

1

(1) 35 points

Transform the given system into a single equation of second order. Then find  $x_1$  and  $x_2$  that satisfy the given initial conditions.

*5/4, 3/4, etc.*

$$\begin{aligned}x_1' &= 1.25x_1 + .75x_2 & x_1(0) &= -2 \\x_2' &= .75x_1 + 1.25x_2 & x_2(0) &= 1\end{aligned}$$

Solution:

Solve the first Differential Equation for  $x_2$ .

$$x_2 = (4/3) * x_1' - (5/3) * x_1$$

Substitute the result into the second equation :  $x_2' = .75x_1 + 1.25x_2$ .

The result is

$$x_1'' - 2.5x_1' + x_1 = 0$$

Thus,

$$x_1 = c_1 e^{t/2} + c_2 e^{(2t)}$$

$$x_2 = -c_1 e^{t/2} + c_2 e^{(2t)}$$

Use the initial conditions and solve for  $c_1$  and  $c_2$ :

$$c_1 = (-3/2)$$

$$c_2 = (-1/2)$$

$$x_1 = -(3/2) * e^{t/2} + (-1/2) * e^{(2t)}$$

$$x_2 = (3/2) * e^{t/2} + (-1/2) * e^{(2t)}$$

*don't use decimal*

*no characteristic equation, etc.?*

*where does this come from? Do you know?*

(2) 35 points

Find the general solution of the given system of equations.

$$x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x$$

Answer:

We assume that  $x = \xi e^r$ , hence we obtain the algebraic system

$$\begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1)$$

So, we find the eigenvalues

$$(1-r)(-2-r)-4=0,$$

$$r = -3, 2$$

*step missing - factoring*

For  $r = -3$  equation (1) becomes

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, the eigenvector  $\xi_1$  can be expressed as the following:  $\xi_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

The eigenvalue  $r = 2$  gives the eigenvector  $\xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Thus the general solution is  $x = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

(3) 45 points

Express the solution of the given initial value problem in terms of a convolution integral.  
 $4y'' + 4y' + 17y = g(t)$     $y(0) = 0$     $y'(0) = 0$

Solution:

We take the Laplace transform of the equation using the initial condition and obtain:

$$4s^2Y(s) + 4sY(s) + 17Y(s) = G(s)$$

Therefore,

$$Y(s) = \frac{G(s)}{4s^2 + 4s + 17}$$

This reduces to

$$Y(s) = \frac{G(s)}{4((s+1/2)^2 + 4)} \text{ or}$$

$$Y(s) = \frac{1}{8} \frac{2G(s)}{(s+1/2)^2 + 4}$$

Which in turn reduces to

$$Y(s) = \frac{1}{8} G(s) L(e^{-\frac{1}{2}t} \sin 2t)$$

If we use the theorem of the convolution integral, we can take the inverse Laplace transform to solve for  $y$ . Thus,

$$y = \frac{1}{8} \int_0^t g(t-\tau) e^{-\frac{\tau}{2}} \sin(2\tau) d\tau$$

(4) 35 points

Find the inverse transform of

$$F(s) = \frac{s - e^{-\pi^* s} * s}{s^2 + 1}$$

Since the inverse transform is linear,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-\pi^* s} * s}{s^2 + 1}\right\}$$

Thus,

$$f(t) = \cos(t) - u_{\pi}(t)(\cos(t) - \pi).$$

Which is also written as,

$$\cos(t), \quad 0 \leq t \leq \pi,$$

$$f(t) = \begin{cases} \pi, & t \geq \pi, \end{cases}$$

*not correct!*

Use the Laplace transform to solve the given initial value problem:

2

$$y'' + y' - 2y = 4 \quad \text{satisfying the initial conditions: } y(0)=2, y'(0)=1$$

**Solution:**

Take the Laplace transform on both sides. You may need a table to determine the transform.

$$s^2 Y(s) - sy(0) - y'(0) + sY(s) + y(0) - 2Y(s) = \frac{4}{s}$$

Combine like terms:

$$[s^2 + s - 2]Y(s) = \frac{4}{s} + 2s - 3$$

Notice that the coefficient in front of  $Y(s)$  is the characteristic equation of the differential equation. Divide the characteristic equation through.

$$Y(s) = \frac{\frac{4}{s} + 2s - 3}{s^2 + s - 2} \longrightarrow Y(s) = \frac{2s^2 - 3s + 4}{s(s+2)(s-1)}$$

Now use partial fractions to expand it:

$$\frac{2s^2 - 3s + 4}{s(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$$

Which yields:

$$2s^2 - 3s + 4 = A(s+2)(s-1) + B(s)(s-1) + C(s)(s+2)$$

$$\text{Letting } s = 0 \text{ gives } -2A = 4 \quad A = -2$$

$$\text{Letting } s = -2 \text{ gives } 6B = 18 \quad B = 3$$

$$\text{Letting } s = 1 \text{ gives } 3C = 3 \quad C = 1$$

Now we solve:

$$Y(s) = \frac{-2}{s} + \frac{3}{s+2} + \frac{1}{s-1}$$

Now find the inverse Laplace using the table found in 6.2 of the textbook. (Hopefully an actual table will be provided on the test.)

$$y = -2 + 3e^{-2t} + e^t$$

Question:

Find the solution of the following initial value problem:

$$y'' + 9y = 3u_{\pi}(t) + 8$$

$$y(0) = 0; y'(0) = 0$$

Answer:

Using the Laplace transforms:

$$y'' + 9y \rightarrow s^2 L\{f\}(s) - sf(0) - f'(0) + 9L\{f\}(s)$$

$$= (s^2 + 9)L\{f\}(s)$$

$$3u_{\pi} + 8 \rightarrow \frac{3e^{-\pi s} + 8}{s}$$

$$(s^2 + 9)L\{f\}(s) = \frac{3e^{-\pi s} + 8}{s}$$

$$L\{f\}(s) = \frac{3e^{-\pi s} + 8}{s(s^2 + 9)}$$

Using partial fractions to split up the equation:

$$\frac{A}{s} + \frac{Bs + C}{s^2 + 9} = \frac{3e^{-\pi s} + 8}{s(s^2 + 9)}$$

$$A(s^2 + 9) + s(Bs + C) = 3e^{-\pi s} + 8$$

$$As^2 + 9A + Bs^2 + Cs = 3e^{-\pi s} + 8$$

$$A + B = 0$$

$$C = 0$$

$$9A = 3e^{-\pi s} + 8$$

$$A = \frac{3e^{-\pi s} + 8}{9}$$

$$B = -\frac{3e^{-\pi s} + 8}{9}$$

$$L\{f\}(s) = \frac{3e^{-\pi s} + 8}{9s} - \frac{s(3e^{-\pi s} + 8)}{9(s^2 + 9)}$$

$$L\{f\}(s) = \frac{3e^{-\pi s}}{9s} + \frac{8}{9s} - \frac{3e^{-\pi s}s}{9(s^2 + 9)} - \frac{8s}{9(s^2 + 9)}$$

Using the inverse Laplace transforms:

$$y = \frac{1}{3}u_{\pi}(t) + \frac{8}{9} - \frac{1}{3}u_{\pi}(t)\cos(3(t - \pi)) - \frac{8}{9}\cos(3t)$$

$$y = \frac{1}{3}u_{\pi}(t)(1 - \cos(3(t - \pi))) + \frac{8}{9}(1 - \cos(3t))$$

do not use this  
in p.f. decomposition.

Use  $\frac{1}{s(s^2 + 9)}$  only,

and afterwards

handle  $3e^{-\pi s} + 8$ —

see your notes

on my lecture.

Otherwise, procedure  
is dangerous/confusing

— how do you know

to write it  
above?

Find a fundamental set of real-valued solutions of the system

$$\mathbf{x}' = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix} \mathbf{x} \quad (1)$$

We assume that

$$\mathbf{x} = \xi e^{rt}$$

Then we get the set of linear algebraic equations.

$$\begin{pmatrix} -1-r & 3 \\ -3 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2)$$

Following, we find the characteristic equation

$$\begin{vmatrix} -1-r & 3 \\ 3 & -1-r \end{vmatrix} = r^2 + 2r + 10 = 0.$$

Therefore we find the eigenvalues to be

$$r_1 = -1 + 3i$$

$$r_2 = -1 - 3i$$

When we substitute  $r_1$  and  $r_2$  into equation (2) we find the corresponding eigenvectors to be

$$\xi^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\xi^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

From this we see that the fundamental set of solutions of equation (1) is

$$x^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1-3i)t}$$

$$x^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1+3i)t}$$

To find a set of real-valued solutions we need to find the real and imaginary parts of either  $x^{(1)}$  or  $x^{(2)}$ .

$$x^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-t} (\cos 3t + i \sin 3t) = e^{-t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + i e^{-t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$$

This gives us

$$u(t) = e^{-t} \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix}$$

$$v(t) = e^{-t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$$

is a set of real valued equations *solutions*.

Solve the initial value problem

$$\begin{aligned}x'_1 &= 6x_1 + 8x_2 \\x'_2 &= x_1 + 4x_2\end{aligned}\tag{1}$$

$$x_1(0) = 1, \quad x_2(0) = 0.\tag{2}$$

In matrix notation the system (1) reads

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\tag{3}$$

The characteristic polynomial of the matrix  $A$  is

$$f(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 6 - \lambda & 8 \\ 1 & 4 - \lambda \end{pmatrix} = (\lambda - 2)(\lambda - 8).\tag{4}$$

Hence, the eigenvalues are  $\lambda_1=2$  and  $\lambda_2=8$ . Substituting the eigenvalues

in  $(A - \lambda I)u = 0$ , we can determine corresponding eigenvectors

$$u_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}\tag{5}$$

Now, the general solution of equation (1) can be presented as

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{8t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -2c_1 e^{2t} + 4c_2 e^{8t} \\ c_1 e^{2t} + c_2 e^{8t} \end{pmatrix}\tag{6}$$

The initial conditions (2) in matrix notation take the form

$$x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\tag{7}$$

It follows from (6),(7) that

$$\begin{aligned} -2c_1 + 4c_2 &= 1 \\ c_1 + c_2 &= 0 \end{aligned} \tag{8}$$

Solving system (8) yields:  $c_1 = -\frac{1}{6}$  and  $c_2 = \frac{1}{6}$ . Inserting  $c_1$  and  $c_2$  into general solution (6) gives the solution to the initial value problem (1), (2) as

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}e^{2t} + \frac{2}{3}e^{8t} \\ -\frac{1}{6}e^{2t} + \frac{1}{6}e^{8t} \end{pmatrix}$$

Use the Laplace transform to solve the initial value problem:

3

$$y'' - 4y' + 8y = 0, y(0) = 1, y'(0) = 0.$$

**Solution:**

To solve this we assume a solution that satisfies the conditions of Corollary 6.2.2. The Laplace transform of the differential equation is

$$s^2 Y(s) - sy(0) - y'(0) - 4sY(s) + 4y(0) + 8Y(s) = 0$$

Using the initial conditions and solving for  $Y(s)$  we have

$$Y(s) = \frac{(s-4)}{(s^2 - 4s + 8)}.$$

We now try to get it in a form that we know so we can easily do the ~~reverse~~ Laplace

transform. It looks like it will fit  $\frac{(s-a)}{((s-a)^2 + b^2)}$  which has an inverse Laplace  $e^{at} \cos(bt)$ . So we first complete the square on the bottom giving

$$Y(s) = \frac{(s-4)}{(s^2 - 4s + 4 - 4 + 8)} = \frac{(s-4)}{((s-2)^2 + 2^2)}$$

Now we need an  $(s-2)$  term on top so we will take a -2 out and split the fraction into

two fractions. The first one will satisfy  $\frac{(s-a)}{((s-a)^2 + b^2)}$  while the second one will satisfy  $\frac{b}{((s-a)^2 + b^2)}$ .

$$\frac{(s-4)}{((s-2)^2 + 2^2)} = \frac{(s-2)}{((s-2)^2 + 2^2)} - \frac{2}{((s-2)^2 + 2^2)}.$$

This fits our two desired forms so doing the inverse Laplace transform we have

$$y = e^{2t} \cos(2t) - e^{2t} \sin(2t).$$

36 points

Find the solution  $y'' + y = \sin 2t$  subject to  $y(0)=0$  and  $y'(0)=0$  using the Laplace Transform

Taking the Laplace Transform of both sides we have  $s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2+4}$  (3 pts).

Substituting in the initial conditions and solving for  $Y(s)$  we have  $Y(s) = \frac{2}{(s^2+1)(s^2+4)}$  (2 pt). We

must decompose the right side of the equation into recognizable Laplace transforms using some

technique like partial fractions. We set the left side equal to a sum of 2 fractions

$\frac{2}{(s^2+1)(s^2+4)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}$  (1 pt). Then we multiply the denominators out  
*— a=c=0 obviously — no terms linear in s on left, so none needed on right*

$2 = (as+b)(s^2+1) + (cs+d)(s^2+4) = (a+c)s^3 + (b+d)s^2 + (a+4c)s + b+4c$  (3 pts). From this

we find the following system of equations  $a+c=0, b+d=0, a+4c=2, b+4c=0$  (3 pts).

Solving this system of equations we find that  $b = \frac{2}{3}$  and  $d = -\frac{2}{3}$  (1 pt). Therefore  
*— and a=? , c=?*

$Y(s) = \frac{2}{3(s^2+1)} - \frac{2}{3(s^2+4)}$  (1 pt). It follows that  $y = \frac{2}{3}\sin(t) - \frac{1}{3}\sin(2t)$  (5pts) is the solution to

$y'' + y = \sin(2t)$ .

Extra credit: What is the specific name applied to the initial data  $y(0)=y'(0)=0$ ? Quiescent

(2pts)

19 points.

Find the eigenvalues and eigenvectors of the matrix  $A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$

**Solution :**

The eigenvalues and the eigenvectors  $x$  satisfy the equation  $(A - \lambda I)x = 0$

$$\begin{vmatrix} 3 - \lambda & -1 \\ 4 & -2 - \lambda \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

The roots of the equation can be found by finding *what equation - antec edant?* (10 points)

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -1 \\ 4 & -2 - \lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = 0 \quad \lambda = 2, 1$$

(5 points)

To find the eigenvectors replace  $\lambda$  by each of the eigenvalues in turn.

For  $\lambda = 2$  we have  $x_1 - x_2 = 0$  and so  $x_1 = x_2$

There corresponding eigenvector will be  $\mathbf{x}^1 = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

(10 points)

For  $\lambda = 1$  we have

$$\begin{vmatrix} 4 & -1 \\ 4 & -1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

(5 points)

There corresponding eigenvector will be  $\mathbf{x}^2 = \begin{vmatrix} 1 \\ 4 \end{vmatrix}$

(10 points)

Find the solution of the given initial value problem

$$\mathbf{x}' = \begin{pmatrix} -3 & 3 \\ 2 & 2 \end{pmatrix} \mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Assume that  $\mathbf{x} = \xi e^{rt}$  and substitute for  $\mathbf{x}$  in the original equation.

This leads us to the system of algebraic equations

$$\mathbf{x} = \begin{pmatrix} -3-r & 3 \\ 2 & 2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We will have a non-trivial solution if and only if the determinant of the coefficient <sup>matrix</sup> is zero. Thus, allowable values of  $r$  are found by solving for the determinant <sup>equation</sup>

$$0 = \begin{vmatrix} -3-r & 3 \\ 2 & 2-r \end{vmatrix} = (-3-r)(2-r) - (3)(2) = -6 + 3r - 2r + r^2 - 6 = r^2 + r - 12 = (r+4)(r-3)$$

$\therefore r_1 = -4; r_2 = 3$

$r_1, r_2$  are the eigenvalues of the coefficient matrix.

Now, let  $r = -4$

$$\begin{pmatrix} -3+4 & 3 \\ 2 & 2+4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\xi_1 + 3\xi_2 = 0 \Rightarrow \xi_1 = -3\xi_2$$

We can now conclude that

$$\xi_{-4} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Do the same for  $r=3$

$$\begin{pmatrix} -3-3 & 3 \\ 2 & 2-3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -6 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$2\xi_1 - \xi_2 = 0 \Rightarrow \xi_1 = \frac{1}{2}\xi_2$$

We can now also conclude that

$$\xi_3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The corresponding solutions of the differential equation are

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-4t} \quad \text{and} \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}.$$

The Wronskian of these solutions is

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \begin{vmatrix} -3e^{-4t} & e^{3t} \\ e^{-4t} & 2e^{3t} \end{vmatrix} = -6e^{-t} - e^{-t} = -7e^{-t}$$

which is never zero. This means that the solutions  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$  form a fundamental set. The general solution of the system is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}.$$

where  $c_1, c_2$  are arbitrary constants.

Now we can solve for these constants using the given initial value condition. Therefore,

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-4(0)} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3(0)} = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

This gives us two equations to solve for the constants

$$2 = -3c_1 + c_2 \quad -4 = c_1 + 2c_2$$

This results in

$$c_1 = \frac{-10}{7}; c_2 = \frac{-8}{7}.$$

Substitute the constants in the general solution to find the initial value solution

$$\mathbf{x}(t) = \frac{-10}{7} \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-4t} + \frac{-8}{7} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

Simplifying this equation results in our final answer

$$\mathbf{x}(t) = \begin{pmatrix} 30/7 \\ -10/7 \end{pmatrix} e^{-4t} + \begin{pmatrix} -8/7 \\ -16/7 \end{pmatrix} e^{3t}.$$

37 points

**Problem 1 (35 Points):**

Express the solution for the following initial value problem in terms of a convolution integral:

4

$$2y'' + 18y = g(t); y(0) = 27, y'(0) = 63$$

**Solution:**

Begin by taking the Laplace transform and solving for the Laplace of  $y$  as follows:

$$\mathcal{L}\{2y'' + 18y\}(s) = \mathcal{L}\{g\}(s)$$

$$\mathcal{L}\{2y''\}(s) + \mathcal{L}\{18y\}(s) = G(s)$$

$$2(s^2 \mathcal{L}\{y\}(s) - sy(0) - y'(0)) + 18\mathcal{L}\{y\}(s) = G(s)$$

$$2(s^2 \mathcal{L}\{y\}(s) - 27s - 63) + 18\mathcal{L}\{y\}(s) = G(s)$$

$$2s^2 \mathcal{L}\{y\}(s) - 54s - 126 + 18\mathcal{L}\{y\}(s) = G(s)$$

$$2s^2 \mathcal{L}\{y\}(s) + 18\mathcal{L}\{y\}(s) = 54s + 126 + G(s)$$

$$(2s^2 + 18)\mathcal{L}\{y\}(s) = 54s + 126 + G(s)$$

$$\mathcal{L}\{y\}(s) = \frac{54s + 126 + G(s)}{2s^2 + 18}$$

$$\mathcal{L}\{y\}(s) = \frac{54s}{2s^2 + 18} + \frac{126}{2s^2 + 18} + \frac{G(s)}{2s^2 + 18}$$

$$\mathcal{L}\{y\}(s) = \frac{27s}{(s^2 + 9)} + \frac{63}{(s^2 + 9)} + \frac{G(s)}{2(s^2 + 9)}$$

$$\mathcal{L}\{y\}(s) = 27 \frac{s}{(s^2 + 9)} + 21 \frac{3}{(s^2 + 9)} + \frac{1}{6} \frac{3}{(s^2 + 9)} G(s)$$

Observe that the general form of the inverse Laplace is:

$$y = 27f(t) + 21h(t) + \frac{1}{6} \int_0^t h(t - \tau)g(\tau)d\tau$$

Then  $h$  and  $f$  can be determined as follows:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2 + 9)}\right\} = \cos(3t)$$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{3}{(s^2 + 9)}\right\} = \sin(3t)$$

Then the solution is:

$$y = 27 \cos(3t) + 21 \sin(3t) + \frac{1}{6} \int_0^t \sin[3(t - \tau)]g(\tau)d\tau$$

**Problem 2 (40 Points):**

Find the solution of the initial value problem:  $y'' + 2y' - 15y = \delta(t - 4)$   $y(0) = 0$ ,  $y'(0) = 0$  recalling that

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t) dt = f(t_0)$$

**Solution:**

To solve the initial value problem, we start by taking the Laplace Transform of the differential equation and plugging in the initial conditions as follows:

$$s^2 Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] - 15Y(s) = e^{-4s}$$

$$Y(s)(s^2 + 2s - 15) = e^{-4s}$$

Solving for  $Y(s)$ , factoring the denominator, and applying partial fractions gives:

$$Y(s) = \frac{e^{-4s}}{(s^2 + 2s - 15)} \rightarrow$$

$$Y(s) = \frac{e^{-4s}}{(s-3)(s+5)} \rightarrow$$

$$Y(s) = \frac{\frac{1}{8}e^{-4s}}{(s-3)} - \frac{\frac{1}{8}e^{-4s}}{(s+5)}.$$

*don't use exp.s  
in p.f. decomp. -  
attack later.*

Then by the inverse Laplace Transform, the solution is:

$$y(t) = u_4(t) \left( \frac{1}{8} e^{3(t-4)} - \frac{1}{8} e^{-5(t-4)} \right) \rightarrow$$

$$y(t) = \frac{1}{8} u_4(t) \left( e^{3(t-4)} - e^{-5(t-4)} \right).$$

**Problem 3 (25 Points):**

Find all eigenvalues and eigenvectors for the following matrix:

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}$$

**Solution:**

We have

$$A - \lambda E = \begin{pmatrix} 1 - \lambda & -2 \\ 2 & 5 - \lambda \end{pmatrix}$$

Then

$$\det(A - \lambda E) = \det \begin{pmatrix} 1 - \lambda & -2 \\ 2 & 5 - \lambda \end{pmatrix} = (1 - \lambda)(5 - \lambda) + 4 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

Therefore the matrix A has only one eigenvalue  $\lambda = 3$ .

Now to find the eigenvectors of the matrix A we have

$$A - 3E = \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}$$

Therefore the system  $(A - 3E)\vec{e} = 0$  is equivalent to the equation  $e_1 + e_2 = 0$

Then vector  $\vec{e} = (1, -1)$  is the eigenvector of the matrix A.

**Problem 4 (25 Points):**

Find the general solution for the following system:

$$X' = \begin{bmatrix} -2 & -1 \\ 3 & -2 \end{bmatrix} * X$$

**Solution:**

1) To find the general solution we assume that  $X = \xi e^{rt}$

2) By substituting for X in the original equation we get the system of algebraic equations that follows:

$$\begin{bmatrix} -2-r & -1 \\ 3 & -2-r \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

3) Equations like this one have a nontrivial solution if and only if the determinant of coefficients is zero. Therefore the allowable values of r are found from the equation of the determinant, which yields the following:

$$\begin{aligned} (-2-r)(-2-r) - 1 &= 0 \\ \Rightarrow 4 + 2r + 2r + r^2 - 1 & \\ \Rightarrow r^2 + 4r + 3 & \text{ Therefore } r = -1 \text{ and } -3 \end{aligned}$$

4) Now we plug into the matrix in step #2 for each r that we found in step #3:

$$\text{For } r = -1 \text{ we get: } \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \text{ which row reduces to } \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

This tells us that  $-\xi_1 + \xi_2 = 0$

$$\text{So } \xi_1 = \xi_2 \text{ if we plug in 1 for } \xi_1 \text{ we get: } \xi^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{For } r = -3 \text{ we get: } \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ which row reduces to } \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

This tells us that  $-\xi_1 + \xi_2 = 0$

$$\text{So } \xi_2 = -\xi_1 \text{ if we plug in 1 for } \xi_1 \text{ we get: } \xi^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

5) Now we just plug everything we found into the general solution form and we get

$$X = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$$

(5)

6.2

45 points

Use the Laplace transformation to solve the initial value problem.

$$y^{(4)} - 4y''' + 6y'' - 4y' + y = 0$$

$$y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 1$$

Solution:

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0) - 4[s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)] + 6[s^2 Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] + Y(s) = 0$$

Substituting the initial value conditions, and solving for  $Y(s)$  yields:

$$Y(s) = \frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2 - 4s + 1}$$

Separating the above equation into partial fractions yields

$$\frac{a}{(s-1)^4} + \frac{b}{(s-1)^3} + \frac{c}{(s-1)^2} + \frac{d}{(s-1)}$$

Setting this equal to  $Y(s)$  and equating the numerators gives us:

$$s^2 - 4s + 7 = a + b(s-1) + c(s-1)^2 + d(s-1)^3$$

Solving for  $a, b, c,$  and  $d,$  and using the table 6.2 yields:

$$y = te^t - t^2 e^t + (2/3)t^3 e^t$$

where did this come from?

? missing a step? factorization?

35 points

12. Find the solution of

$$y'' + 2y' + y = f(t) \quad y(0)=1, y'(0)=0$$

where

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ t-1 & \text{if } 1 \leq t \end{cases}$$

solution:

$$s^2 Y(s) - s + 2sY(s) - 2 + Y(s) = e^{-s}/s^2$$

$$Y(s) = (s-2)/(s+1)^2 + (s+1)^2 e^{-s}/s^2$$

Via partial fractions we have

$$Y(s) = 1/(s+1) - 3/(s+1)^2 + e^{-s}(-2/s + 1/s^2 + 2/(s+1) + 1/(s+1)^2)$$

Hence,

$$y(t) = e^{-t} - 3te^{-t} + (-2 + t - 1 + 2e^{-t+1} + (t-1)e^{-t+1}) u_1(t)$$

$$= e^{-t} - 3te^{-t} + (-3 + t + e^{-t+1} + te^{-t+1}) u_1(t)$$

show how!

good p.f. decomposition

first  
good one I've

seen in  
this type of  
case.

Compute the inverse of the matrix:

25 points

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix}$$

Augmenting the above matrix with the identity matrix yields:

$$\begin{vmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{vmatrix}$$

Reduce the matrix into reduced row echelon form:

$$\begin{vmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{vmatrix} \begin{array}{l} -2I + II \\ -3I + III \\ \end{array}$$

$$\begin{vmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{vmatrix} \begin{array}{l} -1 * II \\ -1 * III \\ (II \leftrightarrow III) \end{array}$$

$$\begin{vmatrix} 1 & 2 & 0 & -5 & 3 & 0 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{vmatrix} \begin{array}{l} -3III + I \\ -3III + II \\ \end{array}$$

$$\begin{vmatrix} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{vmatrix} \begin{array}{l} -2II + I \\ \\ \end{array}$$

The inverse matrix is the matrix on the right side or:

$$\begin{vmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{vmatrix}$$

relevance to O.D.E.'s?  
(It has one - what  
is it?)  
Need to show  
relevance.

(Pt value 45)

Solve the following system of first order linear differential equations for  $x_1$  and  $x_2$ , with the given initial conditions:

$$\begin{aligned} (1) \quad & x_1' = 5/4 x_1 + 3/4 x_2 & x_1(0) = -2, x_2(0) = 1 \\ (2) \quad & x_2' = 3/4 x_1 + 5/4 x_2 \end{aligned}$$

Solution:

Solving the first equation for  $x_2$ :

$$x_2 = 4/3 x_1' - 5/3 x_1$$

Taking the derivative of this yields:

$$x_2' = 4/3 x_1'' - 5/3 x_1'$$

Substituting  $x_2$  and  $x_2'$  into equation two simplifies to:

$$0 = x_1'' - 5/2 x_1' + x_1$$

The characteristic equation for the solution of  $x_1$  is:

$$x_1 = c_1 e^{2t} + c_2 e^{t/2}$$

Using the initial condition for  $x_1$  gives:

$$-2 = c_1 + c_2$$

Taking the derivative of  $x_1$  yields:

$$x_1' = 2 c_1 e^{2t} + (1/2) c_2 e^{t/2}$$

Substituting  $x_1$  and  $x_1'$  back into equation one simplifies to:

$$x_2 = c_1 e^{2t} - c_2 e^{t/2}$$

Using the initial condition for  $x_2$  gives:

$$1 = c_1 - c_2$$

Solving for  $c_1$  and  $c_2$ :

$$c_1 = -3/2$$

$$c_2 = -1/2$$

Thus the final solutions for  $x_1$  and  $x_2$  are:

$$x_1 = (-1/2) e^{2t} + (-3/2) e^{t/2}$$

$$x_2 = (-1/2) e^{2t} + (3/2) e^{t/2}$$

1)(35pts) Express the general solution of the given system of equations in terms of real-valued functions.

6

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$$

**Answer:**

First we find the eigen values of the matrix A.

This is done by taking the determinant of  $(A-rI)$ :

$$\det \begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} = r^2 + r - 6$$

Set the determinant equal to zero and solve for the eigen values:

$$r^2 + r - 6 = 0$$

$$(r-2)(r+3)=0$$

$$r = 2, -3$$

Use the eigen values to find the associative eigen vectors:

**for 2:**

$$\begin{pmatrix} 1-2 & 1 \\ 4 & -2-2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}$$

**row reducing we get:**

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \text{ so the resulting eigen vector is: } \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \xi^{(1)}$$

**now for -3:**

$$\begin{pmatrix} 1+3 & 1 \\ 4 & -2+3 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}$$

**row reducing we get:**

$$\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix} \text{ so the resulting eigen vector is: } \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \xi^{(2)}$$

Therefore the independent solutions are:

$$\mathbf{x} = \mathbf{c}_1 \xi^{(1)} e^{2t} + \mathbf{c}_2 \xi^{(2)} e^{-3t}$$

or, inserting the eigen vectors:

$$\mathbf{x} = \mathbf{c}_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \mathbf{c}_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}.$$

(40 pts)

2) Solve the following IVP.

$$y'' - y' = \cos(2t) + \cos(2t - 12)u_6(t) \quad y(0) = -4, y'(0) = 0$$

Solution:

First rewrite the DE to more easily see the function being shifted.

$$y'' - y' = \cos(2t) + \cos(2(t - 6))u_6(t)$$

The function being shifted is  $\cos(2t)$ . Taking the Laplace transform of everything and plugging in initial conditions gives:

$$\begin{aligned} s^2 Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) &= \frac{s}{s^2 + 4} + \frac{se^{-6s}}{s^2 + 4} \\ (s^2 - s)Y(s) + 4s - 4 &= \frac{s}{s^2 + 4} + \frac{se^{-6s}}{s^2 + 4} \end{aligned}$$

Now solve for  $Y(s)$ .

$$\begin{aligned} (s^2 - s)Y(s) &= \frac{s + se^{-6s}}{s^2 + 4} - 4s + 4 \\ Y(s) &= \frac{s(1 + e^{-6s})}{s(s-1)(s^2 + 4)} - 4 \frac{s-1}{s(s-1)} \\ &= \frac{1 + e^{-6s}}{(s-1)(s^2 + 4)} - \frac{4}{s} \end{aligned}$$

Let  $F(s) = \frac{1}{(s-1)(s^2 + 4)}$ . Expand into partial fractions, then take the inverse Laplace:

$$F(s) = \frac{1}{(s-1)(s^2 + 4)} = \frac{1}{5} \left( \frac{1}{s-1} - \frac{s+1}{s^2 + 4} \right)$$

$$f(t) = \frac{1}{5} \left( e^t - \cos(2t) - \frac{1}{2} \sin(2t) \right)$$

From above, we have:  $Y(s) = (1 + e^{-6s})F(s) - \frac{4}{s}$

$$Y(s) = F(s) + F(s)e^{-6s} - \frac{4}{s}$$

Take the inverse Laplace of  $Y(s)$ . The solution is

$$y(t) = f(t) + u_6(t)f(t-6) - 4$$

with  $f(t)$  given above.

good decomposition  
- second I've seen.

3.)(45pts) For the following system of equations

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{x}$$

a.) find the general solution

b.) find the solution that satisfies the initial condition  $\mathbf{x}_0 = (1, -1)$ .

### Solution

a.) First we must find the Eigen values and vectors for the matrix

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}.$$

To do so we compute the determinate of the following matrix

$$\det[A - \lambda I] = 0 \Rightarrow \det \begin{pmatrix} 2 - \lambda & 1 \\ -1 & 4 - \lambda \end{pmatrix} = 0$$

which yields the characteristic polynomial

$$\lambda^2 - 6\lambda + 9 = 0$$

Since

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

we have a repeated eigenvalue equal to 3.

Next find the associated eigenvector  $\xi_1$ .

To do so replace the value found for  $\lambda$  in the matrix  $[A - \lambda I]$  which yields

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which implies

$$\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Next we must find a second vector  $\eta_1$ . The Equation giving this vector is

*this matrix has no nontrivial null space - hence, an error has occurred.*

*In particular  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .*

$$A\eta_1 = \lambda \eta_1 + \xi_1$$

This yields the following matrix

$$\begin{pmatrix} -1 & 1 & | & 1 \\ -1 & 1 & | & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & | & -1 \\ 0 & 0 & | & 0 \end{pmatrix}$$

not possible with your starting point \*

Yielding the eigenvector

$$\eta_1 = \begin{pmatrix} \zeta \\ 1 \end{pmatrix}$$

Where  $\zeta$  may be chosen to be any number. So if we take  $\zeta = 0$ , we get

$$\eta_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

?  $s - 1 = -1$   
 $\Rightarrow s = 0$   
 where did you get this? here  $s$  must be zero.  
 good thing!

Therefore the two independent solutions are

$$\mathbf{x}_1 = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{x}_2 = e^{3t} \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

The general solution will be

$$\mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

b) In order to find the solution that satisfies the initial condition

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

we have

$$\mathbf{x}_0 = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

this implies  $c_1 = 1$  and  $c_2 = -2$ . Which yields the solution

can't start a sentence with "which..."

$$\mathbf{x}(t) = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + -2e^{3t} \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = e^{3t} \begin{pmatrix} 1-2t \\ -1-2t \end{pmatrix}.$$

4. (40 points)

Problem:

Given that  $y'' + y' - 12y = e^{2t}$   $y(0) = 3$ ,  $y'(0) = 4$ , solve the initial value problem using the Laplace transform.

Solution:

Taking the Laplace transform of both sides gives:

$$s^2 \mathcal{L}\{y\} - 3s - 4 + s \mathcal{L}\{y\} - 3 - 12 \mathcal{L}\{y\} = \frac{1}{s-2}$$

Solving for  $\mathcal{L}\{y\}$  gives:

$$\begin{aligned} \mathcal{L}\{y\} &= \frac{1}{(s-2)(s-3)(s+4)} + \frac{3s+7}{(s-3)(s+4)} \\ &= \frac{-\frac{1}{6}}{s-2} + \frac{\frac{1}{7}}{s-3} + \frac{\frac{1}{42}}{s+4} + \frac{\frac{5}{7}}{s+4} + \frac{\frac{16}{7}}{s-3} = \frac{-\frac{1}{6}}{s-2} + \frac{\frac{17}{7}}{s-3} + \frac{\frac{31}{42}}{s+4} \end{aligned}$$

Taking the inverse Laplace transform gives:

$$y = -\frac{1}{6}e^{2t} + \frac{17}{7}e^{3t} + \frac{31}{42}e^{-4t}$$

(40 pts)

Using the Laplace Transform method, Find the solution to the Differential Equation

$$y'' - 4y = e^{2t}$$

Satisfying the initial conditions

$$y(0) = 0 \quad y'(0) = 1$$

**Solution**

Using the table 6.2.1 on pg. 319 of the text and Corollary 6.2.2 we take the Laplace Transform of the differential equation and we get

$$s^2 Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{1}{s-2}$$

Where  $Y(s) = \mathcal{L}\{y\}$

Then we solve for  $Y(s)$  and we get

$$Y(s) = \frac{1}{(s-2)(s^2-4)} - \frac{1}{(s^2-4)}$$

Then we must use partial fractions to split up the first term into the following form

$$\frac{1}{(s-2)(s^2-4)} = \frac{a}{(s-2)} + \frac{bs+c}{(s^2-4)}$$

We then get

$$as^2 - 4a + bs^2 - 2bs + cs - 2c = 1$$

or

$$a - b = 0, \quad c - 2b = 0, \quad -2c = 1$$

Solving the system of equations yields

$$a = 1, \quad b = 1, \quad c = -\frac{1}{2}$$

Therefore we get

$$Y(s) = \frac{1}{(s-2)} + \frac{s}{(s^2-4)} - \left(\frac{1}{2}\right) \frac{1}{(s^2-4)}$$

From the lines 2, 7, and 8 of Table 6.2.1, the solution of the initial value problem is

$$y = e^{2t} + \cosh(2t) - \frac{\sinh(2t)}{4}$$

(25pts)

Solve the given set of equations:

$$x_1 + 2x_2 - x_3 = 1$$

$$2x_1 + x_2 + x_3 = 1$$

$$x_1 - x_2 + 2x_3 = 1$$

Putting this system in matrix form yields:

$$\left( \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & 1 \end{array} \right)$$

We must not try to get the matrix into rref.

$r_2 - 2r_1$  yields:

$$\left( \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & -3 & 3 & 0 \end{array} \right)$$

Dividing row 2 by -3 yields:

$$\left( \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1/3 \\ 0 & -3 & 3 & 0 \end{array} \right) \Rightarrow r_1 - 2r_2 \text{ yields: } \left( \begin{array}{ccc|c} 1 & 0 & 1 & 1/3 \\ 0 & 1 & -1 & 1/3 \\ 0 & -3 & 3 & 0 \end{array} \right)$$

$r_3 + 3r_2$  yields:

$$\left( \begin{array}{ccc|c} 1 & 0 & 1 & 1/3 \\ 0 & 1 & -1 & 1/3 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

$0x_1 + 0x_2 + 0x_3$  cannot equal 1 so this system has no solution.

— why?

reference  
to linear  
algebra

ordinary  
differential  
equations?

(300rs)

Solve the following differential equation:

$$y'' + 4y' - 5y = g(t)$$

$$y(0) = 0 \quad y'(0) = 0 \quad g(t) = \begin{cases} 1, & 0 \leq t < 2 \\ 0, & 2 \leq t < 5 \\ \frac{t-5}{2}, & 5 \leq t \end{cases}$$

solution:

convert  $g(t)$ :

$$g(t) = 1 - u_2(t) + u_5(t) \left( \frac{t-5}{2} \right)$$

take the laplace transform after substituting  $g(t)$  back in the original equation:

$$s^2 \ell\{y\} + 4s \ell\{y\} - 5 \ell\{y\} = \frac{1 - e^{-2s}}{s} + \frac{e^{-5s}}{s} \left( \frac{1}{2s^2} - \frac{5}{2s} \right)$$

Factor the transform out on the left hand side of the equation to obtain:

$$(s^2 + 4s - 5) \ell\{y\} = \frac{1 - e^{-2s}}{s} + \frac{e^{-5s}}{s} \left( \frac{1}{2s^2} - \frac{5}{2s} \right)$$

Divide the polynomial from both sides;

$$\ell\{y\} = \frac{1 - e^{-2s}}{s(s-1)(s-5)} + \frac{e^{-5s}}{s} \left( \frac{1}{2s^2(s-1)(s-5)} - \frac{5}{2s(s-1)(s-5)} \right)$$

Separate into partial fractions:

$$\ell\{y\} = \left( \frac{1/30}{s+5} + \frac{5/12}{s-1} - \frac{1/5}{s} \right) \left( \frac{1 - e^{-2s}}{s} \right) + \frac{e^{-5s}}{s} \left( \frac{-1/300}{s+5} + \frac{1/12}{s-1} - \frac{2/25}{s^2} - \frac{1/10}{s} - \left( \frac{1/12}{s+5} + \frac{5/12}{s-1} - \frac{1/2}{s} \right) \right)$$

Take the inverse Laplace transform

$$y = \left( \frac{1}{30} e^{-5t} + \frac{1}{6} e^t - \frac{1}{5} \right) (1 - u_2(t)) + u_5(t) \left( \frac{13}{150} e^{-5t} - \frac{1}{3} e^t - \frac{t}{10} + \frac{2t}{50} \right)$$

see the even!

$s^2 + 4s - 5 \neq (s-1)(s-5)$

same type of error.

(30 points)

Find the general solution of the following system.

$$\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \mathbf{x} \quad (1) \quad \text{where } A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

By making the assumption that,  $\mathbf{x} = \varepsilon^{\lambda t}$  what?  
The problem takes the form of

$$\begin{bmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \mathbf{0} \quad (2)$$

In order to find the eigenvectors of (2), I need to find the eigenvalues of its corresponding matrix.

The eigenvalues are  $\lambda = 1, 5$ . why?

For the eigenvalue of 5: the matrix takes these values

$$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \mathbf{0} \longrightarrow \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \mathbf{0}$$

So solving for my eigenvector I get

$$\varepsilon = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and my linearly independent solution is } \mathbf{x}^{(1)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{5t}.$$

Repeating the same steps for the eigenvalue of 1 I get my second linearly independent solution to be,

$$\mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t \text{ — maybe — can't tell}$$

My general solution of (1) is a linear combination of these two linearly independent solutions or:

$$\mathbf{x} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t$$

8

1. Use the Laplace transform to solve the given initial value problem

$$y'' + 2y' + y = 4e^{-t}, \quad y(0) = 2, y'(0) = -1$$

You may reference the following table of known Elementary Laplace Transforms

Table of Elementary Laplace Transforms

| $f(t) = \mathcal{L}^{-1}\{F(s)\}$               | $F(s) = \mathcal{L}\{f(t)\}$          |
|---|---------------------------------------|
| $e^{at}$  | $\frac{1}{s-a}$                       |
| $t^n e^{at}, \quad n = \text{positive integer}$ | $\frac{n!}{(s-a)^{n+1}}, \quad s > a$ |

45 points

Solution:

Take the Laplace transform of the given equation

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 4\mathcal{L}\{e^{-t}\}$$

$$(s^2 \mathcal{L}\{y\} - sy(0) - y'(0)) + 2(s\mathcal{L}\{y\} - y(0)) + \mathcal{L}\{y\} = 4\left(\frac{1}{s+1}\right)$$

Solve for  $\mathcal{L}\{y\}$

$$(s^2 \mathcal{L}\{y\} - 2s - 1) + (2s\mathcal{L}\{y\} - 2(2)) + \mathcal{L}\{y\} = 4\left(\frac{1}{s+1}\right)$$

$$\mathcal{L}\{y\}(s^2 + 2s + 1) = 4\left(\frac{1}{s+1}\right) + 2s + 3$$

$$\mathcal{L}\{y\} = \left(4\left(\frac{1}{s+1}\right) + 2s + 3\right)\left(\frac{1}{(s+1)^2}\right)$$

$$\mathcal{L}\{y\} = 4\left(\frac{1}{(s+1)^3}\right) + 2\left(\frac{s+1}{(s+1)^2}\right) + \frac{1}{(s+1)^2}$$

$$\mathcal{L}\{y\} = 2\left(\frac{2}{(s+1)^3}\right) + \frac{1}{(s+1)^2} + 2\left(\frac{1}{s+1}\right)$$

Take the inverse Laplace transform to solve for y

$$y = 2t^2 e^{-t} + te^{-t} + 2e^{-t}$$

2) Solve the following I.V.P.

$$\underline{x}' = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix} \underline{x}, \quad \underline{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

(45 points)

Solution

We begin with the ansatz:

$\underline{x} = c_1 e^{\lambda_1 t} \xi_1 + c_2 e^{\lambda_2 t} \xi_2$  where  $\lambda$  is an eigen value and  $\xi$  is an associated eigen vector

Therefore we must first find the eigen values for this matrix

$$\det \begin{bmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{bmatrix} = 0$$

$$(-2-\lambda)(4-\lambda) + 5 = 0$$

$$-8 - 2\lambda + \lambda^2 + 5 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = 3, -1$$

So our  $\lambda_1 = 3$  and  $\lambda_2 = -1$

We now need to find corresponding eigen vectors (note there are infinite, we only need one for each eigen value)

We do so by finding an element of the nulspace of the matrix  $(A - \lambda I)$

$$\xi_1 \in \text{nul} \begin{bmatrix} -2-\lambda_1 & 1 \\ -5 & 4-\lambda_1 \end{bmatrix} = \text{nul} \begin{bmatrix} -2-3 & 1 \\ -5 & 4-3 \end{bmatrix} = \text{nul} \begin{bmatrix} -5 & 1 \\ -5 & 1 \end{bmatrix} = \text{nul} \begin{bmatrix} -5 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\xi_2 \in \text{nul} \begin{bmatrix} -2-\lambda_2 & 1 \\ -5 & 4-\lambda_2 \end{bmatrix} = \text{nul} \begin{bmatrix} -2+1 & 1 \\ -5 & 4+1 \end{bmatrix} = \text{nul} \begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix} = \text{nul} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We insert these eigen vales and vectors into our ansatz

$$\underline{x} = c_1 e^{3t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

And now solve the I.V.P.

$$\underline{x}(0) = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 5 & 1 \end{bmatrix} * \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & | & 1 \\ 5 & 1 & | & 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & | & 1 \\ 0 & -4 & | & -2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & | & 1 \\ 0 & 1 & | & 1/2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 1/2 \\ 0 & 1 & | & 1/2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$c_1 = c_2 = 1/2$$

We insert these  $c_1$  and  $c_2$  into our general solution to get our final solution.

$$\underline{x} = 1/2 e^{3t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 1/2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3. Solve the following initial value problem which has repeated eigen values:

$$\mathbf{x}' = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \mathbf{x} \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

**45 Points**

Solution:

First, we start out by finding the eigenvalues. To do this we set  $\det[\mathbf{A} - \lambda \mathbf{I}] = 0$

$$\det \begin{bmatrix} 3-\lambda & 9 \\ -1 & -3-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 = 0 \Rightarrow \lambda = 0$$

Now we have to find the eigenvector:

$$\xi = \text{Nul} \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} = \text{Nul} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$$

Because we only have one eigenvalue and one eigenvector, we have to also find  $\eta$  such that

$$\begin{aligned} (\mathbf{A} - \lambda \mathbf{I})\eta &= \xi \\ \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \eta &= \begin{bmatrix} 3 \\ -1 \end{bmatrix} \Rightarrow 3\eta_1 + 9\eta_2 = 3 \\ &\Rightarrow \eta_1 + 3\eta_2 = 1 \end{aligned}$$

From this we can pick  $\eta$  to be any vector that satisfies this equation.

$$\eta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Now we can set up our two solutions:

$$\begin{aligned} \mathbf{x}_1(t) &= \xi e^{\lambda t} & \mathbf{x}_2(t) &= \xi t e^{\lambda t} + \eta e^{\lambda t} \\ \mathbf{x}_1(t) &= \begin{bmatrix} 3 \\ -1 \end{bmatrix} & \mathbf{x}_2(t) &= \begin{bmatrix} 3 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Now to solve the initial value problem, we will find  $\Phi(t)$  because  $\mathbf{x}(t) = \Phi(t)\mathbf{x}_0$ :

$$\mathbf{X} = \begin{bmatrix} 3 & 3t+1 \\ -1 & -t \end{bmatrix} \quad \mathbf{X}^{-1}(0) = \frac{1}{\det} \text{adj.} = \frac{1}{1} \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\Phi(t) = \mathbf{X}(t)\mathbf{X}^{-1}(0) = \begin{bmatrix} 3 & 3t+1 \\ -1 & -t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3t+1 & 9t \\ -t & 1-3t \end{bmatrix}$$

So the solution is:

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 = \begin{bmatrix} 3t+1 & 9t \\ -t & 1-3t \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2+42t \\ 4-14t \end{bmatrix}$$

— ed !

4) What is an ansatz (i.e. what does “ansatz” mean)?

(15 points)

Ansatz is a German language term often used by physicists and mathematicians. An ansatz is an assumed form for a mathematical function that is not based on any underlying theory or principle.

In other words it's **a really good guess as to the form of the solution.**

(9)

1. (40 points)

Find all of the eigenvalues and eigenvectors of the matrix  $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ .

**Solution:**

To find the eigenvalues, we need to find solutions of the equation

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0. \quad (1)$$

Equation (1) has nonzero solutions if and only if  $\lambda$  is chosen so that

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0. \quad (2)$$

The corresponding matrix of equation (2) is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} =$$

$$(3-\lambda)(-\lambda)(3-\lambda) + 16 + 16 - 4(3-\lambda) - 4(3-\lambda) + 16\lambda = 0. \quad (3)$$

Simplifying equation (3) yields

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0. \quad (4)$$

Factoring equation (4) gives us

$$(\lambda + 1)(\lambda + 1)(\lambda - 8) = 0. \quad (5)$$

From equation (5) we have that the eigenvalues are

$$\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 8.$$

To find the eigenvectors, we find that they must satisfy the matrix from (3) and satisfy equation (1), so for  $\lambda_1 = -1$ , and  $\lambda_2 = -1$ , we have the following matrix

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ which reduces to } \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ so that the}$$

$$\text{eigenvectors are } \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}.$$

Following the same process for  $\lambda_3 = 8$  gives us

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ which reduces to } \begin{pmatrix} 0 & 2 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ so the}$$

$$\text{eigenvector is } \mathbf{x}^{(3)} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

2) (40 points)

Express the general solution in terms of real-valued functions. Then describe the resulting direction field behavior as  $t \rightarrow \infty$ .

Lastly evaluate the solution at the given initial conditions.

$$x' = \begin{pmatrix} 1 & 5 \\ -1 & -3 \end{pmatrix} x \quad (1)$$

$$x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (2)$$

**Solution:**

Using the equations below the eigenvalues can be found and placed into a related equation.

$$\begin{aligned} x &= \xi e^{rt}, & \det(A - rI) &= 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \det \left[ \begin{pmatrix} 1 & 5 \\ -1 & -3 \end{pmatrix} - r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \det \left[ \begin{pmatrix} 1-r & 5 \\ -1 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right] \\ &= (1-r)(-3-r) - (-1)(5) \\ &= r^2 + 2r + 2 \\ r &= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)} \end{aligned} \quad (3)$$

Therefore the eigenvalues are  $r_1 = -1 + i$   
 $r_2 = -1 - i$

*just factor for these  
benign e. values.*

Substituting the eigenvalues into (3),  $\xi_1$  and  $\xi_2$  can be found.

Using  $r_1 = -1 + i$

$$0 = \begin{pmatrix} 1+1-i & 5 \\ -1 & -3+1-i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$0 = \begin{pmatrix} 2-i & 5 \\ -1 & -2-i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\text{Then, } (2-i)\xi_1 + (5)\xi_2 = 0$$

$$(2-i)\xi_1 = (-5)\xi_2$$

$$\text{If } \xi_2 = (2-i), \text{ then } \xi_1 = (5)$$

$$x^{(1)}(t) = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} e^{(-1+i)t}$$

$$x^{(1)}(t) = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} e^{(-1)t} [\cos(t) + i \sin(t)]$$

$$x^{(1)}(t) = e^{-t} \begin{bmatrix} 5 \cos(t) + 5i \sin(t) \\ 2 \cos(t) + 2i \sin(t) - i(\cos(t) + i \sin(t)) \end{bmatrix}$$

$$x^{(1)}(t) = e^{-t} \begin{bmatrix} 5 \cos(t) \\ 2 \cos(t) - \sin(t) \end{bmatrix} + i e^{-t} \begin{bmatrix} \sin(t) \\ 2 \sin(t) - \cos(t) \end{bmatrix}$$

Hence the general solution of the differential equation will be,

$$x = C_1 e^{-t} \begin{bmatrix} 5 \cos(t) \\ 2 \cos(t) - \sin(t) \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} \sin(t) \\ 2 \sin(t) - \cos(t) \end{bmatrix} \quad (4)$$

The function overtime, due to the exponential  $e^{-t}$ , will spiral around and converge towards the t-axis as  $t \longrightarrow \infty$ .

Then evaluate (4) at the given initial conditions.

$$x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = C_1 e^0 \begin{bmatrix} 5 \cos(0) \\ 2 \cos(0) - \sin(0) \end{bmatrix} + C_2 e^0 \begin{bmatrix} \sin(0) \\ 2 \sin(0) - \cos(0) \end{bmatrix}$$

$$x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = C_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Two equations can be found and used to determine  $C_1$  and  $C_2$ .

$$2 = 5C_1 \quad \text{and} \quad -1 = 2C_1 - C_2$$

$$C_1 = \frac{2}{5}, \quad C_2 = -\frac{9}{5}$$

Substituting the above values into (4).

$$x(t) = \left(\frac{2}{5}\right)e^{-t} \begin{bmatrix} 5\cos(t) \\ 2\cos(t) - \sin(t) \end{bmatrix} + \left(-\frac{9}{5}\right)e^{-t} \begin{bmatrix} \sin(t) \\ 2\sin(t) - \cos(t) \end{bmatrix}$$

*not correct*

3) (40 points)

Prove the associative property of convolution integrals exists. Show that

$$F*(G*H) = (F*G)*H$$

#1

Given that:

$$F = \sin t$$

$$G = \cos t$$

$$H = t$$

**Solution**

First let us solve the left side starting with what is in the parenthesis

$$\begin{aligned} G*H &= \int_0^t \cos(t-\tau)\tau d\tau \\ &= -\sin(t-\tau)\tau \Big|_0^t - \int_0^t -\sin(t-\tau)d\tau \\ &= -\sin(t-\tau) \Big|_0^t + \cos(t-\tau) \Big|_0^t \\ &= [(-\sin(0)t + \sin(t)(0))] + [\cos(0) - \cos(t)] \\ &= 0 + (1 - \cos t) \end{aligned}$$

$$G*H = 1 - \cos t$$

Now taking the value of  $G*H$  found in the parenthesis we will find  $F*(\text{of that value})$

*which are you going to do? program #1 or program #2?*

$$F*(G*H) = F*(1-\cos t)$$

$$= \int_0^t \sin(t-\tau) - \sin(t-\tau) \cos \tau d\tau$$

$$= \int_0^t \sin(t-\tau) - \int_0^t \sin(t-\tau) \cos \tau d\tau$$

$$= \cos(t-\tau) \cos \tau \Big|_0^t + \int_0^t \cos(t-\tau) \sin \tau d\tau$$

$$= \cos(t-\tau) \cos \tau \Big|_0^t - \sin(t-\tau) \sin \tau \Big|_0^t - \int_0^t \sin(t-\tau) \cos \tau d\tau$$

$$2 \int_0^t \sin(t-\tau) \cos \tau d\tau = \cos(t-\tau) \cos \tau \Big|_0^t - \sin(t-\tau) \sin \tau \Big|_0^t$$

$$\int_0^t \sin(t-\tau) \cos \tau d\tau = \frac{1}{2} (\cos(t-\tau) \cos \tau \Big|_0^t - \sin(t-\tau) \sin \tau \Big|_0^t)$$

$$= \frac{1}{2} [(\cos(0)\cos(t) - (\cos(t)\cos(0)) - (\sin(0)\sin(t) - (\sin(t)\sin(0)))]$$

$$= \frac{1}{2} (0-0-0-0)$$

$$= 0 \text{ — no way}$$

So the result of  $F*(G*H) = 0$

Now we will solve the right side of the equation starting with what is in the parenthesis:

$$F*G = \int_0^t \sin(t-\tau) \cos \tau d\tau = t \sin t \neq 0$$

We have solved this integral in the previous step – the result is 0 so if  $F*G = 0$  then  $H*(F*G)$  will be 0 because  $H*0 = 0$ .

We find that  $0=0$  - both sides equal 0 - so we have proven the associative property of convolution integrals.

If you get 0 for the conv. of 2 cont. functions, one of them

is zero.

4a. (25 points)

Find the solution to the initial value problem

$$y'' + 2y' + 2y = \delta(t - 3)$$

$$y(0) = 1$$

$$y'(0) = -1.$$

**Solution:**

$$y(t) = u_{-3}(t)e^{-(t+3)} \sin(t+3) + e^{-t} \cos t$$

Take the Laplace Transform of both sides. For the left side Theorem 6.2.1 states that  $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ . For the right side, the Laplace transform for the impulse function is  $\mathcal{L}\{\delta(t - c)\} = e^{-cs}$ . Taking the transform of both sides yield

$$(s^2 + 2s + 2)\mathcal{L}\{y(t)\}(s) - sy(0) - y'(0) - 2y(0) = e^{-3s}. \quad (0.1)$$

Algebraic manipulations find

$$\mathcal{L}\{y\}(s) = \frac{1}{s^2 + 2s + 2} [e^{-3s} + sy(0) + y'(0) + 2y(0)]. \quad (0.2)$$

We can plug the initial data into the equation and simplify it to

$$\mathcal{L}\{y\}(s) = \frac{e^{-3s}}{s^2 + 2s + 2} + \frac{s + 1}{s^2 + 2s + 2}. \quad (0.3)$$

The denominator can be modified by completing the square.  $s^2 + 2s + 2$  can be written as  $s^2 + 2s + 1 + 1 = (s + 1)^2 + 1$ . Now equation 1.3 becomes

$$\mathcal{L}\{y\}(s) = \frac{e^{-3s}}{(s + 1)^2 + 1} + \frac{s + 1}{(s + 1)^2 + 1} \quad (0.4)$$

Each term in the right hand of equation 1.4 can now clearly be seen as the Laplace transforms of common functions. The exponential in the first term is the result of a unit

+3, not -3

step shift. Entering the Laplace transforms, equation 1.4 becomes

$$\Lambda\{y\}(s) = \Lambda\{u_{-3}(t)e^{-1(t+3)}\sin(t+3)\}(s) + \Lambda\{e^{-1t}\cos t\}(s) \quad (0.5)$$

The equation is now ready to be converted back into the normal time domain. The inverse Laplace transforms of both sides yields our solution:

$$y(t) = u_{-3}(t)e^{-(t+3)}\sin(t+3) + e^{-t}\cos t \quad (0.6)$$

4b. (5 points)

Rewrite the solution vertically

**Solution:**

$$y(t) = \begin{cases} e^{-t}\cos t, & t \leq -3 \\ e^{-(t+3)}\sin(t+3) + e^{-t}\cos t, & -3 \leq t \end{cases}$$

$t \leq 0$   
 $0 \leq t \leq 3$   
 $t \geq 3$  } relevant intervals

1. Find the solution of the initial value problem

$$y'' + y = \delta(t - 2\pi) \cos t; \quad y(0) = 0, y'(0) = 1.$$

25 points

Solution

First we will take the Laplace transform of the differential equation,

$$(s^2 + 1)Y(s) = e^{-2\pi s} \cos t$$

Introducing the initial values and solving for  $Y(s)$  gives us,

$$Y(s) = \frac{e^{-2\pi s}}{(s^2 + 1)}$$

To find  $y = \phi(t)$  it is convenient to write  $Y(s)$  as

$$Y(s) = e^{-2\pi s} H(s).$$

Where

$$H(s) = \frac{1}{(s^2 + 1)}.$$

By Theorem 6.3.2 or from Table 6.2.1 in the book,

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + 1^2)} \right\} = \sin t.$$

Hence, providing the shift where  $t \geq 2\pi$ ,

$$y = \phi(t) = \mathcal{L}^{-1} \{Y(s)\} = \sin t + u_{2\pi}(t) \sin(t - 2\pi)$$

Or in another form

$$y = \begin{cases} \sin t, & t < 2\pi \\ \sin(t - 2\pi), & t \geq 2\pi \end{cases}$$

Handwritten notes:  
 $\cos 2\pi = 1$   
 $\delta(t - 2\pi)$   
 $\delta(t - 2\pi)$

10

you didn't use this nontrivial initial data

where did this come from?

no way.

2. Find all eigenvalues and their corresponding eigenvectors of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & -2 \\ 6 & 2 & 1 \end{pmatrix}.$$

45 points

### Solution

If we declare our matrix to be matrix A, then in order to satisfy  $(A - I\lambda)x = 0$ , we have

$$\begin{pmatrix} 1 - \lambda & 0 & 0 \\ 4 & 1 - \lambda & -2 \\ 6 & 2 & 1 - \lambda \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The eigenvalues are shown by the roots of the determinant of  $(A - I\lambda)$ .

The determinant of  $(A - I\lambda)$  can be found by performing the following operation

$$\det \cdot \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 4 & 1 - \lambda & -2 \\ 6 & 2 & 1 - \lambda \end{vmatrix} = (1 - \lambda) \cdot [(1 - \lambda) \cdot (1 - \lambda) - (-2) \cdot (2)] = 0.$$

This can be simplified to the equation  $(1 - \lambda)(\lambda^2 - 2\lambda + 5) = 0$ .

The second set of parenthesis contains a polynomial which can be solved by the quadratic equation to find that  $\lambda_1 = 1 + 2i$  and  $\lambda_2 = 1 - 2i$ . The first set shows that the equation equals zero when  $\lambda_3 = 1$ .

*just factor for benign roots*

For our first eigenvalue  $\lambda_1 = 1 + 2i$  we can find its eigenvector by substituting in our  $\lambda_1$  value, which gives us

$$\begin{pmatrix} -2i & 0 & 0 \\ 4 & -2i & -2 \\ 6 & 2 & -2i \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Our first eigenvector is  $x^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}.$

For our second eigenvalue  $\lambda_2 = 1 - 2i$  we find that

$$\begin{pmatrix} 2i & 0 & 0 \\ 4 & 2i & -2 \\ 6 & 2 & 2i \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Our second eigenvector is  $x^{(2)} = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}.$

For our third eigenvalue  $\lambda_3 = 1$  we find that

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & -2 \\ 6 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Our third eigenvector is  $x^{(3)} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}.$

3. Find the general solution of the given initial value problem:

$$\underline{x}' = \begin{bmatrix} 2 & 5 \\ 0 & 6 \end{bmatrix} \underline{x}, \underline{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

40 pts

**Solution –**

To begin we remember that  $0 = \det(A - \lambda I)$  which gives us:

$$0 = \begin{vmatrix} 2 - \lambda & 5 \\ 0 & 6 - \lambda \end{vmatrix} = (2 - \lambda)(6 - \lambda) - 5(0) = (2 - \lambda)(6 - \lambda)$$

This easily shows us that  $\lambda_1 = 2, \lambda_2 = 6$

We then find our vectors  $\underline{\xi}_1, \underline{\xi}_2$  using the above information.

We have:

$$\underline{\xi}_1 \in \text{Nul}(A - \lambda_1 I) = \text{Nul} \begin{bmatrix} 0 & 5 \\ 0 & 4 \end{bmatrix} = \text{Nul} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \text{Span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

And

$$\underline{\xi}_2 \in \text{Nul}(A - \lambda_2 I) = \text{Nul} \begin{bmatrix} -4 & 5 \\ 0 & 0 \end{bmatrix} = \text{Span} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

The form for our general solution is:

$$\underline{x}(t) = c_1 \underline{\xi}_1 e^{\lambda_1 t} + c_2 \underline{\xi}_2 e^{\lambda_2 t}$$

Putting in our values for  $\lambda_1, \lambda_2, \underline{\xi}_1, \underline{\xi}_2$  we have:

$$\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 5 \\ 4 \end{bmatrix} e^{6t}$$

Then we put in our initial data which gives us:

$$\underline{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 4 \end{bmatrix}, c_1 = \frac{-7}{4}, c_2 = \frac{3}{4}$$

And our final solution:

$$\underline{x}(t) = \frac{-7}{4} \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + \frac{3}{4} \begin{bmatrix} 5 \\ 4 \end{bmatrix} e^{6t}$$

4. Write the solution of the initial value problem in terms of a convolution integral.

$$y'' + 4y' + 4y = g(t) \quad y(0) = 2 \quad y'(0) = -3$$

40 points

### Solution

Take the Laplace of the left and right side;

$$s^2 Y(s) - 2s + 3 + 4sY(s) + 8 + 4Y(s) = L(g)(s)$$

Solve for  $Y(s)$ .

$$s^2 + 4s + 4Y(s) = 2s + 5L(g)(s)$$

$Y(s) = \frac{2s+5}{s^2+4s+4} + \frac{L(g)(s)}{s^2+4s+4}$  Then figure out the Laplace transforms for the different parts.

$$Y(s) = \frac{2s+5}{(s+2)^2} + \frac{L(g)(s)}{(s+2)^2} \quad \text{the left happens to split into two...}$$

$\frac{2s+5}{(s+2)^2} \Rightarrow \frac{2(s+2)}{(s+2)^2} + \frac{1}{(s+2)^2}$  which then is recognized as the following equations when put into time space... domain

$$\text{Laplace Inverse} \left( \frac{2(s+2)}{(s+2)^2} \right) + \text{Laplace Inverse} \left( \frac{1}{(s+2)^2} \right) = 2e^{-2t} + te^{-2t}$$

The right side converts to...

$$\text{Laplace Inverse} \left( \frac{L(g)(s)}{(s+2)^2} \right) = \int_0^t (t-\tau) e^{-2(t-\tau)} g(\tau) d\tau$$

Putting the right hand side with the left side gives the complete solution in terms of the convolution integral.

$$\text{Laplace Inverse} = y = 2e^{-2t} + te^{-2t} + \int_0^t (t-\tau) e^{-2(t-\tau)} g(\tau) d\tau$$

35 points - 1) Use the Laplace transform to solve the given initial value problem :

$$y'' + 2y' + 5 = 0$$

$$y(0) = 0, y'(0) = 2$$

**SOLUTION :**

make the following ansatz :

$$\mathcal{L}(y'') + 2\mathcal{L}(y') + 5\mathcal{L}(y) = 0$$

use the corollary to express  $\mathcal{L}(y'')$  and  $\mathcal{L}(y')$  in terms of  $\mathcal{L}(y)$  :

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + 2(s\mathcal{L}(y) - y(0)) + 5\mathcal{L}(y) = 0$$

which may be written as :

$$(s^2 + 2s + 5)Y(s) - (s+1)y(0) - y'(0) = 0$$

where  $Y(s) = \mathcal{L}(y)$

Substituting in the initial data, we have :

$$(s^2 + 2s + 5)Y(s) - 2 = 0$$

and solving for  $Y(s)$  we have :

$$Y(s) = \frac{2}{s^2 + 2s + 5} = \frac{2}{(s+1)^2 + 4} = \frac{2}{(s+1)^2 + 2^2}$$

The reverse Laplace transform of which is (see table 6.2.1) :

$$e^{-t}\sin(2t)$$

Therefore, the answer is :

$$y = e^{-t}\sin(2t)$$

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not an ansatz

(25 points) 2) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}. \quad (1)$$

The eigenvalues  $\lambda$  and eigenvectors  $\mathbf{x}$  satisfy the equation  $(A - \lambda I)\mathbf{x} = \mathbf{0}$ , or

$$\begin{pmatrix} 1-\lambda & 2 \\ 3 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (2)$$

The eigenvalues are the roots of the equation

$$\det(A - \lambda I) = \lambda^2 - 3\lambda - 4 = 0 \quad (3)$$

Therefore the eigenvalues are  $\lambda_1 = 4$  and  $\lambda_2 = -1$ .

To find the eigenvectors, first substitute the eigen values into one row of (2) to get:

$$-3x_1 + 2x_2 = 0$$

and

$$2x_1 + 2x_2 = 0.$$

Thus, the eigenvectors are

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

and

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

While we say that these are the eigenvectors, it should be remembered that any nonzero multiple of these vectors are also eigenvectors.

3 (40 points)

**Problem:**

Find the inverse Laplace Transform of the given function.

$$F(s) = \frac{e^{-2s}}{s^2 + s - 2}$$

**Solution:**

Note: Right off the bat, you should notice that this question pertains to the Laplace Transform of step functions because of the factor of  $s$  in the exponential.

Use the property that  $u_2(t)f(t-2) = \mathcal{L}^{-1}\{e^{-2s}F(s)\}$  (Theorem 6.3.1) to solve the problem.

Due to Thm. 6.3.1,

where  $\mathcal{L}\{F\}(t) = f(t)$

If  $G(s) = \left(\frac{1}{s^2 + s - 2}\right)$ , and  $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{e^{-2s}G(s)\} = u_2(t)g(t-2)$ .

Then

$$G(s) = \frac{1}{(s+2)(s-1)} = \frac{1}{3}\left(\frac{1}{s-1}\right) - \frac{1}{3}\left(\frac{1}{s+2}\right)$$

and

$$\mathcal{L}^{-1}\{G(s)\} = \frac{1}{3}e^t - \frac{1}{3}e^{-2t} = g(t).$$

Therefore

$$\mathcal{L}^{-1}\{F(s)\} = u_2(t) \left( \frac{1}{3}e^{t-2} - \frac{1}{3}e^{-2t+4} \right). \quad \leftarrow \text{Answer.}$$

(35 pts)  
Consider the vectors  $\mathbf{x}^{(1)}(t) = (t^2, 2t)$  and  $\mathbf{x}^{(2)}(t) = (e^t, e^t)$

- compute the Wronskian of  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$ .
- In what intervals are  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$  linearly independent?
- What conclusion can be drawn about the coefficients in the system of homogeneous differential equations satisfied by  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$ ?
- Find this system of equations and verify the conclusions of part c)

### Solution

- Set  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$  as the columns of a  $2 \times 2$  matrix,  $X$ . Find this matrix's determinant.  
From this, we have  $\det X = W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$

→  $W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = t(t-2)e^t$

- The intervals where  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$  are linearly independent include all values of  $t$  for which  $W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \neq 0$ . Observing that the zeros of this equation

$$W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = t(t-2)e^t$$

are  $t = 0$  and  $t = 2$ ,

→ we conclude that  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$  are linearly independent for all  $t$  except  $t = 0$  and  $t = 2$ .

→ c) We know that at least one of the coefficients must be discontinuous at  $t = 0$  and  $t = 2$ .

- We know that the system of equations has the following form:

$$\mathbf{x}' = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mathbf{x}$$

Inserting each of the solutions,  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$ , we have:

$$\begin{pmatrix} 2t \\ 2 \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$$

and

$$\begin{pmatrix} e^t \\ e^t \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} e^t \\ e^t \end{pmatrix}$$

Solving these matrices, we have

$$At^2 + 2Bt = 2t$$

$$Ct^2 + 2Dt = 2$$

$$Ae^t + Be^t = e^t$$

$$Ce^t + De^t = e^t$$

Solving this system of equations, we have

$$A = 0$$

$$B = 1$$

$$C = (2-2t)/(t^2-2t)$$

$$D = (t^2-2)/(t^2-2t)$$

→ Thus,

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ (2-2t)/(t^2-2t) & (t^2-2)/(t^2-2t) \end{bmatrix} \mathbf{x}$$

Part c) is verified because the bottom two entries are discontinuous at both  $t = 0$  and  $t = 2$ .

After thank, this is a good problem

\* Here fore they are lin. ind. vector-valued

functions over every interval - this regards the question you actually asked, rather than the one you

presumably wanted to ask.

## Problem # 1

---

(20 pts.) Solve the following differential equation (1.1) with associated initial data (1.2) using the methods of Laplace transform and inverse Laplace transform.

$$y'' + 5y = \sinh(t) \quad (1.1)$$

$$y(0) = 4, y'(0) = 0 \quad (1.2)$$

### *Solution*

We begin by taking the Laplace transform of both sides, which yields  $(\mathcal{L}\{y\}s^2 - 4s) + 5\mathcal{L}\{y\} = \frac{1}{s^2 - 1}$ . After rearranging terms, and using partial fractions, this becomes:

$$\mathcal{L}\{y\} = \frac{1}{6(s^2 - 1)} - \frac{1}{6(s^2 + 5)} + \frac{4s}{(s^2 + 5)} \quad (1.3)$$

We now use the inverse Laplace transform with equation (1.3) which yields the following solution to the initial value problem:

$$y = \frac{1}{6} \sinh(t) + \frac{1}{6\sqrt{5}} \left[ \sin(t) + \frac{4}{\sqrt{5}} \right] \cos(t)$$

## Problem # 2

---

(40 pts) Find the general solution of system (2.1)

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{A}\mathbf{x} \quad (2.1)$$

### Solution

We know by theory that the eigenvalues of this matrix are related to the solutions of this system of equations. Thus we start by finding eigenvalues for this matrix. We can find them by solving the equation  $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$  for  $\lambda$ , which yields  $(\lambda - 2)(\lambda + 1)^2$ . Thus  $\lambda_1 = 2$ ,  $\lambda_2, \lambda_3 = -1$ . Substituting  $\lambda_1, \lambda_2$  into  $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$  for  $\lambda$ , we find the following eigenvectors (paired with the corresponding eigenvalues):

$$\lambda_1 : \xi^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \lambda_2 : \xi^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \lambda_3 : \xi^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad (2.2)$$

We can find the eigenvectors for the repeated eigenvalue  $\lambda_2, \lambda_3 = -1$  because the nullspace of the eigenmatrix corresponding to this value is two dimensional. The multiplicity of the eigenvalue is two, hence we can simply apply it to the general solution twice, once for *each* eigenvector.

Thus the general solution of (2.1) using the eigenvectors of (2.2) is:

$$\mathbf{x} = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad (2.3)$$

## Problem # 3

---

(40 pts) Transform the given system of first order linear equations into a single equation of second order. Also find  $x_1$  and  $x_2$  that satisfy the given initial conditions.

$$x_1' = 3x_1 - 2x_2 \quad x_1(0) = 3 \quad (3.1)$$

$$x_2' = 3x_1 - 2x_2 \quad x_2(0) = \frac{1}{2} \quad (3.2)$$

### Solution

Substituting our initial data in (3.1) and (3.2) into the equations (3.1) and (3.2) we find that  $x_1'(0) = 8$  and  $x_2'(0) = 5$  (3.3). This provides us with an extension to our initial data we can use later.

$$x_1'(0) = 8 \text{ and } x_2'(0) = 5 \quad (3.3)$$

First, we seek a solution to  $x_1$ . To do this, we solve (3.1) for  $x_2$ , take the derivative of the result, and then substitute those results into (3.2). This gives us  $(-x_1'' + 3x_1') = 6x_1 - 2(-x_1' + 3x_1)$ . Solving algebraically we find that  $x_1'' - x_1' - 2x_1 = 0$  which is a second order equation that we can solve by the methods of previous chapters with relative ease.

$$x_1 = c_1 e^{2t} + c_2 e^{-t} \quad (3.4)$$

Now we turn our attention to  $x_2$ . For this equation, we solve (3.2) for  $x_1$ , take the derivative of the result, and then substitute those results into (3.1). This gives us  $x_2'' + 2x_2' = -6x_2 + 3(x_2' + 3x_2)$ . Solving algebraically we find that  $x_2'' - x_2' - 3x_2 = 0$  which we can solve as:

$$x_2 = c_1 e^{\frac{1}{2} + \sqrt{13}t} + c_2 e^{\frac{1}{2} - \sqrt{13}t} \quad (3.5)$$

Solving (3.4) and (3.5) using our initial data of (3.1), (3.2), and (3.3) we find that:

$$x_1 = \frac{11}{3} e^{2t} - \frac{2}{3} e^{-t}$$

And

$$x_2 = \frac{11}{6} e^{\frac{1}{2} + \sqrt{13}t} - \frac{4}{3} e^{\frac{1}{2} - \sqrt{13}t}$$

## Problem # 4

(50 pts.) Using what you know about step functions and Dirac delta functions, find the solution of the given initial value problem and then graph from  $t = 0 \rightarrow 3\pi$ .

$$y'' + y = u_0(t) + \delta(t - \pi) \quad y(0) = 0 = y'(0) \quad (4.1)$$

### Solution

First start by taking the Laplace transform of both sides. Remember that  $\mathcal{L}\{u_c\}(t) = \frac{e^{-cs}}{s}$  and  $\mathcal{L}\{\delta(t-c)\}(t) = e^{-cs}$ . This in conjunction with (4.1) yields the following:

$$\mathcal{L}\{y\}(s^2) - sy(0) - y'(0) + \mathcal{L}\{y\} = \frac{1}{s} + e^{-\pi s} - \frac{e^{-2\pi s}}{s} \quad (4.2)$$

From (4.1) we can apply initial data to (4.2) and, after some manipulation, it becomes:

$$\mathcal{L}\{y\} = \frac{1}{s(s^2+1)} + \frac{e^{-\pi s}}{(s^2+1)} + \frac{e^{-2\pi s}}{s(s^2+1)} \quad (4.3)$$

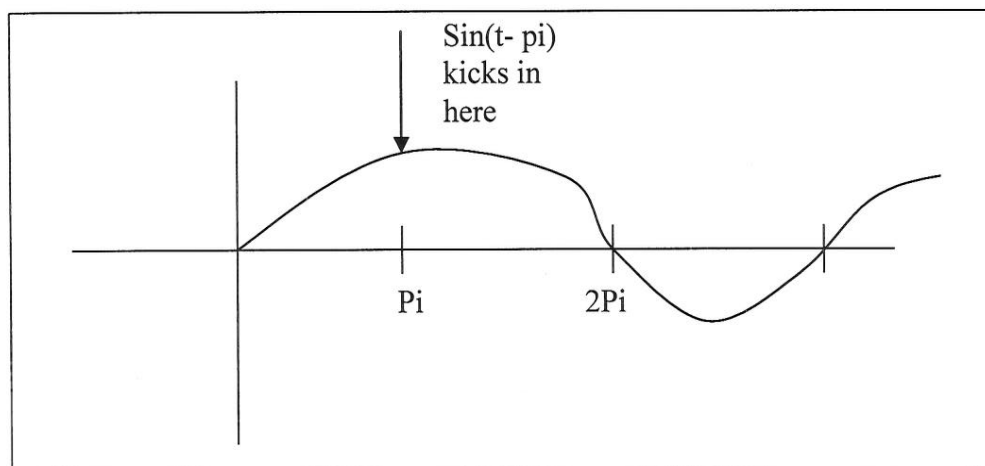
We can use partial fractions to separate  $\frac{1}{s(s^2+1)}$  into  $\frac{-s}{(s^2+1)} + \frac{1}{s}$  which, when applied to (4.3), becomes:

$$\mathcal{L}\{y\} = \frac{-s}{(s^2+1)} + \frac{1}{s} + \frac{e^{-\pi s}}{(s^2+1)} + e^{-2\pi s} \left[ \frac{-s}{(s^2+1)} + \frac{1}{s} \right] \quad (4.4)$$

Solving (3.4) and (3.5) using our initial data of (3.1), (3.2), and (3.3) we find that:

$$Y = 1 - \cos(t) + u_\pi(t) \sin(t - \pi) - u_{2\pi}(t)(1 - \cos(t))$$

Which we graph thus:



1)

Find the general solution of

$$\underline{x}' = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}$$

Solution:

Using the Ansatz that the solution will be of the form  $\underline{x} = \underline{\xi} e^{\lambda t}$ , find the eigenvalues and corresponding eigenvectors of the coefficient matrix.

$$\det(A - \lambda I) = 0$$
$$\det \begin{pmatrix} 1-\lambda & 3 & 2 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda)(1-\lambda) = 0$$
$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 2 \\ \lambda_3 &= 1 \end{aligned}$$

Now find  $\xi^{(1)}$ ,  $\xi^{(2)}$ , and  $\xi^{(3)}$

$$\lambda = 1$$

$$\begin{bmatrix} 0 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x_2 + 2x_3 = 0$$

$$x_2 + x_3 = 0$$

$$3x_2 + 2x_3 = x_2 + x_3$$

$$2x_2 + x_3 = 0$$

$$2x_2 = -x_3$$

$$\text{Thus } \xi^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \text{ and } \xi^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = 2$$

$$\begin{bmatrix} -1 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 3x_2 = 0$$

$$3x_2 = x_1$$

$$\text{Thus } \xi^{(2)} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \text{ and,}$$

$$\underline{x}^{(1)} = C_1 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} e^t, \underline{x}^{(2)} = C_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} e^{2t}, \text{ and } \underline{x}^{(3)} = C_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t.$$

Which leads to the general solution,

$$\underline{x} = C_1 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} e^t + C_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} e^{2t} + C_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t.$$

(40 points)

2)

**Express the solution of the given initial value problem in terms of a convolution integral.**

$$y'' + 2y' + 5y = 2\sin(\alpha t) \quad y(0) = 0, \quad y'(0) = 0$$

**Solution**

**The first step of the solution is to take the Laplace transform of the differential equation:**

$$(s^2 \mathcal{L}\{y\} - s y(0) - y'(0)) + (2s \mathcal{L}\{y\} - y(0)) + 5 \mathcal{L}\{y\} = \frac{2\alpha}{s^2 + \alpha^2}.$$

**Then by plugging in the initial data, we obtain:**

$$s^2 \mathcal{L}\{y\} + 2s \mathcal{L}\{y\} + 5 \mathcal{L}\{y\} = \frac{2\alpha}{s^2 + \alpha^2}.$$

**Then we solve for  $\mathcal{L}\{y\}$  to get:**

$$\mathcal{L}\{y\} = \left( \frac{\alpha}{s^2 + \alpha^2} \right) \left( \frac{2}{s^2 + 2s + 5} \right).$$

**Changing the form of the second term, we can recognize a Laplace transform pair:**

$$\mathcal{L}\{y\} = \left( \frac{\alpha}{s^2 + \alpha^2} \right) \left( \frac{2}{(s+1)^2 + 4} \right) \quad e^{at} \sin(bt) = \frac{b}{(s+a)^2 + b^2}.$$

**Since multiplying two arguments in the Laplace domain is equivalent to taking the convolution of two arguments in the time domain, our solution is:**

$$\mathcal{L}^{-1}\{y\} = \mathcal{L}^{-1} \left( \frac{\alpha}{s^2 + \alpha^2} \right) \left( \frac{2}{(s+1)^2 + 4} \right) = \sin(\alpha t) * e^{-t} \sin(t)$$

$$\mathcal{L}^{-1}\{y\} = \int_0^t \sin(\alpha(t-\tau)) e^{-\tau} \sin(\tau) d\tau = \int_0^t \sin(\alpha\tau) e^{-(t-\tau)} \sin(t-\tau) d\tau$$

(28 points)

3)

Find the inverse Laplace transform of the given function by using the convolution theorem (i.e. in terms of a convolution integral).

$$F(s) = \frac{s^2 - 4s}{(s^2 + 4)(s^2 - 8s + 18)}$$

so don't compute

explicitly using convolution

Solution:

For this solution, the key is to change the argument to a form that will match two Laplace transform pairs. Once we do this, we follow the same steps as the problem above in finding the inverse Laplace transform in terms of a convolution integral. So, with a little mathmagic:

$$F(s) = \frac{s^2 - 4s}{(s^2 + 4)(s^2 - 8s + 18)} = \frac{s(s-4)}{(s^2 + 4)[(s-4)^2 + 2]} = \left( \frac{s}{s^2 + 4} \right) \left( \frac{(s-4)}{(s-4)^2 + 2} \right).$$

Then, we can recognize our Laplace transform pairs:

$$e^{at} \sin(bt) = \frac{s-a}{(s+a)^2 + b^2}, \text{ and } \cos(at) = \frac{s}{s^2 + a^2}.$$

Since multiplying two arguments in the Laplace domain is equivalent to taking the convolution of two arguments in the time domain, our solution is:

$$\mathcal{L}^{-1}\{y\} = e^{2t} \sin(\sqrt{2}t) * \cos(2t) \quad \text{what's } y?$$

$$\mathcal{L}^{-1}\{y\} = \int_0^t e^{2(t-\tau)} \sin(\sqrt{2}(t-\tau)) \cos(\tau) d\tau = \int_0^t e^{2(\tau)} \sin(\sqrt{2}(\tau)) \cos(t-\tau) d\tau = ?$$

(20 points)

4)

Let  $f$  satisfy  $f(t+T) = f(t)$  for all  $t \geq 0$  and for some fixed positive number  $T$ ;  $f$  is said to be periodic with period  $T$  on  $0 \leq t < \infty$ . Show that

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$$

**Solution:**

Using the definition of the Laplace transform we get:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$f$  is periodic with period  $T$  and we therefore get:

$$f(t+T) = f(t)$$

We can then write the improper integral as :

$$\int_0^{\infty} e^{-st} f(t) dt = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t) dt$$

Then we change the variable  $t = r + nT$  and then we get:

$$F(s) = \sum_{n=0}^{\infty} \int_0^T e^{-s(r+nT)} f(r+nT) dr = \sum_{n=0}^{\infty} (e^{-sT})^n \int_0^T e^{-rs} f(r) dr$$

[Since  $f(r+nt) = f(r+(n-1)T) = f(r+(n-2)T) \dots = f(r+T) = f(r)$ ]

$$\sum_{n=0}^{\infty} (e^{-sT})^n \int_0^T e^{-rs} f(r) dr \text{ is a geometric series in the form } \sum_{n=0}^{\infty} au^n$$

$$\text{Where } a = \int_0^T e^{-rs} f(r) dr \text{ and } u^n = e^{-sT}$$

The geometric series converges to  $a/(1-u)$  for  $|u| < 1$

**Therefore we get:**

$$F(s) = \frac{\int_0^T e^{-rs} f(r) dr}{1 - e^{-sT}}$$

**As a result we get the final answer:**

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$$

cool,

*(45 points)*

### Problem 1

Solve the following initial value problem in terms of the convolution integral

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$$y'' + 9y = g(t); \quad y(0) = 2, \quad y'(0) = 3 \quad (1)$$

The first step is to take the Laplace transform of eq. (1) using the initial conditions

$$s^2 Y(s) - 2s - 3 + 9Y(s) = \mathcal{L}\{g(t)\} \quad (2)$$

Solving for  $Y(s)$  gives the following

$$Y(s)(s^2 + 9) = \mathcal{L}\{g(t)\} + 3 + 2s$$

$$Y(s) = \frac{\mathcal{L}\{g(t)\} + 3 + 2s}{s^2 + 9} \quad (3)$$

The next step is to separate the terms and put them in a form that will allow us to more easily identify which inverse Laplace transforms can be applied

$$\mathcal{L}\{g(t)\} \cdot \frac{1}{s^2 + 3^2} + \frac{3}{s^2 + 3^2} + \frac{2s}{s^2 + 3^2} \quad (4)$$

We will now find the inverse transforms of each individual term. For the first term, we factor out  $1/3$  in order to take the inverse Laplace transform

$$\mathcal{L}\{g(t)\} \cdot \frac{1}{3} \cdot \frac{3}{s^2 + 3^2} \quad (5)$$

Using the convolution integral to take the inverse Laplace of eq. (5)

$$\frac{1}{3} \int_0^t \sin 3(t - \tau) g(\tau) d\tau \quad (6)$$

The inverse transform of the second term in eq. (4) is straightforward

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 3^2}\right\} = \sin 3t \quad (7)$$

For the third term in eq. (4), we factor out 2 in order to take the inverse transform

$$\mathcal{L}^{-1}\left\{\frac{2s}{s^2 + 3^2}\right\} = 2 \cdot \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 3^2}\right\} = 2\cos 3t \quad (8)$$

The solution is the sum of the individual inverse Laplace transforms, eqs. (6), (7), and (8)

$$y(t) = 2\cos 3t + \sin 3t + \frac{1}{3} \int_0^t \sin 3(t - \tau) g(\tau) d\tau \quad (9)$$

## Problem 2

Find the inverse Laplace transform of:

$$(1) F(s) = \frac{1}{s^4 \times (s^2 + 1)}$$

*so don't compute?*

using the convolution theorem.

Sol.

The convolution theorem allows us to break (1) into two parts, the product of the parts must still equal (1), but can be written in a way to make it easier to compute the inverse Laplace of (1). The easiest way to write (1) is the product of

$$s^{-4} \text{ and } (s^2 + 1)^{-1}.$$

In order to use the convolution theorem we must take the Laplace of each of the two parts separately. The Laplace of these two function can be found in Table 6.2.1 they are

$$\frac{t^3}{6} \text{ and } \sin(t)$$

Respectively. With the Laplace of the two parts of (1) we can find the inverse Laplace transform of (1) using the convolution theorem and the variable  $q$ .

$$f(t) = \frac{1}{6} \times \int_0^t (t-q)^3 \times \sin(q) dq = ?$$

### Problem 3

Solve for the general solution

$$x' = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix} x.$$

The first step we need to do is take the original matrix and find the Eigen values for it. We do this by inserting a  $-r$  into the diagonal.

$$\begin{pmatrix} 3-r & 1 \\ 4 & 3-r \end{pmatrix}$$

Then multiplying by a  $1 \times 2$  matrix and setting it equal to a  $1 \times 2$  0 matrix we can solve for  $r$  and get the Eigen values.

$$\begin{pmatrix} 1-r & 2 \\ 4 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now we take the determinant of the  $1^{\text{st}}$  matrix and set it equal to 0 to find the Eigen values from which we will get the eigenvectors.

$$(3-r)(3-r) - 4 = 0$$

Multiplying out this becomes:

$$r^2 - 6r + 5 = 0$$

Solving for the roots we get:

$$r = 5 \text{ and } r = 1$$

Now using these values we can solve for the coefficient matrices.

Using the first value of 5 for  $r$  we get the matrix:

$$\begin{pmatrix} -4 & 2 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Which leads us to  $4\xi_1 = 2\xi_2$ . Which gives us a matrix of:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Now using  $r=1$  we get:

$$\begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix} \text{ which leads us simply to a coefficient matrix of } \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Our general solution is then:

$$x = c_1 \times \begin{pmatrix} 1 \\ 2 \end{pmatrix} \times e^{5t}$$

— not possible —  
only has 1 parameter.

where do the 3s come from?

not possible/acceptable  
— you must have  
made a mistake  
probably wrong  
eigenvalue

#### Problem 4

Find the inverse transform of:

$$G(s) = \frac{1}{s^2 - 4s + 5}.$$

By completing the square in the denominator we can write

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2),$$

Where  $F(s) = (s^2 + 1)^{-1}$ . Since  $L^{-1}\{F(s)\} = \sin(t)$ , it follows from Theorem 6.3.2 that  $g(t) = L^{-1}\{G(s)\} = e^{2t} \times \sin(t)$ .

(Theorem 6.3.2 states: If  $F(s) = L\{f(t)\}$  exists for  $s > a \geq 0$ , and if  $c$  is a constant, then  $L\{e^{ct} f(t)\} = F(s-c)$ ,  $s > a+c$ . Conversely, if  $f(t) = L^{-1}\{F(s-c)\}$ .)