Math 334 Midterm II KEY
Fall 2006
sections 001 and 004
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Please do NOT write on this exam. No credit will be given for such work. Rather write in a blue book, or on your own paper, preferably engineering.

1. Determine a lower bound for the radius of convergence of the power series representation of the general solution of the following differential equation about the point $x_0 = 1$:

$$\left(x^2+4\right)y''+xy'+x^2y=0. \tag{1.1}$$

5 points

Solution

Equation (1.1) has singularities at the zeroes of the leading coefficient, which are $x = \pm 2i = 0 \pm 2i$. In the complex plain the distance of the singularities to the expansion point $x_0 = 1 = 1+0i$ is $\sqrt{(1-0)^2+(0\mp 2)^2} = \sqrt{5}$. Thus, even along the real axis, we cannot guarantee a radius of convergence beyond $\sqrt{5}$ without more information.

2. Find a (particular) solution of the following differential equation by the method of undetermined coefficients:

$$y''+y'-2y=2t. \tag{1.2}$$

7 points

Solution

The usual explanation of the ansatz for developing a particular solution to a linear constant coefficient differential equation (with a RHS that is in the null space of a linear constant coefficient differential operator) is to first find a basis for the span of the RHS together with all its derivatives. Then, barring the phenomena of *resonance* (which is that one or more elements of such a basis are in the null space of the specific differential operator in question), one then forms a general element of the space spanned by the basis, which general element constitutes the "method of undetermined coefficients ansatz" for a solution of the equation in question. For the problem at hand, and since the RHS of (1.2) is spanned by the single function t, whose first derivative is the linearly independent function 1, and since subsequent differentiations all produce the zero function (which is dependent on any other function), the relevant basis for a particular solution of (1.2) is, barring resonance, $\{t,1\}$. One soon finds that neither of these is a solution of the homogeneous version of (1.2), so that there is no resonance, and the ansatz for a solution of (1.2) is, together with relevant derivatives,

$$\begin{aligned} y &= At + B \\ y' &= 0t + A \\ y'' &= 0t + 0. \end{aligned} \tag{1.3}$$

Weighted appropriate for the equation (1.2), the equations (1.3) are

$$\begin{aligned} -2y &= -2At - 2B \\ +1y' &= 0t + A \\ +1y'' &= 0t + 0 \end{aligned} \tag{1.4}$$

which sum to

$$y'' + y' - 2y = -2At + (-2B + A)1 = 2t + 0 \cdot 1. \tag{1.5}$$

The latter equation holds uniformly in t if and only if $A = -1$ and $-2B + A = 0 \Rightarrow B = A/2 = -1/2.$ Thus the solution sought is

$$y = At + B = -t - 1/2. \tag{1.6}$$

3. A 5 kilogram mass stretches a spring 1/5 meter. If the mass is set in motion from the equilibrium position at 3 meters per second upward, and there is no damping, determine the displacement $u(t)$ of the mass above the equilibrium position at any subsequent time t. Use that the acceleration of gravity is 49/5 meters per second per second.

9 points

Solution

The relevant version of Newton's second law is

$$0 = mu'' + ku = 5\text{kg}u'' + ku. \tag{1.7}$$

Here we may determine the spring constant k from

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$$k = F/s = ma/s = 5\text{kg}49/5m/s^2/(1/5m) = 5 \cdot 7^2\text{kg}/s^2, \tag{1.8}$$

so that (1.7) is

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$$0 = 5\text{kg}u'' + 5 \cdot 7^2\text{kg}/s^2 u \Leftrightarrow 0 = u'' + 7^2/s^2 u. \tag{1.9}$$

Rendering (1.9) unit-less, by measuring time in seconds, this is

$$0 = u'' + 7^2 u, \tag{1.10}$$

1

the general solution to which being

$$u = A\cos(7t) + B\sin(7t). \tag{1.11}$$

3 2

The initial data specifies that

$$\begin{gathered} u(0) = 0 = A, u'(0) = 3 = 7B \\ \Leftrightarrow \\ A = 0, B = 3/7, \end{gathered} \tag{1.12}$$

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so that the required solution to the initial value problem is

$$u = A\cos(7t) + B\sin(7t) = (3/7)\sin(7t). \tag{1.13}$$

1

4. Find the general solution of the following Euler equation, one that is valid for $x > 0$:

$$x^2y'' + 3xy' + 5y = 0. \tag{1.14}$$

11 points

<u>Solution</u>

The differential equation (1.14) defines a linear differential operator L_x, in terms of which (1.14) can be written $L_x[y] = 0$. On a function $y_r = x^r$ one finds that

$$L_x[y_r] = \left(r(r-1) + 3r + 5\right)x^r = \left((r+1)^2 + 2^2\right)x^r, \tag{1.15}$$

so that complex solutions of (1.14) are clearly then $y_{-1+2i} = x^{-1+2i} = x^{-1}e^{2i\ln x} = x^{-1}\left(\cos(2\ln x) + i\sin(2\ln x)\right)$ and

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$y_{-1-2i} = x^{-1-2i} = x^{-1}e^{-2i\ln x} = x^{-1}\left(\cos(2\ln x) - i\sin(2\ln x)\right)$. Independent complex linear combinations of these linearly independent complex solutions gives the following real-representation of the general solution:

$$y = x^{-1}\left(A\cos(2\ln x) + B\sin(2\ln x)\right). \tag{1.16}$$

5. Solve the following initial value problem:

$$y'' - 10y' + 29y = 0;\ \ y(0) = 1,\ y'(0) = 3. \tag{1.17}$$

12 points

Solution

This linear homogeneous differential equation is associated with the following characteristic (polynomial) and characteristic exponents r:

$$\begin{aligned} &0 = r^2 - 10r + 29 = r^2 - 10r + 25 + 4 = (r-5)^2 - (2i)^2 \\ &\Leftrightarrow \\ &r = 5 \pm 2i. \end{aligned} \tag{1.18}$$

According to the usual theory, a real-representation of the general solution, and its corresponding first derivative, are

$$\begin{aligned} &y = e^{5t}\left(C_1\cos 2t + C_2\sin 2t\right) \\ &\text{and} \\ &y' = e^{5t}\left(\left(5C_1 + 2C_2\right)\cos 2t + \left(-2C_1 + 5C_2\right)\sin 2t\right). \end{aligned} \tag{1.19}$$

Inserting $t = 0$ into (1.19), and using the initial data given in (1.17), one obtains

$$\begin{aligned} &y(0) = C_1 = 1 \\ &\text{and} \\ &y'(0) = 5C_1 + 2C_2 = 3, \end{aligned} \tag{1.20}$$

the solution to which being $C_1 = 1$ and $C_2 = -1$. Thus the solution to the initial value problem is then

$$y = e^{5t}\left(\cos 2t - \sin 2t\right). \tag{1.21}$$

6. Given that $y_1 = t^{-1}$ is a solution of

$$t^2 y'' + 3ty' + y = 0 \tag{1.22}$$

for $t > 0$, find a second, linearly independent solution y_2 of (1.22) for $t > 0$ by making the D'Alembert ansatz $y_2 = vy_1 = vt^{-1}$.

14 points

Solution

Using D'Alembert's ansatz in (1.22) gives

$$0 = t^2 y_2'' + 3ty_2' + y_2 = t^2\left(vt^{-1}\right)'' + 3t\left(vt^{-1}\right)' + vt^{-1} = t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) + vt^{-1}$$
$$= tv'' + v' = tu' + u, \quad 6$$
$$(1.23)$$

and where we used $u = v'$. By any one of a number of standard techniques, one finds that the first order homogeneous equation (1.23) has a nontrivial solution $u = t^{-1} = v' \Leftarrow v = \ln t$. Thus a second, linearly independent solution is

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$$y_2 = vt^{-1} = t^{-1}\ln t. \quad 2 \tag{1.24}$$

7. Find the general solution of the following Euler equation, one that is valid for $x > 0$:

$$x^2 y'' + 5xy' + 4y = 0. \tag{1.25}$$

16 points

Solution

The differential equation (1.25) defines a linear differential operator L_x, in terms of which (1.25) can be written $L_x[y] = 0$. On a function $y_r = x^r$ one finds that

$$L_x[y_r] = \left(r(r-1) + 5r + 4\right)x^r = \left(r+2\right)^2 x^r, \tag{1.26}$$

so that a solution of (1.25) is clearly then $y_{-2} = x^{-2}$. To find the general solution to this second order differential equation we need to find a second, linearly independent solution. Since the ansatz $y_r = x^r$ only produces solutions dependent upon $y_{-2} = x^{-2}$, we must use another ansatz. Fortunately the structure of (1.26), together with the fact that the differential operators $\frac{d}{dr}$ and L_x commute, suggest such an alternative ansatz: applying $\frac{d}{dr}$ to both sides of (1.26), and using the indicated commutivity, one obtains

$$L_x[\frac{d}{dr}y_r] = (r+2)^2 x^r \ln x + 2(r+2)^1 x^r, \tag{1.27}$$

so that $\left.\frac{d}{dr}y_r\right|_{r=-2} = \left. x^r \ln x\right|_{r=-2} = x^{-2}\ln x$ is clearly a second, linearly independent solution of (1.25). Thus the general solution to this linear homogeneous equation is

$$y = (A + B\ln x)x^{-2}. \tag{1.28}$$

8. Find the first 3 nonzero terms in the series representation of each of 2 linearly independent solutions of the equation

$$(x^2+1)y'' - 4xy' + 6y = 0 \tag{1.29}$$

about the point $x_0 = 0$.

18 points

Solution

The point $x_0 = 0$ is a ordinary point, so that the required series solution is a Taylor series: insert $y = \sum_n a_n x^n$ (with the assumption that $a_n = 0$ for $n < 0$, and that the sum is over the integers)in (1.29) to obtain

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$$0 = \sum_n (x^2+1)n(n-1)a_n x^{n-2} - 4xna_n x^{n-1} + 6a_n x^n = \sum_n \left(n(n-1)a_n + (n+2)(n+1)a_{n+2} - 4na_n + 6a_n\right)x^n$$

$\Leftrightarrow$

$$a_{n+2} = -\frac{n(n-1)-4n+6}{(n+2)(n+1)}a_n = -\frac{n^2-5n+6}{(n+2)(n+1)}a_n = -\frac{(n-2)(n-3)}{(n+2)(n+1)}a_n \tag{1.30}$$

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for integer $n \geq 0$, and which is satisfied otherwise since $a_n = 0$ for $n < 0$. Using the recursion in (1.30) we get

$$\begin{aligned}
a_2 &= -\frac{(-2)(-3)}{(2)(1)} a_0 = -3a_0 \qquad 2 \\
a_3 &= -\frac{(1-2)(1-3)}{(1+2)(1+1)} a_1 = -\frac{1}{3} a_1 \qquad 2 \\
a_4 &= -\frac{(2-2)(2-3)}{(2+2)(2+1)} a_2 = 0 \\
a_5 &= -\frac{(3-2)(3-3)}{(3+2)(3+1)} a_3 = 0 \\
&\vdots
\end{aligned} \tag{1.31}$$

so that the series terminates, i.e. the solutions are polynomials. We have

$$\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 = a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3} a_1 x^3 \\
&= a_0 \left(1 - 3x^2\right) + a_1 \left(x - \frac{1}{3} x^3\right)
\end{aligned} \tag{1.32}$$

which indicates the general solution and the required linearly independent solutions (up to “arbitrary number of terms”—certainly three).

9. Find the general solution of the following linear but non-homogeneous differential equation by the method of variation of parameters. Do not use the (memorized) formula/theorem (involving a Wronskian), rather generate the relevant version of the formula afresh by using the “D'Alembert-like” ansatz that leads to that formula.

$$y'' - 5y' - 24y = e^{5t} \tag{1.33}$$

22 points

Solution

The characteristic equation of the homogeneous version of the constant coefficient differential equation (1.33) is

$$0 = r^2 - 5r - 24 = (r-8)(r+3) \quad 4 \tag{1.34}$$

so that the general solution of the corresponding homogeneous equation is

$$y = Ae^{8t} + Be^{-3t}, \tag{1.35}$$

where A and B are independent of t. But allowing the parameters A and B to vary with t in (1.35), we have also an ansatz there for the solution of the non-homogeneous equation (1.33): with such an ansatz one immediately has

$$y' = 8Ae^{8t} - 3Be^{-3t} + \left(A'e^{8t} + B'e^{-3t}\right). \tag{1.36}$$

But this ansatz is "initially consistent with A and B independent of t" if we choose here that

$$A'e^{8t} + B'e^{-3t} = 0, \tag{1.37}$$

so that then (1.36) becomes

$$y' = 8Ae^{8t} - 3Be^{-3t}. \tag{1.38}$$

Differentiating (1.38) gives

$$y'' = 64Ae^{8t} + 9Be^{-3t} + \left(8A'e^{8t} - 3B'e^{-3t}\right). \tag{1.39}$$

Combining these derivatives with the appropriate weights (dictated by the differential equation) we get the ledger

$$\begin{aligned} -24y &= -24Ae^{8t} - 24Be^{-3t} \\ -5y' &= -40Ae^{8t} + 15Be^{-3t} \\ +1y'' &= +64Ae^{8t} + 9Be^{-3t} + \left(8A'e^{8t} - 3B'e^{-3t}\right), \end{aligned} \tag{1.40}$$

and from which it is clear that the differential equation demands that

$$8A'e^{8t} - 3B'e^{-3t} = e^{5t}. \quad 3 \tag{1.41}$$

Combining this with the "consistency ansatz" (1.37) we get

$$\begin{bmatrix} e^{8t} & e^{-3t} \\ 8e^{8t} & -3e^{-3t} \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 0 \\ e^{5t} \end{bmatrix} \tag{1.42}$$

which implies that

$$A' = \frac{\begin{vmatrix} 0 & e^{-3t} \\ e^{5t} & -3e^{-3t} \end{vmatrix}}{\begin{vmatrix} e^{8t} & e^{-3t} \\ 8e^{8t} & -3e^{-3t} \end{vmatrix}} = \frac{-e^{2t}}{-11e^{5t}} = \frac{e^{-3t}}{11}$$

$$B' = \frac{\begin{vmatrix} e^{8t} & 0 \\ 8e^{8t} & e^{5t} \end{vmatrix}}{\begin{vmatrix} e^{8t} & e^{-3t} \\ 8e^{8t} & -3e^{-3t} \end{vmatrix}} = \frac{e^{11t}}{-11e^{5t}} = -\frac{e^{8t}}{11}. \tag{1.43}$$

Solutions to (1.43) include the pair $A = -\frac{e^{-3t}}{33}$, $B = -\frac{e^{8t}}{88}$, so that a solution to (1.33) is, according to (1.35),

$$\begin{aligned} y = Ae^{8t} + Be^{-3t} &= -\frac{e^{-3t}}{33}e^{8t} - \frac{e^{8t}}{88}e^{-3t} = -\frac{1/3+1/8}{11}e^{5t} \\ &= -\frac{8+3}{11\cdot 3\cdot 8}e^{5t} = -\frac{e^{5t}}{24}, \end{aligned} \tag{1.44}$$

and the general solution to (1.33) is

$$y = Ae^{8t} + Be^{-3t} - \frac{e^{5t}}{24}, \tag{1.45}$$

where A and B are (truly) constants now.

10. Solve the initial value problem obtained from combining the differential equation of problem 9 with the initial data $y(0) = 0$, $y'(0) = 0$. In order that errors don't "cascade", I will tell you that $y = Ae^{8t} + Be^{-3t} - \frac{e^{5t}}{24}$ is the general solution of the differential equation of problem 9. (So now if you just write down this solution to 9 without very convincing work, you will get 0 points on problem 9.) Thus, I am just testing if you understand the correct principles needed to construct the solution to the initial value problem given the general solution to the associated differential equation.

20 points

Solution

From the information given we have

$$
\begin{aligned}
y(0) &= 0 = A + B - \frac{1}{24} \quad 4 \\
y'(0) &= 0 = 8A - 3B - \frac{5}{24}, \quad 4
\end{aligned} \tag{1.46}
$$

or, equivalently, the following augmented matrix for the column vector (A, B), which, together with row reduction, is

$$
\begin{bmatrix} 24 & 24 & 1 \\ 8\cdot 24 & -3\cdot 24 & 5 \end{bmatrix} \sim \begin{bmatrix} 24 & 24 & 1 \\ 0 & -11\cdot 24 & -3 \end{bmatrix} \sim \begin{bmatrix} 24 & 24 & 1 \\ 0 & -24 & -3/11 \end{bmatrix} \sim \begin{bmatrix} 24 & 0 & 8/11 \\ 0 & -24 & -3/11 \end{bmatrix}. \tag{1.47}
$$

From (1.47) one has that

$$
\begin{aligned}
A &= 8/(11\cdot 24) \quad 4 = \frac{1}{33} \quad 264 \\
B &= 3/(11\cdot 24) \quad 4 = \frac{1}{88}
\end{aligned} \tag{1.48}
$$

and the solution sought is

$$
y = Ae^{8t} + Be^{-3t} - \frac{e^{5t}}{24} = \frac{1}{24}\left(\frac{8}{11}e^{8t} + \frac{3}{11}e^{-3t} - e^{5t}\right) \tag{1.49}
$$

$$
= \frac{e^{8t}}{33} + \frac{e^{-3t}}{88} - \frac{e^{5t}}{24}
$$