4

Find the general solution of the differential equation

$$y'' + 3y' + 3y = 4e^{2t} + 2\sin t \qquad (1)$$

Solution

The first step is to solve for the solution of the homogeneous equation by setting the right side of the differential equation equal to zero and solve for the roots of the characteristic equation. From the quadratic equation, we find that the roots, r_1 and r_2, are equal to:

$$r_1 = -\frac{3}{2} + i\frac{\sqrt{3}}{2} \quad r_2 = -\frac{3}{2} - i\frac{\sqrt{3}}{2} \qquad (2)$$

So We know now that the solution to the differential equation (1) will be of the form

$$y = c_1 e^{(-3/2)t}\sin\frac{\sqrt{3}}{2}t + c_2 e^{(-3/2)t}\cos\frac{\sqrt{3}}{2}t + Y(t) \qquad (3)$$

And that the particular solution Y will be of the form

$$Y(t) = Ae^{2t} + B\sin t + C\cos t \qquad (4)$$

Taking the first and second derivative of the particular solution gives,

$$Y'(t) = 2Ae^{2t} + B\cos t - C\sin t$$
$$Y''(t) = 4Ae^{2t} - B\sin t - C\cos t \qquad (5)$$

Substituting Y(t), Y'(t), and Y''(t) for y'', y' and y in the differential equation (1) and simplifying gives

$$(13A)e^{2t} + (2B - 3C)\sin t + (3B + 2C)\cos t = 4e^{2t} + 2\sin t \qquad (6)$$

Setting the terms equal of the respective coefficients on both sides of the equations gives

$$13A = 4 \qquad (2B - 3C) = 2 \qquad (3B + 2C) = 0 \qquad (7)$$

Solving the system of equations for A, B, and C gives

$$A = \frac{4}{13} \qquad B = \frac{4}{13} \qquad C = -\frac{6}{13} \qquad (8)$$

Substituting (8) into the particular solution (4), and substituting (4) into the differential equation (1) gives the general solution to the differential equation (1)

$$y = c_1 e^{(-3/2)t}\sin\frac{\sqrt{3}}{2}t + c_2 e^{(-3/2)t}\cos\frac{\sqrt{3}}{2}t + \frac{4}{13}e^{2t} + \frac{4}{13}\sin t - \frac{6}{13}\cos t$$

Find the Solution to the initial value problem.

$4(x-1)^2 - 2(x-1)y' - 4 = 0 \quad y(-1) = 4 \qquad y'(-1) = 8$

<u>Solution</u>

This can be solved by substituting : $y = x^r$. We will ignore the x-1 for now, by replacing it with x_0.

This gives us:

$x^r[4r(r-1) - 2r - 4] = 0$

Expanding this equation we get:

$x^r[4r^2 - 4r - 2r - 4] = 0$

Combining like terms we get:

$x^r[4r^2 - 6r - 4] = 0$

We can now solve for the roots of this equation using the quadratic formula:

$$\frac{6 \pm \sqrt{36-(4)(4)(-4)}}{(2)(4)} \rightarrow \frac{6 \pm 10}{8}$$

Which gives us the roots: $r_1 = 2, r_2 = -\frac{1}{2}$

Replacing x_0 with $|x-1|$, the general solution becomes: $y = c_1|x-1|^2 + c_2|x-1|^{-1/2}$ which we know form the theorem for general solutions to Euler equations. Now we will solve for the initial conditions to find c_1 and c_2.

First: $y(-1) = c_1|-1-1|^2 + c_c|-1-1|^{-1/2} = 4c_1 + \frac{1}{\sqrt{2}}c_2 = 4$. Then take the derivative to y to have two equations to solve for c_1 and c_2.

$y' = 2c_1|x-1| + \frac{1}{2}c_2|x-1|^{-3/2}$ Note $|x-1|^{-1/2} = (1-x)^{-1/2}$ for $x \leq 1$.

$y'(-1) = 2c_1|-1-1| - \frac{1}{2}c_2|-1-1|^{-3/2} = 8$

Solving these two equations simultaneously gives us: $c_1 = \frac{2}{3}, c_2 = 1.88$. The final solution becomes: $y = \frac{2}{3}|x-1|^2 + 1.88|x-1|^{-1/2}$.

A simple series circuit with capacitor $c = 1*10^{-6}$ farads, and an inductor $L = 1$ Henry has an initial charge of 10^{-6} Coulombs and no initial current. Using this information find the charge Q on the capacitor at time t. Same q?

Solution

The voltage drop across a capacitor is $\frac{q}{c}$. The voltage drop across an inductor is $L\frac{dQ}{dt}$ I

where $I = \frac{dQ}{dt}$. Kircoff's Law relates these two the following way: $L\frac{dI}{dt} + \frac{1}{c}q = 0$ or Same Q?

substituting $I = \frac{dQ}{dt}$, $LQ'' + \frac{1}{c}q = 0$. Inserting $c = 1*10^{-6}$ and $L = 1$ we have

$Q'' + 10^{6}q = 0$ with a general solution of $\mu = A\cos(10^{3}t) + B\sin(10^{3}t)$. Our initial conditions are: Initial charge of 10^{-6} Coulombs and no initial current. Since $I = \frac{dQ}{dt}$ we can safely say that $\mu(0) = 10^{-6}$ and $\mu'(0) = 0$. Solving for our initial conditions in the general solution: $10^{-6} = A\cos(10^{3}(0)) + B\sin(10^{3}(0))$ which implies that $10^{-6} = A$.

Now solving for B:

$\mu' = -10^{3}A\sin(10^{3}t) + 10^{3}B\cos(10^{3}t)$

$0 = -10^{3}A\sin(10^{3}(0)) + 10^{3}B\cos(10^{3}(0))$

$0 = B$

Substituting A and B back into the general solution gives us a particular solution of:

$\mu = 10^{-6}\cos(10^{3}t)$ for time t.

(9)

(35 points) 1) suppose that a mass weighing 20lb stretches a spring 4 in air. If the mass is displaced an additional 6 in. and is then set in motion with an initial upward velocity of 4 in per second, determine the position of the mass at any given time t.

Solution :

The spring constant $k = \frac{20lb}{4\ in} = \frac{20lb}{.3ft} = \frac{60ft}{lb}$

The mass $m = \frac{20lb}{32\frac{ft}{sec^2}} = \frac{5}{8}\frac{lb \cdot sec^2}{ft}$

(1)

since the mass is in air, it may be reasonably assumed that there is no damping constant (γ=0).

Therefore, according to Newton's law $(mu'' + \gamma u' + ku = F(t))$, *the equation of motion reduces to:*

$\frac{5}{8}u'' + 60u = 0$ *or*

$u'' + 96u = 0$

The equation for the general solution is $u = A\cos\omega t + B\sin\omega t$, *where* $\omega^2 = \frac{k}{m}$

Therefore, $\omega = \sqrt{\frac{60 \cdot 8}{5}} = 4\sqrt{6}$

so, the general solution is :

$$u = A\cos(4\sqrt{6}\,t) + B\sin(4\sqrt{6}\,t)$$

the solution satisfying the initial conditions $u(0) = \frac{1}{2}ft$ *and* $u'(0) = \frac{-1}{3}\frac{ft}{sec}$, *substituting into the above equation and taking the derivative :*

$$\frac{1}{2} = A\cos(4\sqrt{6}\,(0)) + B\sin(4\sqrt{6}\,(0)) \Rightarrow A = \frac{1}{2}$$

$$-\frac{1}{3} = -(4\sqrt{6})A\sin(4\sqrt{6}\,(0)) + (4\sqrt{6})B\cos(4\sqrt{6}\,(0)) \Rightarrow B = \frac{-1}{(3)4\sqrt{6}} = -\frac{\sqrt{6}}{72}$$

Hence, at any given time (t) *the position of the weight may be expressed as :*

$$u = \frac{1}{2}\cos(4\sqrt{6}\,t) - \frac{\sqrt{6}}{72}\sin(4\sqrt{6}\,t)$$

(35 pts)

Find all singular points of the following equation, and determine whether each is regular or irregular.

$$2x^2(3 - x)^2y'' + (4/x)y' + 2y = 0$$

Solution:

The singular points are the zeros of the first term's coefficient, $2x^2(3 - x)^2$) or possibly singularities of other coefficients.
The zeros are $x = 0$ and $x = 3$

To determine whether $x = 0$ is regular or irregular, compute, with $x_o = 0$:

$$\lim_{x \to x_0} (x - x_o)\, Q(x)/P(x) = \lim_{x \to 0} (x)(4/x)/(2x^2(3 - x)^2)$$

$$= \lim_{x \to 0} 4/(2x^2(3 - x)^2) = \infty$$

$$\lim_{x \to x_0} (x - x_o)^2\, R(x)/P(x) = \lim_{x \to 0} (x)^2 (2)/(2x^2(3 - x)^2)$$

$$= \lim_{x \to 0} 1/(3 - x^2) = \mathbf{1/3}$$

Since only one limit is finite, $x = 0$ is irregular.

To determine whether $x = 3$ is regular or irregular, compute, with $x_o = 3$:

$$\lim_{x \to x_0} (x - x_o)\, Q(x)/P(x) = \lim_{x \to 3} (x - 3)(4/x)/(2x^2(3 - x)^2)$$

$$= \lim_{x \to 3} 2/(x^3(3 - x)) = \infty$$

$$\lim_{x \to x_0} (x - x_o)^2\, R(x)/P(x) = \lim_{x \to 3} (x - 3)^2 (2)/(2x^2(3 - x)^2)$$

$$= \lim_{x \to 3} 1/x^2 = \mathbf{1/9}$$

Since only one limit is finite, $x = 3$ is irregular.

Thus, $x = 0$ and $x = 3$ are both irregular, singular points.

QUESTION (33 pts)
For the differential equation

$$ay'' + by' + cy = 0,$$

determine an equation for the relationship of the (non-negative) constants a, b, and c to satisfy each of the four cases of solutions below.

I.) $y(t) = C_1e^{rt} + C_2e^{st}$

II.) $y(t) = C_1e^{rt} + tC_2e^{st}$

III.) $y(t) = C_1e^{\lambda t}\cos(\mu t) + tC_2e^{\lambda t}\sin(\mu t)$

IV.) $y(t) = C_1\cos(\mu t) + C_2\sin(\mu t)$

SOLUTION
For each case, the relationship between a, b, and c must be determined for the parameters r and s (in cases I and II) and λ and μ (in cases III and IV).

I.) By utilizing the ansatz of $y = e^{rt}$, the root of the characteristic equation, r and s can be determined in the form of

$$r = \frac{-b+\sqrt{b^2-4ac}}{2a}, s = \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

For the solution to be of the form $y(t) = C_1e^{rt} + C_2e^{st}$, r and s must be real. Therefore the square root is a positive number. So

$$b^2 - 4ac > 0.$$

II.) In this case r = s and there is a repeated root. The radical is equal to zero. So

$$b^2 - 4ac = 0.$$

III.) In this case the roots are complex and of the form $\lambda \pm i\mu$. For this to be true,

$$b^2 - 4ac < 0.$$

In addition, $\lambda = \frac{-b}{2a}$ and $\mu = \frac{\sqrt{b^2-4ac}}{2a}$.

IV.) This case the same as case III, with the exception that $\lambda = b = 0$.

(30 points) 4) Determine the lower bound for the radius of convergence of the series solution about the point $X_0=1$ for the differential equation:

$$(x^2 + 4)y'' + xy' + (x^2)y = 0 \qquad (1)$$

Solution:

singularities

Equation (1) has ~~vertical asymptotes~~ at the zeros of $P(x)=x^2 + 4$.

$P(x)=0$ at $x=2i$ and $x=-2i$.

Therefore, the lower bound for the radius of convergence of the series solution about X_0 is the minimum of the distance in the complex plain between 2i and 1 and the distance between -2i and 1.

The distance between 2i and 1 is the square root of (2^2 + 1^2)= square root of 5.

The distance between -2i and 1 is the square root of (-2)^2 + 1^2 = square root of 5.

Thus, the lower bound for the radius of convergence of the series solution about X_0 if square root of 5.

Find the specific solution to the following differential equation based on the included initial data.

$$y'' + 4y = t^2 + 5e^t \text{ , } y(0) = 0, y'(0) = 0$$

First we find ~~the equation of the~~ homogenous version of the differential equation.

$$y'' + 4y = 0$$

We must first find the roots of the characteristic equation.

$$r^2+4=0$$

$$r=\pm\sqrt{4}=\pm 2i$$

This implies that the solution to homogenous equation is:

$$y=c_1\cos 2t+c_2\sin 2t$$

Next we must consider finding the particular solution of the differential equation. For this we will use the method of undetermined coefficients. We will split up the problem into two smaller problems which we will consider separately:

$$y_1'' + 4y_1 = t^2 \text{ , } y_2'' + 4y_2 = 5e^t \quad \text{so that } y = y_1 + y_2$$

We assume the solution of the first differential equation is of the form

$$Y(t)=At^2+Bt+C$$

Differentiate Y(t) twice and then put Y(t) and Y"(t) into the differential equation.

$$Y''(t) = 2A$$

$$2A+4At^2+4Bt+4C=t^2$$

Then we construct a system of equations and solve for A,B,C.

$4A=1 \quad A=1/4$

$4B=0 \quad B=0$

$2A+4C=0 \quad C=-1/8$

The solution is:

$$Y(t) = \frac{t^2}{4} - \frac{1}{8}$$

We will follow a similar approach for the other half of our differential equation.

$$y'' + 4y = 5e^t$$

We will assume our solution is of the form:

$$Y(t) = Fe^t$$

We will twice differentiate it and put Y(t) and Y"(t) into our differential equation and solve for A.

$$Y''(t) = Fe^t$$

$$Fe^t + 4Fe^t = 5e^t$$

$$5F = 5$$

$$F = 1$$

Therefore, our solution is

$$Y(t) = e^t$$

The particular solution of our original differential equation is the sum of the two particular solutions we just found for the "halves" of our differential equation.

$$Y(t) = \frac{t^2}{4} - \frac{1}{8} + e^t$$

Therefore the general solution to the non homogenous differential equation is:

$$y = c_1 \cos 2t + c_2 \sin 2t + \frac{t^2}{4} + \frac{1}{8} + e^t$$

Next, we put in our initial data, differentiate where appropriate, and solve for c_1 and c_2.

$$y(0) = 0$$

$$0 = c_1 + 0 - \frac{1}{8} - 1$$

$$c_1 = \frac{7}{8}$$

$$y'(0) = 0$$

$$y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t + \frac{t}{2} + e^t$$

$$0 = 0 + 2c_2 + 0 + 1$$

$$c_2 = -\frac{1}{2}$$

thanks.

At the end of our exciting mathematical journey we have a specific solution to the differential equation.

$$y = \frac{7}{8}\cos 2t - \frac{1}{2}\sin 2t + \frac{t^2}{4} + \frac{1}{8} + e^t$$

(18 points 1 for each section of the solution)

3.7 Variation of Parameters Extra Credit Problem

Introduction

Theorem 3.7.1 on page 189 explains a few details about nonhomogeneous equations. It states that a particular solution exists, and is of a certain form given a few criteria are met.

1. Functions p, q, and g are continuous on an open interval I
2. Functions y(1) and y(2) are linearly independent solutions of the homogeneous equation y + py + qy = g

This particular solution is of the form

$$Y(t) = -\,y_1(t)\int_0^{\cancel{x}\;t} y_2\, g\,/\,[W(y_1, y_2)](\tau)d\tau + y_2(t)\int_0^{\cancel{x}\;t} y_1\, g\,/\,[W(y_1, y_2)](\tau)d\tau \qquad \text{2 pts}$$

Lets look at an example of this theorem.

The nonhomogeneous is $y'' + 25y = 25\sec^2(5x)$

The r values can be calculated and are r = +5i and -5i

The y_1 and y_2 parts are $y_1 = \sin 5x$ and $y_2 = \cos 5x$ 5 pts

Now, we can calculated $W(y_1, y_2)$

$$W(y_1, y_2) = \begin{matrix} y_1 & y_2 \\ y_1' & y_2' \end{matrix} = \det \begin{matrix} \sin 5x & \cos 5x \\ 5\cos 5x & -5\sin 5x \end{matrix} = -5\sin^2 5x - 5\cos^2 5x = -5 \qquad \text{5 pts}$$

Now we can find the particualer solution with the information we have.

$$Y(t) = -\,\sin 5x \int_0^x \cos 5x\,[25\sec^2(5x)]\;/\,{-5} \quad + \quad \cos 5x \int_0^x \sin 5x\,[25\sec^2(5x)]\;/\,{-5}$$

$$= \sin 5x \int_0^x 5\sec(5x) \quad - \quad + \quad \cos 5x \int_0^x 5\tan 5x\,\sec(5x)$$

$$= \sin 5x\,[\ln \sec 5x + \tan 5x] \;-\; \cos 5x\,[\sec 5x]$$

$$Y(t) = \sin 5x\,[\ln \sec 5x + \tan 5x] \;-\; 1 \qquad \text{5 pts}$$

Finally, we can formulate the solution,

$$y = c_1 \sin 5x + c_2 \cos 5x + \sin 5x\,[\ln \sec 5x + \tan 5x] \;-\; 1 \qquad \text{3pts}$$

Solve the differential equation by means of a power series about the given point x_0. Find the recurrence relation and the first four terms in each of the two linearly independent solutions.

ky'' + xy' + y =0 with x_0=0 and k is an arbitrary constant.

Solution:

We know that y=$\sum_{n=0}^{\infty} a_n x^n$.

By differentiating, we have

y'=$\sum_{n=1}^{\infty} na_n x^{n-1}$ and y''=$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

Since y= $\sum_{n=0}^{\infty} a_n x^n$ = $\sum_{n=-\infty}^{\infty} a_n x^n$ when a_n=0 for every n $\geq$ 0, we can substitute y", y', and y in the equation so that we now have

$$k \sum_{n=-\infty}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=-\infty}^{\infty} na_n x^{n-1} + \sum_{n=-\infty}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} kn(n-1)a_n x^{n-2} + \sum_{n=-\infty}^{\infty} na_n x^n + \sum_{n=-\infty}^{\infty} a_n x^n = 0.$$

for the first term, let n be replaced with n+2 so that we can have the same generic term x^n.

$$\sum_{n=-\infty}^{\infty} k(n+2)(n+1)a_{n+2} x^n + \sum_{n=-\infty}^{\infty} na_n x^n + \sum_{n=-\infty}^{\infty} a_n x^n = 0.$$

Now these summations can be combined to form

$$\sum_{n=-\infty}^{\infty} [k(n+2)(n+1)a_{n+2} + na_n + a_n] x^n = 0.$$

For this equation to be satisfied for all x, the coefficient of each power of x must be zero, therefore

$$k(n+2)(n+1)a_{n+2} + na_n + a_n = 0$$

and solving for a_{n+2} yields

$$a_{n+2} = \frac{-na_n - a_n}{k(n+2)(n+1)} = \frac{-a_n(n+1)}{k(n+2)(n+1)} = \frac{-a_n}{k(n+2)}.$$

So the recurrence relation is

$$a_{n+2} = \frac{-a_n}{k(n+2)}.$$

Now solve for the first few coefficients of a_n.

Let n=0

$$a_2 = \frac{-a_0}{2k}$$

Let n=1

$$a_3 = \frac{-a_1}{3k}$$

Let n=2

$$a_4 = \frac{-a_2}{4k} \Rightarrow \frac{a_0}{4k \cdot 2k}$$

Let n=3

$$a_5 = \frac{-a_3}{5k} \Rightarrow \frac{a_1}{5k \cdot 3k}$$

Let n=4

$$a_6 = \frac{-a_4}{6k} \Rightarrow \frac{-a_0}{6k \cdot 4k \cdot 2k}$$

Let n=5

$$a_7 = \frac{-a_5}{7k} \Rightarrow \frac{-a_1}{7k \cdot 5k \cdot 3k}$$

We can now solve for the linearly independent solutions.

$$y_1 = 1 - \frac{x^2}{2k} + \frac{x^4}{4k \cdot 2k} - \frac{x^6}{6k \cdot 4k \cdot 2k} + \ldots$$

and

$$y_2 = x - \frac{x^3}{3k} + \frac{x^5}{5k \cdot 3k} - \frac{x^7}{7k \cdot 5k \cdot 3k} + \ldots$$

(37 points)

Find the singular point and determine the general solution of the differential equation that is valid in any interval not including the singular point.

$$3x^3 y''+15x^2 y'+12xy=0$$

Solution:

We recognize this equation as an Euler Equation and divide by 3x to get in the desired form.

$$x^2 y''+5xy'+4y=0$$

This equation has a singular point at $x=0$.

This has a general solution $y=x^r$ so we plug this in to the above equation

$$x^2 r(r-1)x^{(r-2)}+5xrx^{(r-1)}+4x^r=0$$

which simplifies to

$$x^r(r(r-1)+5r+4)=0.$$

Because $x=0$ the only way this equation can be zero is if the coefficients of x are zero, so we solve for r by rearranging terms and using the quadratic formula.

$$r(r-1)+5r+4=r^2+4r+4=0$$

then $r_1, r_2=(-4\pm\sqrt{4^2-4*4})/2$.

Since these roots are real and equal we know the equation is in the form

$$y=(c_1+c_2\ln(x))(|x|)^r$$

Therefore the general solution is

$$y=(c_1+c_2\ln(x))(|x|)^{-2}.$$

(37 points)

35 points

(10)

Solve the following initial ~~condition~~ value problem, describe the solution's behavior and find the value of t where the solution is at it's minimum for -1<t<1, simplifying t as far as possible (use exact notation). (12pts)

$$20\ y'' + 80\ y' + 100 = 0$$

$$y'(0) = 12 \qquad y''(0) = 2$$

Solution:

First, simplify the equation by diving out the 20:

$$y'' + 4y' + 5 = 0$$

Then find the characteristic equation:

$$r^2 + 4r + 5$$

Next find the roots of the characteristic equation using the quadratic formula;

$$r = \frac{-4 \pm \sqrt{4^2 - 2\cdot 1\cdot 5}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

So, $\lambda = -2$, and $\mu = 1$

From this the general solution is:

$$y = c_1 \cos(t)\cdot e^{-2t} + c_2 \sin(t)\cdot e^{-2t}$$

Taking the derivative to allow solving for initial data:

$$y' = -2c_1 \cos(t)\cdot e^{-2t} - c_1 \sin(t)\cdot e^{-2t} + c_2 \cos(t)\cdot e^{-2t} - 2c_2 \sin(t)\cdot e^{-2t}$$

and

$$y'' = 3c_1 \cos(t)\cdot e^{-2t} + 4c_1 \sin(t)\cdot e^{-2t} - 4c_2 \cos(t)\cdot e^{-2t} + 3c_2 \sin(t)\cdot e^{-2t}$$

Now inserting initial data

$$y''(0) = 12 = 3c_1 - 4c_2 \qquad y'(0) = 2 = -2c_1 + c_2$$

$$c_1 = -4 \qquad c_2 = -6$$

This results in the equation: $\boxed{y = -4\cos(t)\cdot e^{-2t} - 6\cos(t)\cdot e^{-2t}}$ (6pts)

The behavior of this equation is a decaying oscillation as $t \to \infty$ (4pts)

To find the minimum value of this solution in the given range, the lowest value of t possible where $y' = 0$ will give the desired solution. By inspection, this should be close to t=0.

$$0 = y'(t) = 2\cos(t)\cdot e^{-2t} + 16\sin(t)\cdot e^{-2t}$$

Dividing out the e^{-2t} and separating trigonometric equations:

$$-2\cos(t) = -16\sin(t)$$

Solving for t:

$$\boxed{-\frac{2}{16} = \tan(t) \qquad t = \tan^{-1}\left(\frac{-1}{8}\right)}$$ $= -\tan^{-1}\frac{1}{8}$ (2pts)

35 points

Solve the I.V.P using the method of undetermined coefficients.

$y''+y'-2y = 2t \qquad y(0)=0\,,\ y'(0)=1$

Solution

First we will solve the homogeneous form of the equation

$y''+y'-2y = 0 \Rightarrow r^2+r-2=0 \Rightarrow (r+2)(r-1)=0 \Rightarrow r=-2,1$

Thus the general form of the solution will be

$y = c_1e^t + c_2e^{-2t} + Y(t)$

$Y(t)$ being the particular solution of the nonhomogeneous equation

We want a function Y that will make $Y''(t)+Y'(t)-2Y(t)$ equal to $2t$

We assume Y(t) will be some multiple of t so we get

$Y(t) = At + B$?

$Y'(t) = A$

$Y''(t) = 0$

Substituting for y, y', and y'' we obtain

$A - 2(At+B) = 2t \Rightarrow A - 2B - 2At = 2t$

We then solve for A and B

$-2At = 2t \Rightarrow A = -1$

$A - 2B = 0 \Rightarrow B = -\dfrac{1}{2}$

So the particular solution is

$Y(t) = -t - \dfrac{1}{2}$

Thus the general solution is

$y = c_1^t + c_2e^{-2t} - t - \dfrac{1}{2}$

By applying the initial conditions and solving for c_1 and c_2

$c_1 = 1\,,\ c_2 = -\dfrac{1}{2}$

Then substituting c_1 and c_2 in we obtain the solution to the I.V.P

$y = e^t - \dfrac{1}{2}e^{-2t} - t - \dfrac{1}{2}$

(30 points)

A mass of 2 kg stretches a spring 50cm. The mass is displaced 10 cm upward and released. Find the position of the mass at any given time. Also give the natural frequency and amplitude of the system.

Solution

The general form of the differential equation is $mu'' + \lambda u' + ku$, where k=force/distance and λ= force/velocity. From the problem we see that m=2, λ=0, and k= 2*9.8 N/.5m. So m= 2 and k=39.2, and we plug these constants into our equation which yields $20u'' + 148u = 0$

.

The characteristic equation for this O.D.E. is $2r^2 + 39.2 = 0$. The roots of which are $r = \pm\sqrt{18.1}$.

The general solution of this second degree homogenous differential equation is $u = A\cos(\sqrt{18.1}t) + B\sin(\sqrt{18.1}t)$.
By using our initial data of; u(0)=.1 and $u'(0) = 0$ we can solve for A and B.
u(0)=A(1)+B(0)=1
$u'(0) = -\sqrt{18.1}A(0) + \sqrt{18.1}B(1) = 0$
From these equations we see that A = 1 and B = 0. We plug these values into our general solution to get: $u(t) = \cos(\sqrt{18.1}t)$,which is our position according to time.

The natural frequency of the system is given by the relation $\omega_0 = \sqrt{k/m}$.
So applying this relation we get $\omega_0 = \sqrt{39.2/2} = \sqrt{19.6}$ hertz

The amplitude of the system is given as $R = \sqrt{A^2 + B^2}$
By plugging our constants into this relation it yields: $R = \sqrt{1^2 + 0^2} = 1$
So our amplitude is 1.

(30 pts) Solve the given differential equation (1) by means of a power series around the point X_0. Find the a) recurrence relation and b) the first four terms in each of two linearly independent solutions. Then find c) the general term in each solution.

$2y'' + xy' + 3y = 0, \quad x_0 = 0$ (1)

a)

For this equation P(x) = 2, Q(x) = x, and R(x) = 3 so every point is an ordinary point. We look for a solution in the form of a power series:

$$y = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n = \sum_{n=0}^{\infty} a_n x^n . \qquad (2)$$

Assuming that $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ we substitute in these for y, y', and y''.

$$2\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} + 3\sum_{n=0}^{\infty} a_n x^n = 0 \qquad (3)$$

Now we factor in what lies outside each summation shift the index of summation so that every summation has "x" to the same power.

$$\sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 3 a_n x^n = 0 \qquad (4)$$

Solving for a_{n+2} we find the recurrence relation:

$$a_{n+2} = -[n+3]\, a_n \,/\, 2(n+2)(n+1), \quad n = 0, 1, 2, 3, \dots . \qquad (5)$$

Now we find the successive coefficients first for n=0, then for n=1, and so on.

$a_2 = -3\, a_0 / 2*2*1 = -3\, a_0 / 2*2!$ $\qquad a_3 = -4\, a_1 / 2*3*2 = -4\, a_1 / 2*3!$ (6)

$a_4 = -5\, a_2 / 2*4*3 = 15\, a_0 \,/\, 2*2*4!$ $\qquad a_5 = -6\, a_3 / 2*5*4 = 24\, a_1 / 2*2*5!$

$a_6 = -7\, a_4 \,/\, 2*6*5 = -105\, a_0 \,/\, 2*2*2*6!$ $\qquad a_7 = -8\, a_5 / 2*7*5 = -192\, a_1 / 2*2*2*7!$

Substituting these coefficients into equation (2) we get :

$y_1(x) = 1 - 3/4x^2 + 5/32\, x^4 - 7/384\, x^6 + \dots (-1^n)(2n+1)/\, 2^n(2n)!$ (7)

$y2(x) = x - 1/3x^3 + 1/20\, x^5 - 1/210\, x^7 + \dots (-1^n)(2n+2)/\, 2^n(2n+1)!$ (8)

11

1)(30 points)Find the general solution to the following three 2^{nd} order, homogenous differential equations. Give a short explanation as to what kind of solution the characteristic equation is, ? and also give the general solution to that kind of characteristic equation.

A) Y" -3Y'+Y = 0

B) 2Y" + 2Y' + Y = 0

C) Y" + 2Y' + Y = 0

These problems should be very straight forward and easy. These are to let the student show that they know the solution to the differential equations where the characteristic equation has ~~a~~ solution ~~with~~ constant coefficients, which have real & distinct Complex coefficients, (conjugate) and repeated ~~coefficients~~.

A) Y" -3Y'+Y = 0

given by inspecting what does that mean?

The roots the characteristic equation are: (R-2)(R-1)=0
so the general solution would be of the constant coefficient form:
Y(t) = C1e^(R1t) + C2e^(R2t)
So this general solution is :
Y(t) = C1e^(2t) + C2e^(t)

B) 2Y" + 2Y' + Y = 0

The roots the characteristic equation are: (R-1+2i)(R-1-2i)=0
so the general solution would be of the complex coefficient form:
Y(t) = C1e^(λt)Cos(μt) + C2e^(λt)Sin(μt)
Where λ is the real part, and μ is the value of the imaginary part.
So this general solution is :
Y(t) = C1e^(-t)Cos(2t) + C2e^(-t)Sin(2t)

C) Y" + 2Y' + Y = 0
This has repeated roots, ie (R+1)^2
So the general solution will be a repeated root solution of the form:
Y(t) = C1e^(Rt) + C2te(Rt)
Where R is the repeated ~~solution~~ root (R = -1)
So the general solution is:
Y(t) = C1e^-t + C2te^-t

2)(40points)Problem Statement:
Find the general solution (which includes the particular solution) of the following second order linear nonhomogeneous equation.

$$Y'' - 2Y' - 8 = 4 e^{(-t)}$$

Solution:

Evaluating the roots of the characteristic equation gives

$$r = -2, 4$$

So the general solution so far is Y = c1 e ^ (-2t) + c2 e ^ (4t) + P(t)
where P(t) is ~~the~~ a particular solution

To find ~~the~~ a particular solution of this equation we can use theorem 3.7.1 from the book which says

$$P(t) = -y(t)\int_0^t \frac{y2(s)g(s)}{W(y1, y2)(s)}ds + y2(t)\int_0^t \frac{y1(s)g(s)}{W(y1, y2)(s)}ds$$

To find W(y1,y2), we take the determinant, which is y1*y2'- y1'*y2.

This is W = e^(-2t)*4e^(4t) – (-2)e^(-2t)*e^(4t) = 6 e ^ (2t)

Evaluating P(t) with W = 6 e ^ (2t), y1 = e ^ (-2t) and y(2) = e ^ (4t) yields

P(t) = - 4/5 e ^ (-t).

Thus,

Y(t) = c1 * e ^ (-2t) + c2 * e ^ (4t) - 4/5 e ^ (-t).

3)(40 points) Given that $y_1(t) = t^{-1}$ is a solution of $t^2y'' + 3ty' + y = 0$, $t > 0$, find a second linearly independent solution.

Solution: Let $y_2 = v(t)\, t^{-1}$.

By differentiating, we see that

$$y_2' = v't^{-1} - vt^{-2}\ ,\ y_2'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}.$$

By plugging into the differential equation and simplifying, we see that

$$t^2(v''t^{-1} - 2vt^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) + vt^{-1}$$
$$= tv'' + v' = 0.$$

Letting $u = v'$, we have the first order differential equation

$$tu' + u = 0$$

which has the solution $u = c_1t^{-1}$. Thus,

$$v' = ct^{-1},$$

and

$$v = c \ln t + d.$$

It follows that

$$y_2 = ct^{-1} \ln t + dt^{-1}$$

The second term (dt^{-1}) is a multiple of $y_1(t)$ and can be dropped.
Hence, the second solution is $t^{-1} \ln t$.

4)(40 points) Regular single points

a) $(x^2 - 4)\, y'' + (x - 2) + y = 0$

what's the question?

It should be clear $x = -2$, and $x = 2$ are singular points of the equation.
Dividing the equation by $(x^2 - 4) = (x - 2)^2\,(x + 2)^2$, we find:

$P(x) = 1/(x-2)(x+2)^2$ and $Q(x) = 1/(x-2)^2(x+2)^2$

Testing P(x) and Q(x) at each singular point. In order that $x = -2$ be a regular single point, the factor $x+2$ can appear at most to the first power in the denominator of P(x), and can appear at most to the second power in the denominator of Q(x). We conclude that $x = -2$ is an irregular singular point.

In order for $x = 2$ be a regular single point, the factor $x-2$ can appear at most to the first power in the denominator of P(x) and can appear at most to the second power in the denominator of Q(x) shows that both these conditions are satisfied, $x=2$ is a regular singular point.

b) $x^2(x+1)^2 y'' + (x^2 - 1)y' + 2y = 0$

$$P(x) = \frac{(x-1)}{x^2(x+1)} \quad \text{and} \quad Q(x) = \frac{2}{x^2(x+1)^2}$$

$x=0$ is an irregular point because $(x-0)$ appears to the second power in the denominator of P(x). However, $x=-1$ is a regular single point.

13

Problem (35 Points)

Solve the following initial value problem:

$$y'' - 10y' + 29y = 0 \qquad y(0) = 1 \quad y'(0) = 3$$

Solution

The characteristic equation is

$$r^2 - 10r + 29 = 0$$

Find roots of the equation by using the quadratic equation

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \Rightarrow \qquad \frac{10 \pm \sqrt{10^2 - 4(1)(29)}}{2}$$

Therefore the roots of the characteristic equation are

$$r = 5 + 2i \qquad \text{and} \qquad r = 5 - 2i.$$

We can conclude that the general solution to the differential equation is

$$y = a_1e^{(5+2i)t} + a_2e^{(5-2i)t}$$

Which can be simplified by using the rule of exponents and factoring gout the e^{5t}

$$= a_1e^{5t}e^{2it} + a_2e^{5t}e^{-2it} \qquad = e^{5t}(a_1e^{2it} + a_2e^{-2it})$$

Although this gives the general solution, it is not satisfactory since the solution involves complex exponents. To deal with this we use Euler's formula

$$e^{iq} = \cos q + i \sin q$$

This gives

$$y = e^{5t}[a_1(\cos(2t) + i \sin(2t)) + a_2(\cos(-2t) + i \sin(-2t))]$$

Since the cos(x) is an even function and sin(x) is an odd function, we get

$$y = e^{5t}[a_1(\cos(2t) + i \sin(2t)) + a_2(\cos(2t) - i \sin(2t))]$$

or

$$y = e^{5t}[(a_1 + a_2)\cos(2t) + (a_1 - a_2)i \sin(2t)]$$

Finally let

$$c_1 = a_1 + a_2 \qquad \text{and} \qquad c_2 = i(a_1 - a_2)$$

cool - good explanation

and we get the general solution to be

$$y = e^{5t}[c_1 \cos(2t) + c_2 \sin(2t)]$$

We use the initial values to find the constants. Plug in $y(0) = 1$.

$$1 = 1[c_1(1) + c_2(0)]$$

so that $c_1 = 1$.

Then take the derivative of y and get

$$y' = 5e^{5t}[\cos(2t) + c_2 \sin(2t)] + e^{5t}[-2 \sin(2t) + 2c_2 \cos(2t)]$$

Plugging in $y'(0) = 3$

$$3 = 5[1 + 0] + 1[0 + 2c_2] = 5 + 2c_2$$

Solving for c_2 we get $c_2 = -1$

The final solution is

$$y = e^{5t}[\cos(2t) - \sin(2t)]$$

Question (35 points):

A mass weighing 5kg stretches a spring 20cm. If the mass is set in motion from the equilibrium position at 3m/s and there is no damping, determine the position *u* of the mass at any time *t*.

Answer:

First we need to put the information in the following format:

$m\ddot{u} + ku = 0$

We already know *m,* but we must find *k.*

$F = ks$

$F = ma$

$ma = ks$

$k = {ma}/{s}$

$$k = \frac{(5kg)(9.8\,m/s)}{.20m}$$

$k = 245\,kg/s^2$

Now we have the equation:

$5\ddot{u} + 245u = 0$

Which simplifies to:

$\ddot{u} + 49u = 0$

You must find the roots:

$r^2 + 49 = 0$

$r = \pm i7$

So the equation is:

$u = A\cos(7t) + B\sin(7t)$

Plug in $u(0)=0$:

$0 = A\cos(7(0)) + B\sin(7(0))$

$A = 0$

Take the derivative and plug in u(0)=3:

$u' = -7A\sin(7t) + 7B\cos(7t)$

$3 = -7A\sin(7(0)) + 7B\cos(7(0))$

$3 = 7B$

$B = 3/7$

Plug A and B into the equation and you get the answer:

$u = (0)\cos(7t) + 3/7\sin(7t)$

$u = 3/7\sin(7t)$

40 points

Solve the differential equation $y'' - 4xy' - y = 0$ using a series solution about $x_o = 0$. The assumed solution for the differential equation is:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Solution:

Substitute into the differential equation:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4 \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Relabel each term so it has an x^n.

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - 4 \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Now each sum needs to start at the same value of n; we can do this by removing the *n*=0 terms from the first and third sum:

$$2a_2 - a_0 + \sum_{n=1}^{\infty} (n+1)(n+2) a_{n+2} x^n - 4 \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} a_n x^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty} [(n+1)(n+2) a_{n+2} - 4n a_n - a_n] x^n = 0$$

For this to be true for all values of x, each coefficient of the series must be zero,

$2a_2 - a_0 = 0$

$(n+1)(n+2)a_{n+2} - 4na_n - a_n = 0, \; n = 1,2,3, \ldots$ I don't understand

Notice that since the summation begins at n=1, the second equation is always true for *n*=1,2,3,.. Changing this to begin at *n*=0, we get:

$(n+2)a_{n+2} - a_n = 0, \; n = 0,1,2,3, \ldots$

Solve the recurrence relation for a_{n+2}, then determine the first few coefficients, and try to spot a pattern.

$$a_{n+2} = \frac{a_n}{(n+2)}, n = 0,1,2,3, \ldots$$ $$a_{n+2} = \frac{(4n+1)a_n}{(n+1)(n+2)}$$

$$a_2 = \frac{a_o}{2}$$

$$a_3 = \frac{a_1}{3}$$

$$a_4 = \frac{a_2}{4} = \frac{a_o}{4 * 2}$$

$$a_5 = \frac{a_3}{5} = \frac{a_1}{3 * 5}$$

The pattern above is:

$$y_1(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2 \cdot 4 \cdot 6 \cdots (2k)} = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}$$

$$y_2(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k+1)} = \sum_{k=0}^{\infty} \frac{2k! \, x^{2k+1}}{(2k+1)!}$$

40 points

Find the **general solution** of

$y'' + 2y' - 3y = 10e^{2t}$

We need the solution in the form

$$y = c_1 y_1(t) + c_2 y_2(t) + Y(t) \qquad (1)$$

First solve the homogeneous part

$$y'' + 2y' - 3y = 0 \qquad (2)$$

to find the roots we change the equation to the form

$r^2 + 2r - 3 = 0$

and we find that the roots are at r = -3 and 1 and thus equation (1) changes to

$y = c_1 e^{-3t} + c_2 e^{t} + Y_1(t)$

Now we are looking for a function Y such that

$Y''(t) + 2Y'(t) - 3Y(t) = 10e^{2t}$

To achieve this we assume

$Y(t) = Ae^{2t}$

And we are solving for the coefficient A

$Y'(t) = 2Ae^{2t}$

$Y''(t) = 4Ae^{2t}$

We substitute these into equation (1) for y'', y', and y and we get

$$4Ae^{2t} + 4Ae^{2t} - 3Ae^{2t} = 10e^{2t}$$

We find $A = 2$

And thus $Y_1(t) = 2e^{2t}$

Substitute this into equation (2) and we find the general solution is

$$\boldsymbol{y = c_1 e^{-3t} + c_2 e^{t} + 2e^{2t}}$$

~~#4~~

1

1.) A) *Show that* $y_1(t) = t^{\frac{1}{2}}$ *and* $y_2(t) = t^{-1}$ *form a fundamental set of solutions of* :

$2t^2 y'' + 3ty' - y = 0$ *for* $t > 0$. (1.1)

15 points

Solution:

We can check to see if the given ~~equations~~ functions are solutions simply by substituting them into equation (1.1):

$$y_1'(t) = \frac{1}{2}t^{-\frac{1}{2}} \quad (1.2) \qquad y_2'(t) = -t^{-2} \quad (1.4)$$

$$y_1''(t) = -\frac{1}{4}t^{-\frac{3}{2}} \quad (1.3) \qquad y_2''(t) = 2t^{-3} \quad (1.5)$$

Substituting (1.2) and (1.3) into the equation (1.1) and simplifying yields.

$$2t^2\left(-\frac{1}{4}t^{-\frac{3}{2}}\right) + 3t\left(\frac{1}{2}t^{-\frac{1}{2}}\right) - t^{\frac{1}{2}} = 0 \quad (1.7)$$

$$\left(-\frac{1}{2} + \frac{3}{2} - 1\right)t^{\frac{1}{2}} = 0 \quad (1.8)$$

$$(0)t^{\frac{1}{2}} = 0 \quad (1.9)$$

$$0 = 0 \quad (1.10)$$

therefore y_1 is a solution to the original equation (1.1).

Following the same procedure for y_2:

$$2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} = 0 \quad (1.11)$$

$$(4 - 3 - 1)t^{-1} = 0 \quad (1.12)$$

$$(0)t^{-1} = 0 \quad (1.13)$$

$$0 = 0 \quad (1.14)$$

therefore y_2 is a solution to the original equation(1.1).

To determine if they form a fundamental set of solutions, we need to find the determinant of the Wronskian W of y_1 and y_2.

$$W = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -t^{-\frac{3}{2}} - \frac{1}{2}t^{-\frac{3}{2}} = -\frac{3}{2}t^{-\frac{3}{2}} \neq 0 \quad (1.15)$$

Since the Wronskian is $\neq 0$, then the solutions y_1 and y_2 form a fundamental set of solutions for the original equation. for $t>0$,

B) Solve the following initial value problem:

$$y'' - 6y' + 9y = 0 \qquad (2.1)$$
$$y(0) = -2$$
$$y'(0) = 2$$

20 points

Solution:

We make the assumption that the solution is of the form e^{rt}. Since this is a linear, homogeneous, constant coefficient, second order differential equation, the characteristic equation can be written as $r^2 - 6r + 9 = 0$ (2.2)

Solving equation (2.2) for r produces $\begin{matrix}(r-3)(r-3)=0\\ r=3\end{matrix}$.

This is a repeated root, which means that the solution to equation (2.1) is of the form

$$y = c_1 e^{3t} + c_2 t e^{3t} \qquad (2.3)$$

to solve according to the initial data we need y'. Differentiating (2.3) with respect to t gives us: $y' = 3c_1 e^{3t} + 3c_2 t e^{3t} + c_2 e^{3t}$ (2.4)

using the initial data $y(0) = -2$ and equation (2.3) gives us:

$$-2 = c_1 + 0 \text{ or } c_1 = -2 \qquad (2.5)$$

using the initial data $y'(0) = 2$ and equation (2.4) gives us:

$$2 = 3c_1 + 0 + c_2 \text{ or } 2 = -6 + c_2 \text{ or } c_2 = 8 \qquad (2.6)$$

Substituting the values from (2.5) for c_1 and (2.6) for c_2 gives us the solution to the initial value problem (2.1):

$$y = -2e^{3t} + 8te^{3t} \qquad (2.7)$$

2.) Solve the initial value problem,

$$\frac{1}{2}y'' + y' + y = 2t^2$$
$$y(0) = 1$$
$$y'(0) = 0.$$

30 Points

Solution:

$$Y(t) = 2t^2 - 2t + 1 + e^{-t}\cos(t) + e^{-t}\sin(t)$$?

This nonhomogeneous equation has a general solution of the form

$$Y = Y_1 + c_1 y_1 + c_2 y_2,$$ where, . . .

Begin by looking at the corresponding homogeneous equation:

$$\frac{1}{2}y'' + y' + y = 0.$$

The characteristic equation for this equation is $\frac{1}{2}r^2 + 1r + 1 = 0$. The roots for this are

$$r_1 = -1 + i$$
$$r_2 = -1 - i.$$

Our general solution to the homogeneous equation is then

$$y = c_1 e^{-1+i} + c_2 e^{-1-i}.$$

Because of the complex roots we must introduce a new general solution:

$$e^{(\lambda+\mu i)t} = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t,$$

where λ is the real part and μ is the coefficient of the imaginary part. Just looking at our first root, we can define $\lambda = -1$ and $\mu = 1$. Substituting these into our general solution yields

$$y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t).$$

need Y_1 first - won't work in general

Now, we can use our initial value conditions to find c_1 and c_2.

$$y(0) = c_1 e^{-(0)}\cos((0)) + c_2 e^{-(0)}\sin((0)) = 1.$$

Therefore $c_1 = 1$. The second parameter states

$$y'(0) = (-c_1 + c_2)e^{-(0)}\cos((0)) - (-c_1 + c_2)e^{-(0)}\sin((0)) = 0$$

Therefore,

$$0 = -c_1 + c_2 = -1 + c_2.$$

So $c_2 = 1$. Our solution to the homogeneous equation becomes

$$y = e^{-t}\cos(t) + e^{-t}\sin(t).$$

Now we can look at the nonhomogeneous equation, $\frac{1}{2}y'' + y' + y = 2t^2$. According to the Method of Undetermined Coefficients there is a solution, Y, of the form

$$Y = At^2 + Bt + C.$$

Therefore it must be true that $Y' = At + B$ and $Y'' = A$.

By substituting these expressions into our nonhomogeneous equation, we obtain

$$\frac{1}{2}A + (At + B) + (At^2 + Bt + C) = 2t^2.$$

All like terms must be equal on both sides of the equation so we obtain the following statements:

$$\begin{aligned} &\frac{1}{2}A + B + C = 0 \\ &At + Bt = 0 \\ &At^2 = 2t^2. \end{aligned}$$

Thus,

$$\begin{aligned} A &= 2 \\ B &= -2 \\ C &= 1. \end{aligned}$$

Finally, $Y = 2t^2 - 2t + 1$. Plugging all solutions into our general solution for nonhomogeneous equations, we obtain our answer:

$$Y(t) = 2t^2 - 2t + 1 + e^{-t}\cos(t) + e^{-t}\sin(t).$$

which won't satisfy initial data!

3.) A mass m is put into motion by a force F(t) = $4\cos 2t$. The mass is attached to a spring of with a constant K of 4 slugs. The spring has a dampening constant of .25 lb-

units?

sec/ft.

A) Find a general solution for the movement of the mass m for time t.
30 Points

Solution:

The equation is of the form:

$$\mathrm{m}\, y'' + \gamma y' + ky = 4\cos 2t$$

First we must solve the equation as if it were a homogeneous equation.
So writing the equation again and plugging in the values we know we have:

$$\mathrm{m}\, y'' + .25\, y' + 4y = 0$$

Next we divide by the leading coefficient:

$$y'' + \frac{.25}{m} y' + \frac{4}{m} y = 0$$

To solve the equation we will need to use the characteristic equation, namely:

$$r^2 + \frac{.25}{m} r + \frac{4}{m} = 0$$

Now we solve for the roots of the equation getting:

$$r = \frac{-\frac{.25}{m} \pm \sqrt{\left(\frac{.25}{m}\right)^2 - 4\left(\frac{4}{m}\right)}}{2}$$

via algebraic simplification we arrive at the values of the two roots:

$$r_1 = \frac{-1 + i\sqrt{255}}{8m} \quad , \quad r_2 = \frac{-1 - i\sqrt{255}}{8m}$$

seeing the roots are complex the solution of the homogenous equation is of the form:

$$y_h = c_1\, e^{\frac{-1+i\sqrt{255}}{8m}t} + c_2\, e^{\frac{-1-i\sqrt{255}}{8m}t}$$

however for simplicity sake we will remove the complex numbers from the exponents making the homogenous solution:

$$y_h = e^{-t}\left(c_1 \cos\frac{\sqrt{255}}{8m}t + c_2 \sin\frac{\sqrt{255}}{8m}t\right)$$

Now that we have attained the homogeneous solution we need to solve the differential equation with set equal to $4\cos 2t$ rather then 0. To remind you the equation is:

$$y'' + \frac{.25}{m}y' + \frac{4}{m}y = \frac{4}{m}\cos 2t$$

To solve this equation we will employ the method of undetermined coefficients- namely we will guess the solution of the form:

$$y = A\cos 2t + B\sin 2t$$

Differentiating once and then twice:

$$y' = -2A\sin 2t + 2B\cos 2t$$
$$y'' = -4A\cos 2t - 4B\sin 2t$$

And plugging into the differential equation we have:

$$(-4A\cos 2t - 4B\sin 2t) + \frac{.25}{m}(-2A\sin 2t + 2B\cos 2t) + \frac{4}{m}(A\cos 2t + B\sin 2t) = \frac{4}{m}\cos 2t$$

Which becomes:

$$-4A\cos 2t - 4B\sin 2t - \frac{1}{2m}A\sin 2t + \frac{1}{2m}B\cos 2t + \frac{4}{m}A\cos 2t + \frac{4}{m}B\sin 2t = \frac{4}{m}\cos 2t$$

Combining cosines and sines:

$$\cos 2t\left(\frac{1}{2m}B + \frac{4}{m}A - 4A\right) + \sin 2t\left(\frac{4}{m}B - \frac{1}{2m}A - 4B\right) = \frac{4}{m}\cos 2t$$

from this we can obtain two separate equations of sines and cosines to solve for A and B:

$$\cos 2t\left(\frac{1}{2m}B + \frac{4}{m}A - 4A\right) = \frac{4}{m}\cos 2t$$

$$\sin 2t\left(\frac{4}{m}B - \frac{1}{2m}A - 4B\right) = 0$$

solving each equation simultaneously first we get rid of the trig functions and simplify:

$$\frac{A(4-4m)}{m} + \frac{B}{2m} = 4/m$$

$$\frac{B(4\text{-}4m)}{m} - \frac{A}{2m} = 0$$

Using the second equation we solve for B obtaining:

$$B = \frac{A}{8-8m}$$

We now plug B into the first equation to solve for A

$$\frac{A(4-4m)}{m} + \frac{A}{2m(8-8m)} = \frac{4}{m}$$

simplifying:

$$\frac{4mA(4-4m)^2 + Am}{4m^2(4-4m)} = \frac{4}{m}$$

continuing to simplify:

$$A(4m(4-4m)^2 + m) = 16m(4\text{-}4m)$$

$$A = \frac{4}{(4-4m)}$$

And lastly we plug the known value for A back into the value for B obtaining:

$$B = \frac{2}{(4-4m)^2}$$

Now we plug the values for A and B back into to original equation y for the particular solution making it:

$$Y = \frac{4}{(4-4m)}\cos 2t + \frac{2}{(4-4m)^2}\sin 2t$$

The final step to obtaining the general solution to the movement of mass m to all t is to combine the homogeneous solution and the particular solution giving us:

$$Y = e^{-t}\left(c_1 \cos\frac{\sqrt{255}}{8m}t + c_2 \sin\frac{\sqrt{255}}{8m}t\right) + \frac{4}{(4-4m)}\cos 2t + \frac{2}{(4-4m)^2}\sin 2t$$

Which is the solution to part A.

B) What is the Amplitude of the oscillations of the mass – spring system?
15 Points

Solution:

To find the Amplitude R we use the equation:

$$R = \frac{F_0}{\Delta} \text{ where } \Delta = \sqrt{(m^2(\omega_0^2 + \omega^2)^2 + \gamma^2\omega^2}$$

- parameters mean what? used in which context? the presenture? No! The formula is irrelevant to the present context, at least as stated.

Solving for Δ by plugging in known values (remember $\omega_0^2 = \frac{k}{m}$ and $\omega = 2$) we get:

$$\Delta = \sqrt{(m^2\left(\frac{4}{m} + 4^2\right)^2 + \left(\frac{1}{4}\right)^2 4}$$

solving further:

$$\Delta = \frac{\sqrt{16(1+4m)^2 + 1}}{2}$$

so finally we plug Δ back into the equation for R (remembering that $F_0 = \frac{4}{m}$) and we get:

$$R = \frac{8}{\sqrt{16(1+4m)^2 + 1}}$$

Which is the solution to part B.

C) What mass m is required to maximize the amplitude of oscillations of the mass-spring system?
10 Points

Solution:

In order to get the answer we must use the equation for amplitude that we found in the previous question and maximize it in respect for m. The best way to do that is to realize that the Amplitude ($\frac{F_0}{\Delta}$) can be maximized by minimizing Δ. So taking that function we need to take a derivative of Δ so we can find critical points in order to minimize.

$$\Delta = \sqrt{m^2(\frac{4}{m}+4)^2+\frac{1}{4}}$$

$$\Delta' = \frac{(4m-4m^2)(4-8m)}{\sqrt{16(1+4m)^2+1}}$$

– but this is not relevant, at least as the problem is stated.

Since we are looking for zeroes of the function and since this function will only have zeroes when the top is equal to zero we can just look at that.

$$(4m-4m^2)(4-8m)=0$$

From this we get two separate equations namely:

$$(4m-4m^2) = 0$$

$$(4-8m) = 0$$

Solving these equations simultaneously:

$$m(4-4m) = 0$$

$$\text{m} = 0,\ \text{m} = 1$$

$$(4-8m)=0$$

$$\text{m} = \frac{1}{2}$$

Now it is necessary to analyze the function near the critical points to determine maxima or minima.

$$\Delta'\ (\text{-}1) > 0$$

$$\Delta'(\frac{1}{4}) < 0$$

$$\Delta'(\frac{3}{4}) > 0$$

$$\Delta'(2) < 0$$

looking at the values above there is a minima at m = $\frac{1}{2}$. Thus:

$m = \frac{1}{2}$ maximizes the amplitude and is the solution to C.

4). Solve the given differential equation by means of the power series about the given point x_0. Find the recurrence relationship and the first four terms of the two linearly independent solutions.

$$y'' - xy = 0 \qquad x_0 = 0$$

30 Points

Solution:

Given the series solutions

$y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substitute the initial the series into the differential equation.

$$0 = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n$$

Combine the x into second summation.

$$0 = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$0 = 2a_2 x^{2-2} + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$0 = 2a_2 + \sum_{n=0}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$0 = 2a_2 + \sum_{n=0}^{\infty} \{(n+3)(n+2) a_{n+3} x^{n+1} + a_n x^{n+1}\}$$

Solutions of the summations are

$2a_2 = 0$

$a_2 = 0$

$$\sum_{n=0}^{\infty} \{(n+3)(n+2) a_{n+3} x^{n+1} + a_n x^{n+1}\} = 0$$

$(n+3)(n+2) a_{n+3} = -a_n$, $n \geq 0$

$a_{n+3} = {-a_n}/{(n+3)(n+2)}$

$a_0 =$ unknown

$a_1 =$ unknown

For n=0, 1, 2, 3, …

$a_3 = -\frac{1}{6}a_0$, $a_4 = -\frac{1}{12}a_1$, $a_5 = -\frac{1}{20}a_2 = 0$, $a_6 = -\frac{1}{30}a_3 = \frac{1}{180}a_0$,

$a_7 = -\frac{1}{42}a_4 = \frac{1}{504}a_1$, $a_8 = -\frac{1}{56}a_5 = 0$, $a_9 = -\frac{1}{72}a_6 = -\frac{1}{12960}a_0$,

$a_{10} = -\frac{1}{90}a_7 = \frac{1}{45360}a_1$

General Equation:

$$y(t)= a_0 + a_1 x - \frac{1}{6}a_0 x^3 - \frac{1}{12}a_1 x^4 + 0x^5 + \frac{1}{180}a_0 x^6 + \frac{1}{504}a_1 x^7$$

$$+ 0x^8 - \frac{1}{12960}a_6 x^9 + \frac{1}{45360}a_1 + \ldots.$$

When $a_0 = 0$ and $a_1 = 1$

$$y_1(t) = x - \frac{1}{12}x^4 + \frac{1}{504}x^7 + \frac{1}{45360}x^{10} + \ldots..$$

When $a_1 = 0$ and $a_0 = 1$

$$y_2(t) = 1 + -\frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12960}x^9 + \ldots..$$

2

Problem 1 (45 Points):

Give the general solution for $y'' - 5y' - 24y = e^{5t}$ using the method of variation of Parameters.

Solution:

Theorem 3.7.1 says that a solution of the non-homogeneous equation $y'' + p(t)y' + q(t)y = g(t)$ is
$Y(t) = y_1(t)\int_{t_0}^{t} u'_1(s)ds + y_2(t)\int_{t_0}^{t} u'_2(s)ds$ *
Where $u'_1(t) = -\dfrac{y_2(s)g(s)}{W(y_1, y_2)(s)}$ and $u'_2(t) = \dfrac{y_1(s)g(s)}{W(y_1, y_2)(s)}$. **

The characteristic equation for the homogeneous solution is $r^2 - 5r - 24 = 0$ or
$(\cancel{x}\,r - 8)(\cancel{x}\,r + 3) = 0$. Therefore the general homogeneous solution is $y_c(t) = c_1 e^{8t} + c_2 e^{-3t} \cancel{= 0}$

Replacing the constants with functions $u(t)$ and differentiating yields the following:
$y = u_1(t)e^{8t} + u_2(t)e^{-3t}$ — why are you doing this?
$y' = 8u_1(t)e^{8t} - 3u_2(t)e^{-3t} + u'_1(t)e^{8t} + u'_2(t)e^{-3t}$ Where we'll say $y' = u'_1(t)e^{8t} + u'_2(t)e^{-3t} = 0$ I thought
$y'' = 64u_1(t)e^{8t} + 9u_2(t)e^{-3t} + 8u'_1(t)e^{8t} - 3u'_2(t)e^{-3t}$ you wanted to use formula * and **

Plugging these values back into the original differential equation yields:
$\left(64u_1(t)e^{8t} + 9u_2(t)e^{-3t} + 8u'_1(t)e^{8t} - 3u'_2(t)e^{-3t}\right) - 5\left(8u_1(t)e^{8t} - 3u_2(t)e^{-3t}\right) - 24\left(u_1(t)e^{8t} + u_2(t)e^{-3t}\right) = e^{5t}$
This simplifies to: $8u'_1(t)e^{8t} - 3u'_2(t)e^{-3t} = e^{5t}$.

This gives a system of equations for which we can solve:
$e^{5t} = 8u'_1(t)e^{8t} - 3u'_2(t)e^{-3t}$
$0 = u'_1(t)e^{8t} + u'_2(t)e^{-3t}$

$u'_1(t)$ and $u'_2(t)$ can be solved by substitution or by theorem 3.7.1. Both methods yield:
$u'_1(t) = \dfrac{e^{-3t}}{11}$ and $u'_2(t) = -\dfrac{e^{8t}}{11}$

Integrating using theorem 3.7.1 gives us the general solution:
$$y(t) = e^{8t}\left(-\frac{e^{-3t}}{33}\right) + e^{-3t}\left(-\frac{e^{8t}}{88}\right) + c_1e^{8t} + c_2e^{-3t}.$$

This simplifies to a general solution of:
$$y(t) = -\frac{e^{5t}}{24} + c_1e^{8t} + c_2e^{-3t}.$$

Problem 2 (30 Points):

Find the general solution of the given differential equation that is valid in any interval not including the singular point.

$(x - 2)^2y'' + 5(x - 2)y' + 8y = 0$

Solution:

1) This is a perfect application of Theorem 5.5.1 (p.277), which states that the general solution of the Euler equation ($x^2y'' + \alpha xy' + \beta y = 0$) in any interval not containing the origin is determined by the roots r_1 and r_2 of the equation: $F(r) = r(r - 1) + \alpha r + \beta = 0$

To make our equation fit the Euler equation we need to substitute t for $(x - 2)$, which gives us:
$t^2y'' + 5ty' + 8y = 0$

2) Now let's find the roots:

$$F(r) = r(r - 1) + 5r + 8 = 0$$
$$= r^2 + 4r + 8$$

To find the roots we'll have to use the quadratic equation:

$$\frac{-4 \pm \sqrt{4^2 - 4(1)(8)}}{2}$$

Remember if the roots are complex then $r_1, r_2 = \lambda \pm i\mu$

real and imaginary parts of the

So in this case the roots are $\lambda = -2$ and $\mu = 2$

3) Since the roots are complex we know (according to theorem 5.5.1, p.277) ~~we know~~ that

$$y = |x|^{\lambda} [c_1\cos(\mu\ln|x|) + c_2\sin(\mu\ln|x|)]$$

And when we plug back in we get:

$$y = |t|^{-2} [c_1\cos(2\ln|t|) + c_2\sin(2\ln|t|)]$$

Now substitute (x-2) back into the equation for t and we get:

$$y = (x-2)^{-2} [c_1\cos(2\ln|x - 2|) + c_2\sin(2\ln|x - 2|)]$$

And after distributing the $(x-2)^{-2}$ we get:

$$y = c_1(x-2)^{-2}\cos(2\ln|x - 2|) + c_2(x-2)^{-2}\sin(2\ln|x - 2|)$$

And we're done.

Problem 3 (45 Points):

For the following second order homogenous differential equation:

$2x^2y'' - xy' + (1+x)y = 0$

for a series solution about which x_0?

(1) Find the corresponding Euler's Equation. Then, (2) assume the answer to the equation is of the form $y = \sum_{n=0}^{\infty} a_n x^{r+n}$ and find the Indicial equation. (3) Using the indicial equation, find the exponents at the singularity. Finally, (4) find the recurrence relation for both roots.

Solution:

(1)

$y'' - \frac{x}{2x^2}y' + \frac{(1+x)}{2x^2}y = 0$ so $p(x) = -\frac{x}{2x^2}$, $q(x) = \frac{(1+x)}{2x^2}$

$xp(x) = -\frac{x^2}{2x^2} = -\frac{1}{2}$ and $x^2q(x) = \frac{x^2(1+x)}{2x^2} = \frac{(1+x)}{2}$

$xp(0) = p_0 = -\frac{1}{2}$ and $x^2q(0) = q_0 = \frac{(1+0)}{2} = \frac{1}{2}$

Plug p_0 and q_0 into Euler's Equation and get $x^2y'' - \frac{1}{2}xy' + \frac{1}{2}y = 0$

Multiply everything by 2: $2x^2y'' - xy' + y = 0$ $\Rightarrow r = ?$ (strange order of requests)

(2)

Assume the answer is of the form: $y = \sum_{n=0}^{\infty} a_n x^{r+n}$

Therefore, $y' = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1}$ and $y'' = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}$

Substitute the assumed answer form into the original equation

$$2x^2 \sum_{n=0}^{\infty} a_n (r+n)(r+n-1)x^{r+n-2} - x\sum_{n=0}^{\infty} a_n (r+n)x^{r+n-1} + (1+x)\sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

$$\sum_{n=0}^{\infty} 2a_n (r+n)(r+n-1)x^{r+n} - \sum_{n=0}^{\infty} a_n (r+n)x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + x\sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

$$\sum_{n=0}^{\infty} 2a_n (r+n)(r+n-1)x^{r+n} - \sum_{n=0}^{\infty} a_n (r+n)x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1} = 0$$

$$\sum_{n=0}^{\infty} 2a_n (r+n)(r+n-1)x^{r+n} - \sum_{n=0}^{\infty} a_n (r+n)x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} = 0$$

$$2a_0(r)(r-1)x^r - a_0(r)x^r + a_0x^r$$

$$+ \sum_{n=1}^{\infty} 2a_n (r+n)(r+n-1)x^{r+n} - \sum_{n=1}^{\infty} a_n (r+n)x^{r+n} + \sum_{n=1}^{\infty} a_n x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} = 0$$

$$(2r(r-1)-r+1)a_0x^r+\sum_{n=1}^{\infty} 2a_n(r+n)(r+n-1)x^{r+n}-a_n(r+n)x^{r+n}+a_nx^{r+n}+a_{n-1}x^{r+n}=0$$

Indicial equation (because a_0x^r cannot equal 0): $2r(r-1)-r+1$ (which is the same as if we solved the above Euler's Equation

(3)

Exponents at the singularity (roots to above equation): $r_1 = 1,\ r_1 = \frac{1}{2}$

(4)

Now return to equation above Indicial equation

$$(2r(r-1)-r+1)a_0x^r+\sum_{n=1}^{\infty}[2a_n(r+n)(r+n-1)-a_n(r+n)+a_n+a_{n-1}]x^{r+n}=0$$

"Set the coefficient of x = 0" — what does this mean?

$[2(r+n)(r+n-1)-(r+n)+1]a_n+a_{n-1}=0$

$$a_n=-\frac{a_{n-1}}{2(r+n)(r+n-1)-(r+n)+1} \text{ for } n\geq 1$$

Plug in each root independently to above equation

$r = r_1 = 1$

$$a_n=-\frac{a_{n-1}}{2(r+1)(r+1-1)-(r+1)+1}$$ — what happened to n's?

$$a_n=-\frac{a_{n-1}}{2(r+1)(r)-r}$$

$$a_n=-\frac{a_{n-1}}{2r^2+r}$$ — should vanish at n=0 for r=1.

$r = r_1 = \frac{1}{2}$

$$a_n=-\frac{a_{n-1}}{2(r+\frac{1}{2})(r+\frac{1}{2}-1)-(r+\frac{1}{2})+1}$$ — what happened to n's?

$$a_n=-\frac{a_{n-1}}{(2r+1)(r-\frac{1}{2})-r+\frac{1}{2}}$$

$$a_n=-\frac{a_{n-1}}{2r^2-r}$$

#4 (35 Points)

Suppose a mass weighs 64 lbs stretches a spring 4 ft. If there is no damping and the spring is stretched an additional foot and set in motion with an upward velocity of $\sqrt{8}$ ft/sec, find the equation of motion of the mass.

Note $\gamma^2 - 4km < 0$: $u(t) = e^{\wedge}(\frac{-\gamma t}{2m})$ (Acosμt +Bsinμt)

Solution

Mechanical Vibrations: $mu''(t) + \gamma u'(t) + ku(t) = F_{external}$; $m, \gamma, k, \geq 0$ $mg - kL = 0$

m = mass,
k = spring force proportionality constant,
γ = damping force proportionality constant

Weight = mg: $m = \frac{Weight}{g} = \frac{64}{32} = 2$

$mg - kL = 0$ implies $k = \frac{mg}{L} = \frac{64}{4} = 16$

Now using the Mechanical Vibrations equation

Since $\gamma = 0$ we get u(t) = (Acosμt +Bsinμt)

So our equation looks like

$2u''(t) + 16u(t) = 0$ simplify
$u''(t) + 16u(t) = 0$
$u(0) = 1$, $u'(0) = -\sqrt{8}$
$r^2 + 8 = 0$; $r^2 = -8$; $r = \sqrt{-8} = i\sqrt{8} = 0 + i\sqrt{8}$

So inserting back into the equation we get $u(t) = e^{\wedge}(\frac{-\gamma t}{2m})$ (Acosμt +Bsinμt) we get

$u(t) = (A\cos\sqrt{8}\,t + B\sin\sqrt{8}\,t)$
$u(0) = 1$: $1 = (A\cos(0) + B\sin(0)) = A$
$u'(t) = -\sqrt{8}\,A\cos\sqrt{8}\,t + \sqrt{8}\,B\sin\sqrt{8}\,t$
$u'(0) = -\sqrt{8}$: $-\sqrt{8} = -\sqrt{8}\,A\sin(0) + \sqrt{8}\,B\cos(0)$
$B = -1$

And the final solution is $u(t) = \cos\sqrt{8}\,t - \sin\sqrt{8}\,t$

3

1) **(40 points)**
Find the solution of the given initial value problem. Use the fact that it is an Euler equation to help you.

$x^2 y'' + 3xy' + 5y = 0, \qquad y(1) = 1, \qquad y'(1) = -1,$

Answer:

Substituting $y=x^r$ in the equation gives

? ?

$$x^r[r(r-1) + 3r + 5] = x^r(r^2 - 2r + 5).$$

We find the roots of that equation to be

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-4(1)(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm i4}{2} = -1 \pm i2.$$

Remembering that

$x^r = e^{r \ln x}$ when $x>0$ and r is real, we can use this equation to define x^r when r is complex. Then

$$x^{\lambda + i\mu} = x^\lambda e^{i\mu \ln x} = x^\lambda[\cos(\mu \ln x) + i \sin(\mu \ln x)], ^{*} \qquad x > 0.$$

However it is more straightforward to write the general solution in the form

$$y = c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x), \qquad x > 0.$$

this form can be argued from *.

So our solution is

$$y = c_1 x^{-1} \cos(2 \ln x) + c_2 x^{-1} \sin(2 \ln x),$$

By using our initial value data y(1)=1 and y'(1)=-1, we solve for the constants

$1 = c_1 1^{-1} \cos(2 \ln 1) + 0, \qquad c_1 = 1, \qquad c_2 = 0,$

Then our final solution is

$$y = x^{-1} \cos(2 \ln x).$$

2) Solve this equation using the Series Solution …

$$y'' + k^2x^2y = 0 \quad x_0 = 0 \quad k = constant$$

"by giving the first three iterates" 40 ~~15~~ point problem

Step one: differentiate the ~~equation~~ expression ($\sum_{n=0}^{\infty} a_n (x - x_0)$) twice

Once: $n\sum_{n=0}^{\infty} a_n (x - x_0)^{n-1}$ where $x_0 = 0$ which will be left out.

Twice: $n(n-1)\sum_{n=0}^{\infty} a_n x^{n-2}$

+12 pts

Then put the series notation into the equation.

$$n(n-1)\sum_{n=0}^{\infty} a_n x^{n-2} + k^2x^2\sum_{n=0}^{\infty} a_n (x) = 0$$

Then you have to transfer the terms so that they can add…
The first equation needs an n+4 inserted into each n

$(n+4)(n+3)\sum_{n=0 \; -4}^{\infty} a_{n+4} x^{n+2}$ Where the other equation becomes (this when you smash the x^2 in it)

$k^2\sum_{n=0}^{\infty} a_n (x^2)$ then we put the two together and then we can solve for a_{n+4}

+5ish pts.

so the equation becomes:

$$(n+4)(n+3)\sum_{n=0 \; -4}^{\infty} a_{n+4} x^{n+2} + k^2\sum_{n=0}^{\infty} a_n (x^2) = 0$$

Solving for a_{n+4} looks like this…

$(n+4)(n+3)a_{n+4} + k^2 a_n = 0$

then becomes;

o,k,

$$a_{n+4} = \frac{-k^2 a_n}{(n+4)(n+3)}$$

so to get the first three iterates by plugging in n=0,1,2,3

+16 pts

$a_4 = \frac{-k^2 a_0}{4\times3}$ $\quad a_5 = \frac{-k^2 a_1}{5\times4}$ $\quad a_6 = \frac{-k^2 a_2}{6\times5}$ where $a_2 = 0 = a_3$ so the ones we should do is a_8 and a_9

$a_8 = \frac{-k^2 a_4}{8\times7}$ $\quad a_9 = \frac{-k^2 a_5}{9\times8}$ where $a_4 = \frac{-k^2 a_0}{4\times3}$ and $a_5 = \frac{-k^2 a_1}{5\times4}$ making

$a_8 = \frac{k^4 a_0}{8\times7\times4\times3}$ $\quad a_9 = \frac{k^4 a_1}{9\times8\times5\times4}$

combining the whole kit and caboodle gives;

+5 pts.

$y_1(x) = 1 - \frac{k^2x^4}{4\times3} + \frac{k^4x^8}{8\times7\times4\times3} + \cdots$ "the x is to the 4th power because a2 is zero which is x^2 and a3 =0 where x^3"

$y_2(x) = x - \frac{k^2x^5}{5\times4} + \frac{k^4x^9}{9\times8\times5\times4} + \cdots$

3) **Problem** –

30 pt problem

Find the solution of the given initial value problem:

$$y'' + 4y' + 6 = 0, y(0) = 2, y'(0) = 1$$

Solution –

We begin by letting $y = e^{rt}$ which gives us $y' = re^{rt}$ and $y'' = r^2 e^{rt}$. Then we put these into the initial value problem and factor out the e^{rt} which gives us:

$$r^2 + 4r + 8 = 0.$$

We then find the roots of this equation for the solution. Using the quadratic equation we get:

$$r = \frac{-4 \pm \sqrt{16 - 32}}{2} \text{ or } r = -2 \pm 2i$$

The solution equation is given by $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$. We put our different values for r into this equation and get the solution to be:

$$y = c_1 e^{(-2+2i)t} + c_2 e^{(-2-2i)t}$$

We can then use Euler's formula, $e^{it} = \cos t$ and the solution is then:

$$y = c_1 e^{-2t} \cos 2t + c_2 e^{-2t} \sin 2t$$

To find our values c_1 and c_2, we now differentiate and then plug the initial values in our equations.
Differentiating we have:

$$y' = -2c_1 e^{-2t} \cos 2t - 2c_1 e^{-2t} \sin 2t - 2c_2 e^{-2t} \sin 2t + 2c_2 e^{-2t} \cos 2t$$

When we plug in the values we are left with the two equations:

$$2 = c_1$$
$$1 = -2c_1 + 2c_2$$

Giving us $c_1 = 2$ and $c_2 = \frac{5}{2}$ and a final solution of:

$$y = 2e^{-2t} \cos 2t + \frac{5}{2} e^{-2t} \sin 2t$$

4)A 16lb bowling ball stretches a spring 8/9 ft. If the bowling ball is moved an additional 6 inches upwards from equilibrium position and set in motion with a downward velocity of 1ft/sec. Assuming there is no damping, find the displacement u(t) at any time t. (40 points)

Solution:

The first thing that we must do is determine the initial value problem by determining the coefficients m and k of the general equation mu"(t)+ku = 0.

We need to change our weight in pounds to mass by the following conversion:

$m = \frac{W}{g}$ which gives us a mass of $m := \frac{16\text{lb}}{32\text{ft}\cdot\text{s}^{-1}}$ $m = 0.5\frac{\text{lb}\cdot\text{s}}{\text{ft}}$.

In order to find the value of k we use the relationship

$$k = \frac{mg}{L} \qquad k := \frac{16\text{lb}}{\frac{8}{9}\text{ft}} \quad \text{so} \quad k = 18\frac{\text{lb}}{\text{ft}}.$$

The first initial condition are based on the additional length stretched from the equilibrium position which is 6 inches or .5ft, but it is negative because upward direction is the negative direction. The other initial condition is based on the velocity that the mass is set in motion which is 1ft/sec.

At this point we have enough information in order to determine the initial value problem which is,

.5u"(t) + 18u(t) = 0, u(t) = -.5 and u'(t) = 1.

The natural frequency is given by the equation,

$\omega_0 := \sqrt{\frac{k}{m}}$ o.k. so our value of ω is $\omega_0 := \sqrt{\frac{18}{.5}}$ and $\omega_0 = 6$.

This gives us our general solution u(t) and its derivative u'(t) of

$$u(t) = c_1\cdot\cos(6t) + c_2\cdot\sin(6t) \quad \text{and} \quad u'(t) = -6\cdot c_1\cdot\cos(6t) + 6\cdot c_2\cdot\sin(6t.$$

By applying our initial conditions we have,

u(0) = -1/2 = c1 so $c_1 = \frac{-1}{2}$

and u'(0) = 1 = 6(c2)cos(0) = 6(c2) so $c_2 = \frac{1}{6}$.

This gives us our equation for displacement of the bowling ball at any time t as,

$\boxed{u(t) = \frac{-1}{2}\cdot\cos(6t) + \frac{1}{6}\cdot\sin(6t)}$ in feet.

5

1. (40pts) Use the method of variation of parameters to find a particular solution Y(t) to the differential equation:

$$4y'' - 4y' + y = 16e^{(t/2)};$$

$$\text{which is } y'' - y' + (1/4)y = 4e^{(t/2)};$$

Answer) First find the general solution to ~~this equation.~~ ~~start~~ the homogeneous version of this equation,

The "roots of the differential equation" are found with the homogeneous equation.

$$y'' - y' + y/4 = 0;$$

To find the roots we can use the coefficients and say:

$$r^2 - r + 1/4 = 0;$$

Solving for r we get:

$$(r - ½)(r - ½) = 0;$$

$$r = ½;$$

Since we have a double root, the general ~~equation~~ solution is:

$$y(t) = Ce^{(t/2)} + Dte^{(t/2)} + Y(t);$$

From theorem 3.7.1 we know :

a & b? - meaning of them here?

$$Y(t) = -a(t)\int((b(s)g(s))/W(a,b)(s))ds + b(t)\int((a(t)g(s))/W(a,b)(s))ds$$

We will choose the integrals to be from 0 to t, a(t) is e^(t/2) and b(t) is te^(t/2).

W(a,b)(t) is the Wronskian of functions a and b

$$W(a,b)(t) = ab' - a'b;$$

$$W(e^{(t/2)}, te^{(t/2)}) = e^{(t/2)}*(e^{(t/2)} + t*(1/2)*e^{(t/2)}) - te^{(t/2)}*(1/2)*e^{(t/2)}$$

$$= e^t;$$

So,

$$Y(t) = -e^{(t/2)}\int((se^{(s/2)}*4e^{(s/2)})/e^s)ds + te^{(t/2)}\int((e^{(s/2)}*4e^{(s/2)})/e^s)ds;$$

Solving the integrals form 0 to t we get:

Y(t) = -e^(t/2)∫4sds + te^(t/2)∫4ds;

or Y(t) = -2*t^2*e^(t/2) + 4*t^2*e(t/2) ;

or Y(t) = 2(t^2)e^(t/2) ; check?

This is the particular solution Y(t).

2)(40pts) Method of Undetermined Coefficients
Problem:
Find the particular solution to the following equation using the method of undetermined coefficients

$$y'' - 3y' - 4y = 3e^{2x} + 2\sin(x) - 8e^{-x}.$$

Solution
First we notice that we can split the equation in to the following three equations:

(1) $y_1'' - 3y_1' - 4y_1 = 3e^{2x}$
(2) $y_2'' - 3y_2' - 4y_2 = 2\sin(x)$
(3) $y_3'' - 3y_3' - 4y_3 = -8e^{-x}$.

where then $y = y_1 + y_2 + y_3$

Next we find the roots of the characteristic equation $r^2 - 3r - 4$.

$r = -1$ and $r = 4$.

We will now proceed to find particular solutions for equations 1- 3 beginning with equation one.

Particular solution to Equation (1):

The particular solution is given as

$$y_1 = Ae^{2x}.$$

If we plug it into the equation (1) we get

$$4Ae^{2x} - 6Ae^{2x} - 4Ae^{2x} = 3e^{2x},$$

Which implies A = -1/2, or

$$y_1 = \frac{-1}{2} e^{2x}.$$

Particular solution to Equation (2):

The particular solution is given as

$$y_2 = A\cos(x) + B\sin(x).$$

If we plug it into the equation (2), we get

$$(-A\cos(x) - B\sin(x)) - 3(-A\sin(x) + B\cos(x)) - 4(A\cos(x) + B\sin(x)) = 2\sin(x),$$

which implies

$$(-A - 3B - 4A)\cos(x) + (-B + 3A - 4B)\sin(x)$$

and

$$-5A - 3B = 0$$
$$3A - 5B = 2.$$

Solving for A and B give

$$A = \frac{3}{17} \text{ and } B = -\frac{5}{17}$$

and

$$y_2 = \frac{3}{17}\cos(x) + -\frac{5}{17}\sin(x).$$

The Particular solution to Equation (3):

The coefficent of the exponent is equal to one of th roots therefore

$y_3 = x^1(Ae^{-x})$ is the particular solution.

If we plug into the equation (3), we get

$$A(x-2)e^{-x} - 3A(-x+1)e^{-x} - 4Axe^{-x} = -8e^{x},$$

Which implies $A = \frac{8}{5}$ and

$$y_3 = \frac{8}{5} x e^{-x}.$$

Finally a particular solution to the original equation is

$$y = \frac{-1}{2}e^{2x} + \frac{3}{17}\cos(x) + -\frac{5}{17}\sin(x) + \frac{8}{5}xe^{-x}.$$

3) (30pts)Determine $\varphi''(X_0)$, $\varphi'''(X_0)$, and $\varphi^4(X_0)$ for the given point X_0 if $y = \varphi(X_0)$ is a solution of the given initial value problem.

$3y''+4y'-9y=0$; $y(1)=3$, $y'(1)=6$, $X_0=1$

which notation?

Solution:

By substitution:
$3y''(1)+4(6)-9(3)=0 \Rightarrow 3y''(1)=3 \Rightarrow y''(1)=1$
Taking the derivative of both sides yields:
$3y'''+4y''-9y'=0$
By substitution:

$$3y'''(1)+4y''(1)-9y'(1)=0 \Rightarrow 3y'''(1)=50 \Rightarrow y'''(1)=\frac{50}{3}$$

Taking the derivative of both sides again yields:
$3y^4+4y'''-9y''=0$
By substitution:

$$3y^4(1)+4y'''(1)-9y''(1)=0 \Rightarrow 3y^4(1)=-\frac{173}{3} \Rightarrow y^4(1)=-\frac{173}{9}$$

4) (40pts) Find the first four terms in each portion of the series solution around $x_0 = 0$ for the following differential equation.

$(x^2+1)y''-4xy'+6y=0$

Solution

We know that:

$p(x)=x^2+1 \qquad\qquad p(0)=1\neq 0$

So $x_0 = 0$ is an ordinary point for this differential equation. We first need the solution and its derivatives,

$$y(x)=\sum_{n=0}^{\infty}a_n x^n \qquad y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1} \qquad y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}$$

Plug these into the differential equation.

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 4x\sum_{n=1}^{\infty} na_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

Now, break up the first term into two so we can multiply the coefficient into the series and multiply the coefficients of the second and third series in as well.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 4na_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

We will only need to shift the second series down by two to get all the exponents the same in all the series.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} 4na_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

Start the third series at *n=0* because that term is just zero. Like wise the terms in the first series are zero for both *n=1* and *n=0* and so we could start that series at *n=0*. All the series will now start at *n=0* and we can add them up without stripping terms out of any series.

$$\sum_{n=0}^{\infty} \left[n(n-1)a_n + (n+2)(n+1)a_{n+2} - 4na_n + 6a_n\right]x^n = 0$$

$$\sum_{n=0}^{\infty} \left[\left(n^2-5n+6\right)a_n + (n+2)(n+1)a_{n+2}\right]x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n-2)(n-3)a_n + (n+2)(n+1)a_{n+2}\right]x^n = 0$$

Now set coefficients equal to zero.

$$0 = (n-2)(n-3)a_n + (n+2)(n+1)a_{n+2}, \quad n = 0,1,2,\ldots$$

Solving this gives,

$$a_{n+2} = -\frac{(n-2)(n-3)a_n}{(n+2)(n+1)}, \qquad n = 0,1,2,\ldots$$

Now, we plug in values of *n*.

$$n=0: \qquad a_2 = -3a_0$$

$$n=1: \qquad a_3 = -\frac{1}{3}a_1$$

$$n=2: \qquad a_4 = -\frac{0}{12}a_2 = 0$$

$$n=3: \qquad a_5 = -\frac{0}{20}a_3 = 0$$

Now, from this point on all the coefficients are zero. In this case both of the series in the solution will terminate.

The solution is,

$$y(x) = a_0\{1-3x^2\} + a_1\left\{x - \frac{1}{3}x^3\right\}$$ - cool

6

Problem # 1

Sea world, in their large tank, uses a system to stimulate the growth of algae. The algae grow at an increased rate when a certain frequency is transmitted through the water. The way the Sea World employees generate this frequency is by the oscillation of a tungsten weight attached to a spring. The 5 kilogram mass is hung on a spring under water, and stretches the spring 10 centimeters. The weight is pulled an additional 5 centimeters and released with an initial velocity of zero. The water around the tungsten mass exerts a viscous resistance of 20 Newtons when the mass has a velocity of 2 meters/second.

a) (10 pts.) Write a second order linear differential equation that represents this mechanism.

Solution

Since this is not a forced vibration, it is modeled using a homogenous equation. The general formula for a vibration system is $mu'' + \gamma u' + ku = F(x)$. We know from the given data that $m = 5\ \text{Kg}$ and $\ell = 10\ \text{cm} = .10\ \text{m}$. Thus $k = \frac{m}{\ell} = \frac{5\ \text{Kg}}{.10\ \text{m}} = 50\ \frac{\text{Kg}}{\text{m}}$. The viscosity constant is defined as force per velocity, and so $\gamma = \frac{20\ \text{N}}{2\ \frac{\text{m}}{\text{s}}} = 10\ \frac{\text{N} \cdot \text{s}}{\text{m}}$. As stated before, the vibration is not being forced, and so $F(t) = \cos 2t$. The initial position $U(t_0)$ is 5 cm or .05 m. Since velocity at $t_0 = 0$ is 0, $U'(t_0) = 0$. Substituting into the general equation, we obtain this equation:

$$5u'' + 10u' + 50u = \cos 2t \tag{1.1}$$

b) (40 pts.) Find a solution to the initial value problem in part a.

Solution

why? did you specify variation of parameters?

First we need to solve the homogenous version of this equation. Make the ansatz that $u = c_1 e^{rt} + c_2 e^{rt}$. Applying this and its derivatives to the differential equation, we can develop the characteristic polynomial

$$r^2 + 2r + 10 = 0 \tag{1.5}$$

Using the quadratic equation to solve for the roots of (1.5), we find that $r = -2 \pm 6i$. We know from in-class and in-book derivations, that this gives us the general equation for the homogenous form $U_H(t) = c_1 e^{-2t} \cos(6t) + c_2 e^{-2t} \sin(6t)$.

Now we can attempt to solve for the particular solution using the method of undetermined coefficients. We make the ansatz that the particular solution, $U_p(t)$, is of the form:

$$U_p(t) = A\cos(2t) + B\sin(2t) \tag{1.2}$$

$$U_p'(t) = -2A\sin(2t) + 2B\cos(2t) \tag{1.3}$$

$$U_p''(t) = -4A\cos(2t) - 4B\sin(2t) \tag{1.4}$$

Substituting (1.2), (1.3), and (1.4) into (1.1), we get the following

$$(-20A + 20B + 50A)\cos(2t) + (-20B - 20A + 50B)\sin(2t) = \cos(2t) \tag{1.5}$$

1.5 yields a system of equations that we can solve using standard algebraic methods and find that $A = \frac{3}{13}$ and $B = \frac{2}{13}$. Thus the general solution becomes

$$U(t) = c_1 e^{-2t}\cos(6t) + c_2 e^{-2t}\sin(6t) + \frac{3}{13}\cos(2t) + \frac{2}{13}\sin(2t) \tag{1.6}$$

Now we turn our attention to c_1 and c_2. First, when t is 0, then (1.6) becomes $U(t) = .05 = c_1(1)\,(1) + c_2(1)(0) + 3/13(1) + 2/13(0)$, or $\frac{-47}{260} = c_1$. Next, when t is 0, then we can say that

$U'(t) \overset{?}{=} -2\frac{-47}{260}(1)(1) - 6\frac{-47}{260}(1)(0) - 2c_2(1)(0) + 6c_2(1)(1) - \frac{6}{13}(0) + \frac{4}{13}(1)$ or $\frac{114}{65} = c_2$.

Substituting back into the general equation, or answer is given by (1.7)

$$U(t) = \frac{-47}{260}e^{-2t}\cos(6t) + \frac{114}{65}e^{-2t}\sin(6t) + \frac{3}{13}\cos(2t) + \frac{2}{13}\sin(2t) \tag{1.7}$$

Problem # 2

same as problem 1

For a show at SeaWorld a worker puts a 4 pound fish on a spring, and dangles it into the dolphin tank. The fish stretches the spring 2 inches. To get the Dolphins excited the worker pulls the fish down another 6 inches to make the fish bounce up and down. The water in the tank exerts a viscous resistance of 6 lb when the fish is moving 3 ft/sec. SeaWorld's research and development team needs the initial value problem to distinguish the difference in behavior patterns in wild and domestic dolphins.

(20 pts.) Find the initial value problem with the initial conditions.

Solution

For the purposes of this problem, the acceleration of gravity can be estimated so that $g = 32\dfrac{ft}{\sec^2}$. English pounds are a measure of force, and so mass is obtained by dividing weight by the force of gravity, g. Thus the mass m is given by $m = \dfrac{w}{g} = \dfrac{4lb.}{32\dfrac{ft}{\sec^2}} = \dfrac{1}{8}slug$. The spring constant is found by dividing the length of the spring when weighted down into the weight applied. We know that when a $4lb.$ is applied to the spring, then the spring stretches by $2in.$ Hence the spring constant k is given by $k = \dfrac{F}{\ell} = \dfrac{4lb.}{\dfrac{2}{12}in.} = 24\dfrac{lb}{ft}$.

Substituting into the general form of this type of equation, we get $\dfrac{1}{8}U''(t) + 2U'(t) + 24U(t) = 0$ which simplifies to:

$$U''(t) + 16U'(t) + 192U(t) = 0\text{, where } U(0) = \frac{1}{2} \text{ and } U'(0) = 0$$

Problem # 3

For the following problems use the following equation:

$$xy'' + (2-x)y' - 7y = 0 \tag{3.1}$$

a) (15 pts.) List any singular points for the problem, and indicate whether they are regular or not.

Solution

To find the singular points, we need to have the equation in a form resembling $y'' + p(x)y' + q(x)y$ $=0$. To obtain this, we divide the equation by x and obtain the following equation:

$$y'' + \frac{(2-x)}{x}y' - \frac{7}{x}y = 0 \tag{3.2}$$

Now it is plain that the only points of discontinuity (singular points) occur when $x = 0$. To determine if these are regular points, we take the limits of $xp(x)$ and $x^2q(x)$ as x approaches 0. Thus $\lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{(2-x)}{x} = 2$ and likewise, $\lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\left(-\frac{7}{x}\right) = 0$. Since both of the limits are finite, then we know that the point $x = 0$ is a <u>regular</u> singular point.

b) (10 pts.) Determine what values of r (if any) are needed in order to find a non-trivial solution to the problem (i.e. – find the roots of the indicial equation).

Solution

We know from the previous part that $\lim_{x\to 0} xp(x) = 2$. This not only proves the regularity of the singular point, but provides us with the term p_0 which can be used to build an Euler equation that, in combination with a series solution ansatz, will help us find appropriate values of r for finding non-trivial solutions. The equation we'll use is

$$x^2y'' + xp_0y' = 0 \tag{3.3}$$

From experience we know that by making the ansatz $y = x^r$, we can determine r from $r(r-1) + p_0r + q_0 = 0$. Solving for r, we find that

$$r = 0, 2.$$

c) (25 pts.) Determine a recurrence relation in r and n, and then use if to find the general series solution representation. Show at least five terms in open form of the solution, and if possible represent the series in closed form.

Solution

We start by making the ansatz $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. Substituting y and its derivatives into (3.1) we get the following equation

$$x\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{n+r-2} + (2-x)\sum_{n=0}^{\infty}(r+n)a_n x^{n+r-1} - 7\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

which can be simplified (after combination and bound shifting) to:

$$\sum_{n=0}^{\infty}\left\{\left[(r+n+1)(r+n)+2(r+n+1)\right]a_{n+1} - \left[(r+n)+7\right]a_n\right\}x^{n+r} = 0$$

since the recurrence relation is found by requiring the coefficient of each term to be zero, we find that the recurrence relation is

$$\frac{\left[(r+n)+7\right]a_n}{\left[(r+n+1)(r+n)+2(r+n+1)\right]} = a_{n+1} \tag{3.4}$$

Now we can use (3.4) to find the first five terms of the series solution for each value of r. The different series produced by the differing values of r will produce two linearly independent solutions.

For $r = 0$, $a_1 = \frac{7}{2}a_0$, $a_2 = \frac{8\cdot 7}{6\cdot 2}a_0$, $a_3 = \frac{9\cdot 8\cdot 7}{12\cdot 6\cdot 2}a_0$, $a_4 = \frac{10\cdot 9\cdot 8\cdot 7}{20\cdot 12\cdot 6\cdot 2}a_0$ which means

$$y_1 = \left[1 + \frac{7}{2}x + \frac{8\cdot 7}{6\cdot 2}x^2 + \frac{9\cdot 8\cdot 7}{12\cdot 6\cdot 2}x^3 + \frac{10\cdot 9\cdot 8\cdot 7}{20\cdot 12\cdot 6\cdot 2}x^4 + \ldots\right] \tag{3.5}$$

For $r = 2$, $b_1 = \frac{9}{12}b_0$, $b_2 = \frac{10\cdot 9}{20\cdot 12}b_0$, $b_3 = \frac{11\cdot 10\cdot 9}{30\cdot 20\cdot 12}b_0$, $b_4 = \frac{12\cdot 11\cdot 10\cdot 9}{42\cdot 30\cdot 20\cdot 12}b_0$ thus

$$y_2 = \left[1 + \frac{9}{12}x + \frac{10\cdot 9}{20\cdot 12}x^2 + \frac{11\cdot 10\cdot 9}{30\cdot 20\cdot 12}x^3 + \frac{12\cdot 11\cdot 10\cdot 9}{42\cdot 30\cdot 20\cdot 12}x^4 + \ldots\right] \tag{3.6}$$

Since no closed form is obvious, our general solution is simply defined as:

$$y_g = a_0 y_1 + b_0 y_2$$

Problem # 4

Use the following equation for parts a, b, and c:

$$4y'' - 12y' + 9y = e^{3t} \tag{4.1}$$

a) (10 pts.) For the equation below find a fundamental set of solutions for the corresponding homogeneous equation of (4.1).

Solution:

For the corresponding homogeneous equation we can develop the characteristic polynomial $4r^2 - 12r + 9 = 0$ which factors into $(2r-3)^2$ therefore the roots to the characteristic polynomial are $\frac{3}{2}$, and $\frac{3}{2}$ because they are both real we have a fundamental set of the form (after applying reduction of order to resolve the duplication):

$$y_1 = e^{\frac{3}{2}t} \text{ and } y_2 = te^{\frac{3}{2}t}$$

b) (10 pts.) Now find a particular solution for this differential equation using variation of parameters and the set of solutions you found.

Solution:

Using Theorem 3.7.1 in the book we have — so $Y = ?$

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}ds + y_2\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)}ds \tag{4.2}$$

where $W = e^{\frac{6}{2}t} + \frac{3}{2}te^{\frac{6}{2}t} - \frac{3}{2}te^{\frac{6}{2}t} = e^{3t}$

c) (15 pts.) Find the general solution of the differential equation and verify it using method of undetermined coefficients.

Solution:

Applying (4.2) we have $Y(t) = \frac{-e^{\frac{3}{2}t}}{4}\int_{t_0}^{t} se^{\frac{3}{2}s}ds + \frac{te^{\frac{3}{2}t}}{4}\int_{t_0}^{t} e^{\frac{3}{2}s}ds = \frac{1}{9}e^{3t}$

which makes our general solution $y(t) = c_1y_1(t) + c_2y_2(t) + \frac{1}{9}e^{3t}$

Verify with the Method of undetermined coefficients. First, make the ansatz:

$$y(t) = Ae^{3t} \tag{4.3}$$

$$y'(t) = 3Ae^{3t} \tag{4.4}$$

$$y''(t) = 9Ae^{3t} \tag{4.5}$$

By inserting eqns. (4.3), (4.4), and (4.5) into (4.1), we find that

$$g(t) = e^{3t} = (36 - 36 + 9)Ae^{3t}.$$

Solving for A, we find that $A = \frac{1}{9}$, which when applied to (4.3) gives the particular solution:

$$y_p(t) = \frac{1}{9}e^{3t}$$

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(1) Find the general solution of the given differential equation

$$4y''-12y'+(\alpha+3)y=0$$

Also, find the solution when α = -3, 6, 15

35 points

Solution

- divide the entire problem by 4 to get y'' by itself

$$y''-3y'+\frac{(\alpha+3)}{4}y=0$$

- turn the DE into a polynomial to solve for the roots (using the quadratic formula)

$$r^2-3r+\frac{(\alpha+3)}{4}=0$$

$$r=\frac{3\pm\sqrt{9-4\left(\frac{\alpha+3}{4}\right)}}{2}$$

- the general solution is as follows

$$y=c_1e^{\frac{3+\sqrt{6-\alpha}}{2}t}+c_2e^{\frac{3-\sqrt{6-\alpha}}{2}t}$$

- Plug in for different values of α

- α = -3:

$$r=0,\ 3$$

$$y=c_1+c_2e^{3t}$$

- α = 6:

$$r=\frac{3}{2}$$

$$y=c_1e^{\frac{3}{2}t}+c_2te^{\frac{3}{2}t}$$

- α = 15:

$$r=\frac{3\pm 3i}{2}$$

$$y=c_1e^{\frac{3}{2}t}\cos\frac{3}{2}t+c_2e^{\frac{3}{2}t}\sin\frac{3}{2}t$$

2)

Find the general solution of the given differential equation.

$y'' + 4y = 4\cos 2t$

40 points

Solution:

Solve the homogenous equation: $y'' + 4y = 0$

The characteristic equation is $r^2 + 4 = 0$

Solve for r:

$r^2 = -4$

$r = \pm 2i$

Therefore, $\mu = 2$ and $\lambda = 0$

The general solution to the homogenous equation is:

$y = c_1 \sin 2t + c_2 \cos 2t$

Let the general solution to the non-homogenous differential equation be $y(t) = y + Y$

Y is found by **The Method of Undetermined Coefficients**

Assume Y is of the form $A\sin 2t + B\cos 2t$, but this is a repetition of y

Therefore, assume Y is of the form $At\sin 2t + Bt\cos 2t$

Then, $Y' = 2At\cos 2t + A\sin 2t - 2Bt\sin 2t + B\cos 2t$

And, $Y'' = 2A\cos 2t - 4At\sin 2t + 2A\cos 2t - 2B\sin 2t - 4Bt\cos 2t - 2B\sin 2t$

Using $Y'' + 4Y = 4\cos 2t$,

$2A\cos 2t - 4At\sin 2t + 2A\cos 2t - 2B\sin 2t - 4Bt\cos 2t - 2B\sin 2t + 4At\sin 2t + 4Bt\cos 2t = 4\cos 2t$
$4A\cos 2t - 4Bt\cos 2t - 4B\sin 2t - 4At\sin 2t + 4At\sin 2t + 4Bt\cos 2t = 4\cos 2t$
$4A\cos 2t - 4B\sin 2t = 4\cos 2t$

Therefore, $A = 1$ and $B = 0$

$Y = 1t\sin 2t + 0t\cos 2t$

$Y = t\sin 2t$

The general solution to the non-homogenous differential equation is $y(t) = y + Y$:

$\boxed{y(t) = c_1 \sin 2t + c_2 \cos 2t + t\sin 2t}$

3)

Assume we have a 1 kg mass which stretches a certain spring 40 cm. The mass is in a system with a viscous damper which exerts a force of 3 Newtons when the velocity of the mass is 3m/s. The mass is then released from the equilibrium position with an initial velocity of 1 m/s.

a) What is the natural frequency (ω_o) of the undamped spring/mass system?
b) What is an equation describing the position of the mass at any time t?

35 points

First we assign the different parameters to their variables in the general form of the equation.

$\mathbf{m}\, u''(t) + \gamma\, u'(t) + k\, u(t) = F$

$\mathbf{m}=1$ kg $\quad \gamma=3N/(3m/s)=1Ns/m \quad k=1kg/.4m=2.5kg/m$

F=0 (there is no driving force)

For part a) we know that $\omega_o=(k/m)^{1/2}=(2.5)^{1/2}$

$\boxed{\text{a) } \omega_o=(2.5)^{1/2}}$

In part b) we assume the solution is of the form $u=e^{rt}$

We take the first and second derivatives

$u' = r\, e^{rt}$

$u'' = r^2\, e^{rt}$

We plug these back into the original equation

$\mathbf{m}\, r^2 e^{rt} + \gamma\, r\, e^{rt} + k\, e^{rt} = 0$

We factor out the e^{rt}

$(\mathbf{m}\, r^2 + \gamma\, r + k)\, e^{rt} = 0$

Because $e^{rt} \neq 0$

$(\mathbf{m}\, r^2 + \gamma\, r + k) = 0$

We solve for r

$r=\{-\gamma\pm(\gamma^2-4\mathbf{m}k)^{1/2}\}/2\mathbf{m}$

$r=\{-1\pm(1-4(1)(2.5))^{1/2}\}/2$

$r= -1/2\pm 3i/2$

For this equation $\lambda= -1/2$ and $\mu=3/2$

Thus

$u= e^{-(1/2)t}\ \{ C_1\cos(3/2\ t)+ C_2\sin(3/2\ t)\}$

Now we have to solve for C_1 and C_2

We know that $u'=v$ and $v(0)=1m/s$

Also $u(0)=0$

$u(0)=0=1(C_1)$

Therefore $C_1=0$

$u'= -1/2\ e^{-(1/2)t}\{-3/2\ C_1\sin(3/2\ t)+ 3/2\ C_2\cos(3/2\ t)$

$u'(0)=1= -1/2(3/2\ C_2)$

$C_2= -4/3$

Thus our final answer for b) is

$\boxed{\text{b) } u= -4/3e^{-(1/2)t}\sin(3/2\ t)}$

4. The following equation has a regular singular point at x=0. Determine the roots of the characteristic equation and find the first four terms and the general term the solution using the larger root.

for the associated Euler equation, i.e. the indicial eq.

$$x^2y''+xy'+(x-2)y=0$$

40 points

Solution:

The answer will be in the form: $y=\sum_{n=0}^{\infty}a_nx^{n+r}$

but this can be thought of as: $y=\sum_{n=-\infty}^{\infty}a_nx^{n+r}$, as long as $a_n=0$ for n<0

So we differentiate this twice and then plug them into the original equation:

$$y'=\sum_{n=-\infty}^{\infty}(n+r)a_nx^{n+r-1}$$

$$y''=\sum_{n=-\infty}^{\infty}(n+r-1)(n+r)a_nx^{n+r-2}$$

$$\sum_{n=-\infty}^{\infty}x^2(n+r-1)(n+r)a_nx^{n+r-2}+x(n+r)a_nx^{n+r-1}-2a_nx^{n+r}+xa_nx^{n+r}=0$$

We simplify this by multiplying in the x's and on the last term, the exponent of x becomes n+r+1 so we get it in the same form as the others by subtracting 1 from n in the exponent of x and the subscript of a_n getting:

$$\sum_{n=-\infty}^{\infty}(n+r-1)(n+r)a_nx^{n+r}+(n+r)a_nx^{n+r}-2a_nx^{n+r}+a_{n-1}x^{n+r}=0$$

Now we factor:

$$\sum_{n=-\infty}^{\infty}\{[(n+r-1)(n+r)+(n+r)-2]a_n+a_{n-1}\}x^{n+r}=0$$

In order for this to be equal to zero, all the coefficients of x^{n+r} must be zero so:

$$[(n+r-1)(n+r)+(n+r)-2]a_n+a_{n-1}=0$$

$$[(n+r-1)(n+r)+(n+r)-2]a_n=-a_{n-1}$$

So for n=0:

$$[r(r-1)+r-2]a_0=-a_{-1}$$

But for n<0, $a_n=0$ so a_{-1}=0:

$$[r(r-1)+r-2]a_0=0$$

We don't want a_0=0 because that gives us a zero solution so we set the rest equal to zero:

$$r(r-1)+r-2=0$$

$$r^2-2=0$$

$r = \pm\sqrt{2}$

So now we want to find the solution for the larger root ($r = \sqrt{2}$) so we plug that in to the equation for a_n:

$\left[(n+\sqrt{2}-1)(n+\sqrt{2})+(n+\sqrt{2})-2\right]a_n = -a_{n-1}$

Which simplifies to:

$\left[(n+\sqrt{2})^2 - 2\right]a_n = -a_{n-1}$

So for $n \geq 1$:

$$a_n = \frac{-a_{n-1}}{(n+\sqrt{2})^2 - 2}$$

And a_0 is free

Now we find a_1, a_2, and a_3 in order to find the first four terms of the solution and then find the general term:

$$a_1 = \frac{-a_0}{(1+\sqrt{2})^2 - 2} = \frac{-a_0}{1+2\sqrt{2}}$$

$$a_2 = \frac{-a_1}{(2+\sqrt{2})^2 - 2} = \frac{a_0}{(1+2\sqrt{2})(4+4\sqrt{2})}$$

$$a_3 = \frac{-a_2}{(3+\sqrt{2})^2 - 2} = \frac{-a_0}{(1+2\sqrt{2})(4+4\sqrt{2})(9+6\sqrt{2})}$$

$$a_n = \frac{(-1)^n a_0}{(1+2\sqrt{2})(4+4\sqrt{2})(9+6\sqrt{2})\cdots(n^2+2n\sqrt{2})}$$

Now we plug these back into our original equation for the solution ($y = \sum_{n=-\infty}^{\infty} a_n x^{n+r}$) setting a_0=1 and we get one solution for the differential equation:

$$y_1 = x^{\sqrt{2}}\left[1 + \frac{-x}{(1+2\sqrt{2})} + \frac{x^2}{(1+2\sqrt{2})(4+4\sqrt{2})} + \frac{-x^3}{(1+2\sqrt{2})(4+4\sqrt{2})(9+6\sqrt{2})} + \ldots + \frac{(-1)^n x^n}{(1+2\sqrt{2})(4+4\sqrt{2})(9+6\sqrt{2})\cdots(n^2+2n\sqrt{2})} + \ldots\right]$$

8

1. Find ~~the~~ a fundamental set of solutions for the equation with the given initial condition.

$$2y'' + 2y' - 4y = 0\,,\; t_o = 0$$

20 points

Once found, show that the solutions are in fact a fundamental set by determining their Wronskian.

15 points

<u>Solution</u>:

Theorem 3.2.5 states

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

– not a complete statement of a theorem.

If p(t) and q(t) are both continuous on an open interval I, and the interval contains the initial point. There exist ~~equations~~ solutions y_1 and y_2 that satisfy the following initial conditions.

$$y_1(t_o) = 1, \qquad y_1'(t_o) = 0$$

and

$$y_2(t_o) = 0, \qquad y_2'(t_o) = 1$$

That form ~~the~~ a fundamental solution set.

$$2y'' + 2y' - 4y = 0$$

Reduces to the following

$$y'' + y' - 2y = 0$$
$$p(t) = 1$$
$$q(t) = -2$$

Since this equation is homogeneous with constant coefficients, both y_1 and y_2 will be of the form $y = e^{rt}$.

To solve this we must solve the characteristic equation as follows,

$$r^2 + r - 2 = 0$$
$$(r+2)(r-1) = 0$$
$$r = 1, -2$$

We can use these values to solve for both y_1 and y_2

$$y_1(t) = C_1 e^t + C_2 e^{-2t}$$
$$y_1(t_o) = 1$$
$$y_1(0) = C_1 e^0 + C_2 e^{-2(0)}$$
$$1 = C_1 + C_2$$
$$y_1'(t) = C_1 e^t - 2C_2 e^{-2t}$$
$$y_1'(t_o) = 0$$
$$y_1'(0) = C_1 e^0 - 2C_2 e^{-2(0)}$$
$$0 = C_1 - 2C_2$$
$$C_1 = \frac{2}{3}$$
$$C_2 = \frac{1}{3}$$
$$y_1(t) = \frac{2}{3}e^t + \frac{1}{3}e^{-2t}$$

$$y_2(t) = K_1 e^t + K_2 e^{-2t}$$
$$y_2(t_o) = 0$$
$$y_2(0) = K_1 e^0 + K_2 e^{-2(0)}$$
$$0 = K_1 + K_2$$
$$y_2'(t) = K_1 e^t - 2K_2 e^{-2t}$$
$$y_2'(t_o) = 1$$
$$y_2'(0) = K_1 e^0 - 2K_2 e^{-2(0)}$$
$$0 = K_1 - 2K_2$$
$$K_1 = \frac{1}{3}$$
$$K_2 = -\frac{1}{3}$$
$$y_2(t) = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$$

Now that we have y_1 and y_2 we will find the Wronskian,

$$W = y_1 y_2' - y_1' y_2$$
$$y_1(t) = \frac{2}{3}e^t + \frac{1}{3}e^{-2t}$$
$$y_1'(t) = \frac{2}{3}e^t - \frac{2}{3}e^{-2t}$$
$$y_2(t) = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$$
$$y_2'(t) = \frac{1}{3}e^t + \frac{2}{3}e^{-2t}$$
$$\left(\frac{2}{3}e^t + \frac{1}{3}e^{-2t}\right)\left(\frac{1}{3}e^t + \frac{2}{3}e^{-2t}\right) - \left(\frac{2}{3}e^t - \frac{2}{3}e^{-2t}\right)\left(\frac{1}{3}e^t - \frac{1}{3}e^{-2t}\right) =$$
$$\left(\frac{2}{9}e^{2t} + \frac{5}{9}e^{-t} + \frac{2}{9}e^{-4t}\right) - \left(\frac{2}{9}e^{2t} - \frac{4}{9}e^{-t} + \frac{2}{9}e^{-4t}\right) = e^{-t}$$

This will never be zero therefore y_1 and y_2 form the fundamental solution set for this equation.

2. A mass that weighs 8lb stretches a spring 6in. The system is acted on by an external force of 8sin8t lb. If the mass is pulled down 3in and then released,

a) Determine the position of the mass at any time.

30 points

b) Determine the first four times at which the velocity of the mass is zero.

15 points

Solution:

Mass(m) = 8/32 = 1/4 $\text{lb-sec}^2/\text{ft}$
Spring Constant (k) = Weight/extension = 8/(6/12)= 16 lb/ft
External Force (F) = 8sin(8t)

a)

The corresponding differential equation for the motion should be in the form:

$$mu'' + ku = F$$

Therefore, by substituting the given information in the equation yields,

$$(1/4)u'' + (16)u = 8\sin(8t)$$

Simplification gives us,

$$u'' + (64)u = (32)\sin(8t)$$

$$u_c = C_1\cos(8t) + C_2\sin(8t)$$

We assume that the solution is in the form:

$$U(t) = t(A\cos(8t) + B\sin(8t))$$

Resonance occurs.

$$U'(t) = -8tA\sin(8t) + 8tB\cos(8t) + A\cos(8t) + B\sin(8t)$$

$$\begin{aligned} U''(t) &= -64tA\cos(8t) -64tB\sin(8t) -8A\sin(8t) + 8B\cos(8t) -8A\sin(8t) +8B\cos(8t) \\ &= -64tA\cos(8t) -64tB\sin(8t) + 16B\cos(8t) -16A\sin(8t) \end{aligned}$$

We substitute U into our original differential equation.

$$U'' + 64U = -64tA\cos(8t) - 64tB\sin(8t) + 16B\cos(8t) - 16A\sin(8t) + 64tA\cos(8t) + 64tB\sin(8t)$$

$$= 16B\cos(8t) - 16A\sin(8t) = 8\sin(8t)$$

$A = -2 \quad , \quad B = 0$

Therefore u(t) becomes,

$$u(t) = (C_1 - 2t)\cos(8t) + C_2\sin(8t)$$

Initial Conditions:

$$u(0) = 3 \ \& \ u'(0) = 0$$

Using the given initial conditions to find the coefficients C_1 & C_2,

$$C_1 = (\tfrac{1}{4})$$

$$u'(t) = -8C_1\sin(8t) - 2\cos(8t) + 2t\sin(8t) + 8C_2\cos(8t)$$

$$u'(0) = 0 = -2 + 8C_2 \qquad C_2 = (1/4)$$

Therefore, the differential equation becomes:

$$u(t) = (\cos(8t) + \sin(8t) - 8t\cos(8t))/4 \text{ ft}$$

b)

The velocity will be zero when,

$$u'(t) = -8\sin(8t) + 8\cos(8t) + 64t\sin(8t) - 8\cos(8t) = 0$$

$$u'(t) = (8t-1)8\sin(8t) = 0$$

$8t - 1 = 0$, $t = 1/8$ s
$\sin(8t) = 0 \quad t = \pi/8$, $(2\pi/8)$ and $(3\pi/8)$ s

3. The Legendre Equation is as follows:

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \qquad \text{where } \alpha = \text{a constant.}$$

(a) Determine a lower bound for the radius of convergence of series solutions for this equation about the points x = 5 and x = -5.

10 points

(b) Find the recurrence relation of the Legendre Equation by letting $y = \sum_{n=0}^{\infty} a_n x^n$.

20 points

Solution:

(a) First we must determine where the two terms p and q are analytic, where:

$$p = \frac{-2x}{(1-x^2)} \qquad \text{and} \qquad q = \frac{\alpha(\alpha+1)}{(1-x^2)}$$

Both equations have a minimum radius of convergence related to the roots of $(1-x^2)$, which are $r_{1,2} = \pm 1$. The minimum radius of convergence about the point x = 5 is the minimum distance from 5 to +1 (since it is closer), which is 4. The minimum radius of convergence about the point x = -5 is the minimum distance from -5 to -1 (since it is closer), which is also 4.

(b) By substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the Legendre Equation, we get:

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$0 = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} \alpha(\alpha+1) a_n x^n$$

By doing a little math to shift the indices and collecting coefficients of like powers, we get the Legendre Equation into the following form:

$$0 = \left(2\cdot1\cdot a_2 + \alpha(\alpha+1)a_0\right)x^0 + \left(3\cdot2\cdot a_3 - 2\cdot1\cdot a_1 + \alpha(\alpha+1)a_1\right)x^1 +$$

$$\sum_{n=2}^{\infty}\left\{\left[(n+1)(n+2)\right]a_{n+2} - \left[n(n-1)+2n+\alpha(\alpha+1)\right]a_n\right\}x^n$$

Since we are only interested in the recurrence relation, we can ignore the first two terms of the above equation, and set the quantity inside the curly brackets equal to zero to get:

$$0 = \left[(n+1)(n+2)\right]a_{n+2} - \left[n(n-1)+2n+\alpha(\alpha+1)\right]a_n$$

$$\left[(n+1)(n+2)\right]a_{n+2} = \left[n(n+1)+\alpha(\alpha+1)\right]a_n$$

By solving for a_{n+2}, we get the recurrence relation:

$$a_{n+2} = \frac{\left[n(n+1)+\alpha(\alpha+1)\right]}{\left[(n+1)(n+2)\right]}\,a_n$$

4. Solve the differential equation:

$$(2-x^2)y''+4xy'-3y=0 \tag{1}$$

by means of a power series about $x_0=0$. Find the recurrence relation and write the first four terms in each of the two linearly independent solutions.

20 points

Solution:

First, we assume y will be of the form $y=\sum_{n=0}^{\infty}a_n x^n$. This means that $y'=\sum_{n=0}^{\infty}na_n x^{n-1}$ and $y''=\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2}$ as well. Plugging these into our differential equation **(1)**, we can write:

$$(2-x^2)\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2}+4x\sum_{n=0}^{\infty}na_n x^{n-1}-3\sum_{n=0}^{\infty}a_n x^n=0 \tag{2}$$

Distributing through, we get:

$$2\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty}n(n-1)a_n x^n+4\sum_{n=0}^{\infty}na_n x^n-3\sum_{n=0}^{\infty}a_n x^n=0 \tag{3}$$

Now we change the index on the first series so that all our terms will have a common factor, x^n.

$$2\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}n(n-1)a_n x^n+4\sum_{n=0}^{\infty}na_n x^n-3\sum_{n=0}^{\infty}a_n x^n=0$$

So we set the sum of all the coefficients of x^n equal to 0:

$$2(n+2)(n+1)a_{n+2}-n(n-1)a_n+4na_n-3a_n=0 \quad , \ n\ge 2$$

Thus, solving for a_{n+2} our recurrence relation is:

$$a_{n+2}=\frac{a_n[n(n-5)+3]}{2(n+1)(n+2)} \qquad , \qquad n\ge 2$$

Because our relation relates every other term to each other, we will have two solutions, one for odd n's and one for even n's. They are as follows:

$$y_1=1+\frac{3}{4}x^2-\frac{3}{32}x^4+\frac{3}{1920}x^6\ldots \qquad y_2=x-\frac{1}{12}x^3+\frac{1}{160}x^5+\frac{1}{4480}x^7\ldots$$