :

$$\frac{dy}{dx} = \frac{(x+4)^2}{3+y^2} \quad (\textit{seperable})$$

$$(3+y^2)\,dy = (x+4)^2 dx$$

$$(3+y^2)\,dy = (x^2+8x+16)\;dx$$

$$\int(3+y^2)\,dy = \int(x^2+8x+16)\;dx$$

$$3y + \frac{y^3}{3} = \frac{x^3}{3} + 4x^2 + 16x + C$$

$$\boldsymbol{y^3 + 9y - x^3 + 12x^2 + 48x = C}$$

***solution is defined* for $\mathbb{R}$**

y=y(x) or x=x(y)? — makes a difference.

(*60 points*) 2 : a tank holds 100 gallons of brine solution containing 20lb of salt. At $t=0$, fresh water enters the tank at a rate of $5\dfrac{\text{gal}}{\text{min}}$ and leaves the tank at the same rate.

a) write a differential equation to model the amount of salt in the tank at any given time (t).

b) how much salt will remain in the tank after $t = 30$ minutes?

Solution :

$$\frac{dQ}{dt} + \frac{Q}{20} = 0$$

$$\frac{dQ}{dt} + \left(\frac{1}{20}\right)Q = 0$$

$$Q = Ce^{-\frac{t}{20}}$$

$$\boldsymbol{Q = 20e^{-\frac{t}{20}}}$$

$$Q(30) = 20e^{-\left(\frac{30}{20}\right)}$$

$$\boldsymbol{Q(30) = 4.46lb}$$

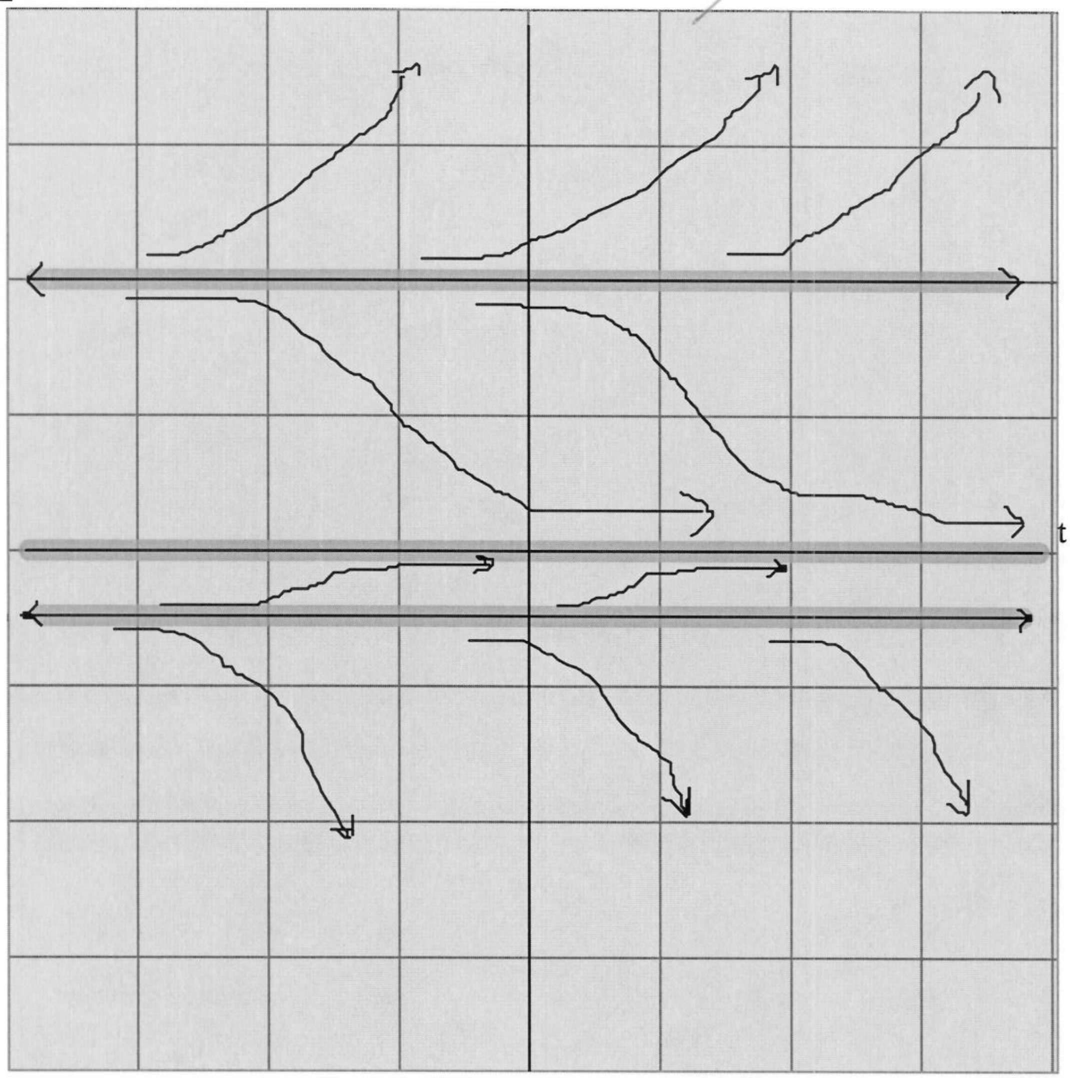
English

(20 *points*) 3 : $\frac{dy}{dt} = y(y-4)(y+1)$

a) *sketch a graph of y versus t for the given function.*
b) *determine the equillibrium points*
c) *state whether equillibrium points are stable* or *unstable.*

why?

Solution :



equillibrium points : y = −1, 0, 4

y = 0 is stable, y = −1, 4 are unstable

(30 *points*) 4 : $\left(\frac{y}{x} + 3x\right)dx + (\ln x - 1)dy = 0$

a) *determine* **if** *exact.*

b) **if** *exact, find the solution.*

Solution :

$$My = \frac{d}{dy}\left(\frac{y}{x} + 3x\right)$$

$$My = \frac{1}{x}$$

$$Nx = \frac{d}{dx}(\ln x - 1)$$

English

$$Nx = \frac{1}{x}$$

$\boldsymbol{My = Nx}$ (***exact***)

$$\int M\,dx + \int N\,dy = 0$$

$$\int\left(\frac{y}{x} + 3x\right)dx + \int(\ln x - 1)dy = 0$$

$$y\ln x + \frac{3x^2}{2} + y\ln x - y = C$$

$$\boldsymbol{2y\ln x + \frac{3x^2}{2} - y = C}$$

Question 1: (50 pts)

You run a very nice little paint shop in Des Moines Iowa. Everyone knows that the favorite color in Des Moines is pink. To please your customers you boast that you have the largest variety of different shades of pink. To accomplish this you start with 250 gallons of white paint in a 300 gallon tank, and then add a solution of white paint mixed with 10 grams red dye per gallon to the tank at a rate of 4 gallons per minute. At the instant the tank fills up (t=0), you begin draining the tank at a rate of 6 gallons per minute so you can bottle the paint to sell. Of course you stop adding paint at the instant the tank is drained. At what instant do you have "tickled pink" coming out of the drain, which is defined as white paint mixed with 6.5 grams of red dye per gallon? Also show the equation that you came up with.

Solution:

change of

The rate of dye in the tank is:
dQ/dt = 4 (gal/min)*10 (gram/gal) – Q(t)(grams)*6(gal/min) / (300-2t)(gal)

the (300-2t) term represents how much paint is in the tank at time t<=0 — explain

look at book to incorporate text into equations (or vice-versa)

in more readable form it is:
dQ/dt = 40 – 3*Q(t)/(150-t)

move it to see the term with P(t)

dQ/dt + 3*Q(t)/(150-t)= 40

so $\mu = e^{\wedge}(\int -3/(150-t)dt) = (t-150)^{\wedge}-3$

Multiply both sides by μ and get
(t-150)^-3 * dQ/dt -4(t-150)^-4Q(t)=40(t-150)^-3

Integrating both sides you get
$(t-150)^{\wedge}\text{-}3 * Q(t) = \int (40 * (t-150)^{\wedge}-3)dt = -20(t-150)^{\wedge}-2 + C$

Solve for Q(t)

Q(t) = -20(t-150) + C(t-150)^3

We can solve that initially, the amount of red dye in grams is 500 in the tank, so using that, we solve for C.

Q(0) =-20*-150 + C(-150)^3 so C = 1/1350

Plugging in C….
Q(t) = (t-150)^3/1350 – 20(t-150)

Now to find out what time we have 6.5 grams of dye per gallon, we divide Q(t) which is grams total at time t, by 300-2t, which is gallons in tank at time t.

Q(t) = ((t-150)^3/1350 – 20(t-150))(300-2t)=6.5

t = 52.79 minutes, or 52 minutes, 47 seconds. ?

Question 2: (30 pts)

Solve: $(\cos x \sin x - x y^2)\, dx + y\,(1-x^2)\, dy = 0$
subject to $y(0) = 2$

antecedents?

Solution:

The equation is exact since dM/dy = -2xy = dN/dx (those d's are dels) good

Now it df/dy = y (1-x^2), then

f (x,y) = y^2 / 2 * (1 – x^2) + h (x). And then

df/dx = -xy^2 + h'(x) = cos x sin x – xy^2.

The last equation implies
h'(x) = cos x sin x
h(x) = ∫ cos x (- sin x dx) = - 1 / 2 (cos x)^2

y^2/2 (1 – x^2) – ½ (cos x) ^2 = c,
y^2 (1-x^2) – (cos x)^2 = c   c = 2c

The initial condition y=2 when x=0 demands that 4(1) – (cos (0))^2 = c or c = 3. Thus a solution of the problem is

y^2 (1-x^2) – (cos x)^2 = 3

Question 3: (40 pts)

-50 ?

Today is Little Timmy's birthday. As an inventor he is concerned about the pollution in Utah Lake. Right now the lake has 1 gm of pollution per gallon. Assume that the lake has a billion gallons of water in it, that 1,000 gallons of fresh water flow in every day from the Provo River, and that 1,000 gallons of polluted lake water escape everyday. Timmy has invented a patented 1 gallon water purification bucket, that when scooped releases clean water back into the lake. Timmy wants the lake to have 0.01gm pollution per gallon instead. How many gallons of water must he purify per day, to have the lake reach its new cleanness goal by his next birthday?

Solution:

dP/dt = - rate_out

dP/dt = (- (1000 + b) gal/day) / (1,000,000,000 gal) * P gm

(dP/dt) / P = (- (1000 + b) gal/day) / (1,000,000,000 gal)

ln |P| = - (1000 + b) / 1,000,000,000 * t + C

P(t) = C e ^ (- (1000 + b) / 1,000,000,000 * t)

P(t) = P0 e ^ (- (1000 + b) / 1,000,000,000 * t)

1/100 P0 = P0 e ^ (- (1000 + b) / 1,000,000,000 * t)

ln (1 / 100) = - (1000 + b) / 1,000,000,000 * t

1,000,000,000 * ln (1 / 100) / t = - (1000 + b)

b = - 1,000,000,000 * ln (1 / 100) / t – 1000

with t = 365, b = 12, 615,904.6 buckets per day

dialogue

Question 4: (30 pts)

A tank with a capacity of 100 gallons initially holds 50 gallons of a brine solution containing 1 lb of salt per gallon. A brine solution of 2 lb of salt per gallon is entering the tank at a rate of 4 gal/min. The well-mixed solution is leaving the tank at a rate of 2 gal/min. How much salt is in the tank at the moment it starts to overflow?

This is the
3rd
"solution"
problem
(among 4)

Solution:

$$\frac{dQ}{dt} = 4 \times 2 - \frac{2 \times Q}{50 + 4t - 2t}$$ – explain

$Q(0) = 50$

?

Then $Q(t) = \frac{4t^2+200t+1250}{t+25}$

When it starts to overflow, $100 = 50 + 4t - 2t$ and we find that $t = 25$.
Hence, $Q(25) = 175$.