Math 334 Midterm I
Fall 2006
sections 001 and 004
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1. Solve the following initial value problem:

$$\frac{dy}{dt} = 1 + t^2 + y^2 + t^2 y^2, \; y(0) = 1. \tag{1.1}$$

10 points

Solution

The equation separates to

$$\frac{dy}{1+y^2} = \left(1+t^2\right)dt. \tag{1.2}$$

Thus, since for any constant C

$$\frac{dy}{1+y^2} = d\arctan y, \text{ and } \left(1+t^2\right)dt = d\left(t + \frac{t^3}{3} + C\right), \tag{1.3}$$

one obtains from (1.2) the integrated equations

$$\arctan y = t + \frac{t^3}{3} + C \Rightarrow y = \tan\left(t + \frac{t^3}{3} + C\right). \tag{1.4}$$

Using the initial data from (1.1) in the first of equations (1.4) we have

$$\frac{\pi}{4} = \arctan 1 = 0 + \frac{0^3}{3} + C = C, \tag{1.5}$$

and then from (1.4) the solution

$$y = \tan\left(t + \frac{t^3}{3} + \frac{\pi}{4}\right). \tag{1.6}$$

2. Prove that the following differential equation is exact and then find an expression of its general solution.

$$\left(3ye^x + x - 1\right)dx + \left(3e^x - y^2 + 4\right)dy = 0. \tag{1.7}$$

10 points

<u>Solution</u>

The equation (1.7) is, by definition, *exact* if the left-hand side is the differential of a (continuously differentiable) function (of two variables x and y, in some region of the x-y plane, etc., etc.), i.e. if there is a function $\psi(x,y)$ such that

$$d\psi(x,y) = \left(3ye^x + x - 1\right)dx + \left(3e^x - y^2 + 4\right)dy. \tag{1.8}$$

But we have, by definition,

$$d\psi(x,y) = \psi_x(x,y)dx + \psi_y(x,y)dy, \tag{1.9}$$

so that the equation (1.8) is the (potentially) over-determined system of equations

$$\psi_x(x,y) = 3ye^x + x - 1, \text{ and } \psi_y(x,y) = 3e^x - y^2 + 4. \tag{1.10}$$

This over-determined pair of equations is consistent (or integrable) if $\left(\psi_x\right)_y = \left(\psi_y\right)_x$, i.e. if

$$\left(3ye^x + x - 1\right)_y = \left(3e^x - y^2 + 4\right)_x. \tag{1.11}$$

(1.11) holds true, so that the equation (1.7) is indeed exact, because either side of (1.11) is $3e^x$.

As for developing the function $\psi(x,y)$, and then (an expression for) the solution of (1.7), one notes that the equations (1.10) demand, respectively, that

$$\psi(x,y) = 3ye^x + \frac{x^2}{2} - x + f(y), \text{ and } \psi(x,y) = 3ye^x - \frac{y^3}{3} + 4y + g(x), \tag{1.12}$$

for some initially rather arbitrary functions $f(y)$, and $g(x)$, but which two statements are not contradictory if, for example, we choose these functions to be $f(y) = -\frac{y^3}{3} + 4y$, and $g(x) = \frac{x^2}{2} - x$. With these choices, either equation in (1.12) gives

$$\psi(x,y) = 3ye^x + \frac{x^2}{2} - x - \frac{y^3}{3} + 4y. \tag{1.13}$$

(1.13) is NOT the general solution to the (exact) differential equation (1.7). It is not even a specific solution. Rather (1.13) defines a “potential (function) for the solution.” Using it one notes that (1.7) can be written as

$$d\psi(x,y) = d\left(3ye^x + \frac{x^2}{2} - x - \frac{y^3}{3} + 4y\right) = 0, \tag{1.14}$$

the general solution to which is obviously

$$3ye^x + \frac{x^2}{2} - x - \frac{y^3}{3} + 4y = C. \tag{1.15}$$

3. Form the Picard iterates $\phi_1(t), \phi_2(t), \ldots, \phi_n(t), \ldots$ for the initial value problem

$$\frac{dy}{dt} = 2(y+1),\ y(0) = 0, \tag{1.16}$$

by first defining $\phi_0(t) \equiv 0$, and then, for $n = 0,1,2,\ldots$, making the recursive definitions

$$\phi_{n+1}(t) := \int_0^t 2(\phi_n(s)+1)ds. \tag{1.17}$$

Specifically, we mean for you to obtain a closed form expression for $\phi_n(t)$ — by, say, calculating a few Picard iterates from (1.17), enough of these first ones to see (and write down) the pattern for general/arbitrary n. Now let

$$\phi(t) := \lim_{n\to\infty} \phi_n(t) \tag{1.18}$$

(formally—do not express the sum in closed form) and show that, at least formally (don’t worry about convergence of the infinite series stemming from definition (1.18)), that $\phi(t)$ solves the initial value problem (1.16) (don’t try to show that it solves the integral equation related to (1.17)).

15 points

<u>Solution</u>

(1.17), together with $\phi_0(t) \equiv 0$, gives

$$\phi_1(t) := \int_0^t 2(\phi_0(s)+1)ds = \frac{2^1}{1}s^1\Big|_0^t = \frac{2^1}{1}t^1,$$

$$\phi_2(t) := \int_0^t 2(\phi_1(s)+1)ds = \frac{2^2}{1\cdot 2}s^2 + \frac{2^1}{1}s^1\Big|_0^t = \frac{2^1}{1}t^1 + \frac{2^2}{1\cdot 2}t^2, \quad (1.19)$$

$$\phi_3(t) := \int_0^t 2(\phi_2(s)+1)ds = \frac{2^3}{1\cdot 2\cdot 3}s^3 + \frac{2^2}{1\cdot 2}s^2 + \frac{2^1}{1}s^1\Big|_0^t = \frac{2^1}{1}t^1 + \frac{2^2}{1\cdot 2}t^2 + \frac{2^3}{1\cdot 2\cdot 3}t^3,$$

so that the general, or n^{th}, iterate is evidently

$$\phi_n(t) = \sum_{k=1}^{n} \frac{2^k}{k!}t^k. \quad (1.20)$$

A candidate solution of the initial value problem (1.16) is thus

$$\phi(t) := \lim_{n\to\infty}\phi_n(t) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{2^k}{k!}t^k =: \sum_{k=1}^{\infty}\frac{2^k}{k!}t^k. \quad (1.21)$$

Assuming convergence of (1.21), we certainly have $\phi(0) = \sum_{k=1}^{\infty}\frac{2^k}{k!}0^k = 0,$ and, importantly,

$$\begin{aligned}\frac{d\phi}{dt} - 2(\phi+1) &= \frac{d}{dt}\sum_{k=1}^{\infty}\frac{2^k}{k!}t^k - 2(\sum_{k=1}^{\infty}\frac{2^k}{k!}t^k + 1) = \sum_{k=1}^{\infty}\frac{2^k}{(k-1)!}t^{k-1} - 2\sum_{k=1}^{\infty}\frac{2^k}{k!}t^k - 2 \\ &= \sum_{k=0}^{\infty}\frac{2^{k+1}}{k!}t^k - 2\sum_{k=1}^{\infty}\frac{2^k}{k!}t^k - 2 = \frac{2^{0+1}}{0!}t^0 + 2\sum_{k=1}^{\infty}\frac{2^k}{k!}t^k - 2\sum_{k=1}^{\infty}\frac{2^k}{k!}t^k - 2 \quad (1.22) \\ &= 2 + 2\sum_{k=1}^{\infty}\frac{2^k}{k!}t^k - 2\sum_{k=1}^{\infty}\frac{2^k}{k!}t^k - 2 = 0.\end{aligned}$$

(formally—we do not check convergence).

4. Use the existence and uniqueness theorem to find x_0's for which the (linear) initial value problem

$$\frac{dy}{dx} = 3x^{-1}y,\ y(x_0) = y_0 \quad (1.23)$$

certainly has a unique solution $y = y(x)$ persisting for an open interval (of x's) containing x_0. Now find the general solution of the differential equation. Use this general solution to investigate the possibility of none or more than one solution passing through the initial point (x_0, y_0). Your investigation should not contradict the theorem, but should render additional insights not mentioned by the theorem.

13 points

Solution

The “right-hand side of the differential equation” is the function $3x^{-1}y$, and its first partial derivative with respect to the dependent variable is the function $3x^{-1}$. The pair of functions $3x^{-1}y$ and $3x^{-1}$ are continuous in any region of the x-y plane not containing the line $x=0$. Hence the theorem guarantees that for each $x_0 \neq 0$, the initial value problem (1.23) has a unique solution $y = y(x)$ persisting for an open interval (of x's) containing x_0. The theorem makes no claim about existence or uniqueness of solutions of (1.23) when $x_0 = 0$. On the other hand, separation of the differential equation of (1.23) gives the general solution

$$y = Cx^3 \tag{1.24}$$

which certainly gives rise to a unique solution $y = y(x)$ of the initial value problem (1.23) provided $x_0 \neq 0$ (here we proceed naively and assume the general solution gives all solutions, which is not true in general). Indeed, in such case, such a solution persists for all real x's, not just for those avoiding $x=0$ (but if we cross $x=0$, uniquess is a problem—see next case and consider the possibility of smoothly gluing together piecewise versions of (1.24)). On the other hand, if $x_0 = 0$, then (1.24) says that, if there is a solution $y = y(x)$ to the initial value problem, then $y_0 = y(0) = 0$. But in such case there are many solutions—one for each choice of C. Finally, by way of emphasis, if $x_0 = 0$ yet we choose in initial value problem (1.23) $y_0 = y(0) \neq 0$, then (1.24) suggests that the initial value problem has no solutions.

5. After a time $t=0$, a solution of constant concentration C_{in} grams solute per liter solvent enters a (perfect) stirring tank at a constant rate of R_{in} liters per minute. The well-stirred mixture exits the tank at a constant rate of R_{out} liters per minute. Suppose the solute takes no volume in solution. If the tank contains V_0 liters of fluid at a time $t=0$, write down a (self-contained) differential equation for the time evolution of the grams of solute $Q(t)$ accumulated in the tank at time t, one that is valid for as long as the tank is not empty or overflowing. Then, assuming there are Q_0 grams of solute in the tank at $t=0$, show that the formula

$$Q(t) = \left(\frac{V_0}{V_0 + t(R_{in} - R_{out})} \right)^{\frac{R_{out}}{R_{in}-R_{out}}} Q_0 + C_{in} \left\{ V_0 + t(R_{in} - R_{out}) - (V_0)^{\frac{R_{in}}{R_{in}-R_{out}}} \left(V_0 + t(R_{in} - R_{out})\right)^{\frac{-R_{out}}{R_{in}-R_{out}}} \right\}$$
(1.25)

solves the associated initial value problem for the relevant period of time. (I'm not asking you to solve the initial value problem, which would be difficult, but rather am asking you only to show that (1.25) solves the associated differential equation and renders the relevant initial value.)

15 points

Solution

By stoichiometric or "chain-rule" reasoning, one has

$$\begin{aligned} \frac{dQ}{dt} &= \left(\frac{dQ}{dt}\right)_{total} = \left(\frac{dQ}{dt}\right)_{in} - \left(\frac{dQ}{dt}\right)_{out} = \left(\frac{dQ}{dV}\right)_{in}\left(\frac{dV}{dt}\right)_{in} - \left(\frac{dQ}{dV}\right)_{out}\left(\frac{dV}{dt}\right)_{out} \\ &= C_{in}R_{in} - C_{out}R_{out} = R_{in}C_{in} - R_{out}\frac{Q}{V}, \end{aligned} \tag{1.26}$$

where the fluid tank volume $V = V(t)$ is specified by

$$\frac{dV}{dt} = R_{in} - R_{out}, \ V(0) = V_0, \tag{1.27}$$

the latter (trivial) initial value problem having the unique solution

$$V = V_0 + t(R_{in} - R_{out}). \tag{1.28}$$

Thus the required, "self-contained" differential equation is

$$\frac{dQ}{dt} = R_{in}C_{in} - R_{out}\frac{Q}{V_0 + t(R_{in} - R_{out})}. \tag{1.29}$$

In an effort to show that (1.25) satisfies the relevant initial value problem, we first note that (1.25) yields that

$$Q(0) = \left(\frac{V_0}{V_0 + 0(R_{in} - R_{out})}\right)^{\frac{R_{out}}{R_{in}-R_{out}}} Q_0 + C_{in}\left\{V_0 + 0(R_{in} - R_{out}) - (V_0)^{\frac{R_{in}}{R_{in}-R_{out}}}\left(V_0 + 0(R_{in} - R_{out})\right)^{\frac{-R_{out}}{R_{in}-R_{out}}}\right\}$$
$$= (1)^{\frac{R_{out}}{R_{in}-R_{out}}} Q_0 + C_{in}\left\{V_0 - (V_0)^{\frac{R_{in}}{R_{in}-R_{out}}}(V_0)^{\frac{-R_{out}}{R_{in}-R_{out}}}\right\} = Q_0 + C_{in}\left\{V_0 - (V_0)^{\frac{R_{in}-R_{out}}{R_{in}-R_{out}}}\right\}$$
$$= Q_0 + C_{in}\left\{V_0 - (V_0)^1\right\} = Q_0, \tag{1.30}$$

so that formula (1.25) gives the correct initial data. Differentiating (1.25), and using it in the result, we get

$$\frac{dQ}{dt} = -\frac{R_{out}}{R_{in} - R_{out}}(R_{in} - R_{out})\frac{V_0}{\left(V_0 + t(R_{in} - R_{out})\right)^2}\left(\frac{V_0}{V_0 + t(R_{in} - R_{out})}\right)^{\frac{R_{out}}{R_{in}-R_{out}}-1} Q_0$$
$$+C_{in}\left\{R_{in} - R_{out} - \frac{-R_{out}}{R_{in} - R_{out}}(R_{in} - R_{out})(V_0)^{\frac{R_{in}}{R_{in}-R_{out}}}\left(V_0 + t(R_{in} - R_{out})\right)^{\frac{-R_{out}}{R_{in}-R_{out}}-1}\right\}$$
$$= -\frac{R_{out}\left(\frac{V_0}{V_0 + t(R_{in} - R_{out})}\right)^{\frac{R_{out}}{R_{in}-R_{out}}} Q_0}{V_0 + t(R_{in} - R_{out})} + C_{in}\left\{R_{in} - R_{out} + R_{out}(V_0)^{\frac{R_{in}}{R_{in}-R_{out}}}\left(V_0 + t(R_{in} - R_{out})\right)^{\frac{-R_{in}}{R_{in}-R_{out}}}\right\}$$
$$= R_{in}C_{in} - \frac{R_{out}}{V_0 + t(R_{in} - R_{out})}\left\{\begin{array}{l}\left(\frac{V_0}{V_0 + t(R_{in} - R_{out})}\right)^{\frac{R_{out}}{R_{in}-R_{out}}} Q_0 \\ +C_{in}\left\{V_0 + t(R_{in} - R_{out}) - (V_0)^{\frac{R_{in}}{R_{in}-R_{out}}}\left(V_0 + t(R_{in} - R_{out})\right)^{\frac{-R_{out}}{R_{in}-R_{out}}}\right\}\end{array}\right\}$$
$$= R_{in}C_{in} - \frac{R_{out}}{V_0 + t(R_{in} - R_{out})}Q, \tag{1.31}$$

which is the relevant differential equation (1.29).

6. Show that the following differential equation is "homogeneous", meaning that the right-hand side can be expressed as a function of only the ratio y/x:

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}. \tag{1.32}$$

Now solve the differential equation (1.32) by introducing a new variable v to replace y via the relation

$$y = vx. \tag{1.33}$$

(The resulting differential equation in v and x will be separable.)

10 points

Solution

The equation can be written as

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} = 1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 =: F(y/x) \tag{1.34}$$

Differentiating (1.33) gives, using (1.34),

$$F(v) = F(y/x) = \frac{dy}{dx} = \frac{dv}{dx}x + v$$
$$\Rightarrow$$
$$\frac{dv}{dx}x = F(v) - v$$
$$\Rightarrow$$
$$d\ln|x| = \frac{dx}{x} = \frac{dv}{F(v) - v} = \frac{dv}{1+v^2} = d\arctan v, \tag{1.35}$$
$$\Rightarrow$$
$$y/x = v = \tan(\ln|x| + c)$$
$$\Rightarrow$$
$$y = x\tan(\ln|x| + c).$$

7. Clearly state, and solve, an initial value problem that you wish I had put on the test.

No more than 15 points

8. Carefully state the (nonlinear) existence and uniqueness theorem for a single first order ODE.

15 points

Solution

Consider the initial value problem

$$y'(t) = f(t, y), y(t_0) = y_0. \qquad (1.36)$$

Suppose $f(t, y)$ and $f_y(t, y)$ are both continuous in an open rectangle $\left(t_{-1}, t_{+1}\right) \times \left(y_{-1}, y_{+1}\right)$ containing the point $\left(t_0, y_0\right)$. Then there exists an $h > 0$ such that (1.36) has a unique, continuously differentiable solution $y = \phi(t)$ persisting over the t interval $\left(t_0 - h, t_0 + h\right)$ (potentially much smaller than the interval $\left(t_{-1}, t_{+1}\right)$).

9. Carefully state the linear existence and uniqueness theorem for a single first order ODE. Explain in general terms how it is proven.

15 points

Solution

Consider the initial value problem

$$y'(t) = p(t)y + q(t), y(t_0) = y_0. \qquad (1.37)$$

Suppose $p(t)$ and $q(t)$ are both continuous in an open interval $\left(t_{-1}, t_{+1}\right)$ containing the point t_0. Then (1.37) has a unique, continuously differentiable solution $y = \phi(t)$ persisting over the t interval $\left(t_{-1}, t_{+1}\right)$.

10. Clearly state, and solve, any type of problem from our text that you wish I had put on the test.

No more than 15 points