Math 334 Test 1 KEY
Spring 2010
Section: 001

Instructor: Scott Glasgow
Dates: May 10 and 11.

Do NOT write on this problem statement booklet, except for your indication of following the honor code just below. No credit will be given for work written on this booklet. Rather write in a blue book. Also write your name, course, and section number on the blue book.

Honor Code: After I have learned of the contents of this exam by any means, I will not disclose to anyone any of these contents by any means until after the exam has closed:

Signature:

1) Sketch the direction field of the ODE

$$\frac{dS}{dt} = 2S - 2\,, \tag{1.1}$$

with special attention to "nullclines", i.e. points where the direction field is horizontal.

(4 pt.s)

Solution: We make a vector field with "x-axis" time t, and "y-axis" S, the slopes of each vector at each point (t,S) given by $2S-2$. Below is a Mathematica rendition generated by the following command:

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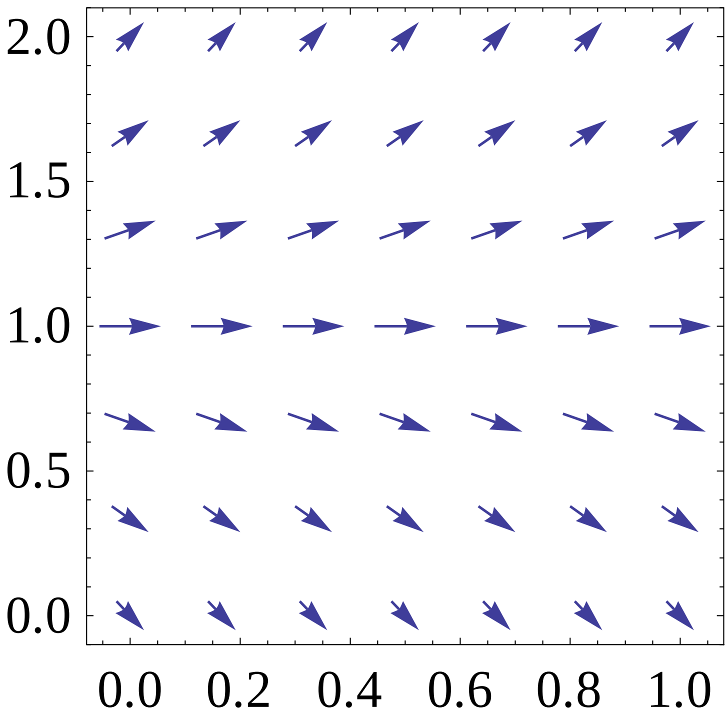
VectorPlot[{1,2y − 2}/Sqrt[1^2 + (2y − 2)^2],{x,0,1},{y,0,2},{VectorPoints →
7,VectorScale → .05}]
```

(Division by the radical is necessary to keep all the vectors roughly the same length.)



2) True or false, the following equation is a effectively an *ordinary* differential equation (ODE):

$$\frac{\partial}{\partial x} y(x,t) = t\, y(x,t). \tag{1.2}$$

Either way, explain.

Hint: If you can solve it by the methods of this course, it is an ODE, rather than a PDE.

(3 pt.s)

Solution: True: Despite the presence of a partial derivative, (1.2) is effectively an ODE. The variable t is effectively a parameter since no (partial) derivatives with respect to it appear. The general solution to it, achieved by integrating factor or separation, is

$$y(x,t) = C(t)\exp(tx), \tag{1.3}$$

where $C(t)$ is an arbitrary function of any and all parameters, including t, but not the "bona-fide variable" (of integration) x.

3) True or false, the equation (1.2) is a nonlinear differential equation. Either way, explain.

(3 pt.s)

Solution: False: The equation is linear because it can be expressed as $F(y_x, y; t, x) = 0$, where $F(y_x, y; t, x)$ is (affine) linear in its first 2 arguments.

4) True or false, the following differential equation is linear:

$$\left(\frac{\partial^2}{\partial x^2} y(x,t)\right)\left(\frac{\partial}{\partial x} y(x,t)\right) = t\, y(x,t). \tag{1.4}$$

Either way, explain.
(3 pt.s)

Solution: False: The equation is nonlinear because it can be expressed as $F(y_{xx}, y_x, y; t, x) = 0$, where $F(y_{xx}, y_x, y; t, x)$ is nonlinear in it first 3 arguments.

5) What is the order of equation (1.4)? Explain.

3 pt.s)

Solution: The order is second because the highest order derivative present is second order.

6) Fill in the blank:

Let the functions $f(t, y)$ and <u>blank </u>be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval

$t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$ there is a unique solution $y = \phi(t)$ of the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0. \tag{1.5}$$

3 pt.s)

<u>Solution:</u> blank= $f_y := \partial_y f := \partial f / \partial y$.

7) Fill in the blank:

If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$\frac{dy}{dt} + p(t)y = g(t) \tag{1.6}$$

for each t in <u>blank</u>, and also satisfies the initial condition $y(t_0) = y_0$, where y_0 is any arbitrary prescribed initial value.

3 pt.s)

<u>Solution:</u> blank= $I := \{t : \alpha < t < \beta\}$.

8) Solve the I.V.P.

$$\frac{dy}{dt} = f'(t), \; y(t_0) = y_0.$$

(3 pt.s)

<u>Solution:</u> Integration of both sides of the equation gives

$$\begin{aligned} y(t) - y_0 &= y(t) - y(t_0) = y(s)\Big|_{s=t_0}^{s=t} = \int_{t_0}^{t} \frac{dy}{ds}(s)ds = \int_{t_0}^{t} f'(s)ds = f(s)\Big|_{s=t_0}^{s=t} \\ &= f(t) - f(t_0), \end{aligned} \tag{1.7}$$

hence

$$y(t) = y_0 + f(t) - f(t_0). \tag{1.8}$$

9)

Solve the I.V.P.

$$\frac{dy}{dt} = ay + b,\ y(t_0) = y_0,$$

in terms of $y_0\,(\neq -b/a, a \neq 0)$ without using an integrating factor.
(14 pt.s)

Solution: Separation is about the only remaining recourse:

$$\begin{aligned}&\frac{dy}{dt} = ay + b = a\left(y + b/a\right) \Leftrightarrow \frac{dy}{y + b/a} = adt \Leftrightarrow \ln\left|y + b/a\right| = at + C \Leftrightarrow \\ &y + b/a = C'e^{at} \Leftrightarrow y = -b/a + C'e^{at}.\end{aligned} \tag{1.9}$$

So then

$$y_0 = y(t_0) = -b/a + C'e^{at_0} \Leftrightarrow C' = \left(y_0 + b/a\right)e^{-at_0}, \tag{1.10}$$

and the solution to the I.V.P. is

$$\begin{aligned}y &= -b/a + C'e^{at} = -b/a + \left(y_0 + b/a\right)e^{-at_0}e^{at} = -b/a + \left(y_0 + b/a\right)e^{a(t-t_0)} \\ &= y_0 e^{a(t-t_0)} + b\frac{e^{a(t-t_0)} - 1}{a}.\end{aligned} \tag{1.11}$$

10)
Solve the I.V.P.

$$\frac{dy}{dt} = -\frac{1}{1-t}y + 1 - t,\ y(0) = y_0. \tag{1.12}$$

(18 pt.s)

Solution: We first put the equation in the standard form for calculating an integrating factor:

$$\frac{dy}{dt} + \frac{1}{1-t}y = 1 - t. \tag{1.13}$$

Thus an integrating factor is

$$e^{\int \frac{1}{1-t}dt} = e^{-\ln(1-t)} = e^{\ln(1-t)^{-1}} = \frac{1}{1-t} \tag{1.14}$$

use of which giving

$$\frac{1}{1-t}\frac{dy}{dt}+\frac{1}{(1-t)^2}y=1\Leftrightarrow\frac{d}{dt}\frac{y}{1-t}=1\Leftrightarrow\frac{y}{1-t}=t+c. \tag{1.15}$$

The initial data specifies that

$$\frac{y_0}{1-0}=0+c\Leftrightarrow c=y_0 \tag{1.16}$$

so that the solution to the I.V.P. is

$$\frac{y}{1-t}=t+y_0\Leftrightarrow y=(1-t)(t+y_0). \tag{1.17}$$

11)
What does the “existence and uniqueness theorem for linear equations” (EUTL) say about the solutions of (1.12) specifically?
(8 pt.s)

Answer: The theorem asserts that a solution persists about the initial point $t=0$ for as long as the coefficient functions are continuous. For (1.12) then, it specifically states that a solution will persist in the interval $(-\infty,1)$, since $(-\infty,1)$ contains the initial point $t=0$, and one of the coefficients is discontinuous (indeed does not exist) at $t=1$.

12)
Explain why the less experienced student may think that your formula obtained for the solution of (1.12) contradicts the EUTL.
(4 pt.s)

Answer: The less experienced student may note that the solution obtained actually solves the IVP (at least in a limiting sense—assuming in particular that $\frac{1-t}{1-t}$ "=" 1 for all t) over the interval $(-\infty,+\infty)\neq(-\infty,1)$.

13)
Explain why the more experienced student sees no contradiction between your formula and the EUTL.
(6 pt.s)

Answer:

The more experienced student simply notes that $(-\infty,+\infty)\supseteq(-\infty,1)$: the theorem does not claim that solutions will definitely not persist beyond an interval in which the coefficient functions are continuous. It simply makes no claim there (but certainly suggests that one ought to be able to find an example I.V.P. for which the solution persists no longer than claimed).

14)
Find the general solution of the DE

$$\left(1+y+\cos(x-y)\right)dx+\left(1+x-\cos(x-y)\right)dy=0.$$

(12 pt.s)

Solution: Note that the equation is exact,

$$\frac{\partial}{\partial y}\left(1+y+\cos(x-y)\right)=1+\sin(x-y)=\frac{\partial}{\partial x}\left(1+x-\cos(x-y)\right) \qquad (1.18)$$

and, so, perform the compatible integrations

$$\begin{aligned}
&\psi_x(x,y)=1+y+\cos(x-y)\\
&\psi_y(x,y)=1+x-\cos(x-y)\\
&\Leftrightarrow\\
&\psi(x,y)=x+xy+\sin(x-y)+f(y)\\
&\psi(x,y)=y+xy+\sin(x-y)+g(x)\\
&\Leftarrow f(y)=y, g(x)=x.
\end{aligned} \qquad (1.19)$$

So then we can take

$$\psi(x,y)=x+y+xy+\sin(x-y), \qquad (1.20)$$

and the O.D.E. is integrated with the expression

$$x+y+xy+\sin(x-y)=C\ . \qquad (1.21)$$

15)
Find the general solution of the “nearly exact” DE

$$\left(3x^2y^2+4x^3y\right)dx+\left(4x^3y+3x^4\right)dy=0.$$

(20 pt.s)

Solution: Note that the equation is not exact,

$$\frac{\partial}{\partial y}\left(3x^2y^2+4x^3y\right)=6x^2y+4x^3\neq 12x^2y+12x^3=\frac{\partial}{\partial x}\left(4x^3y+3x^4\right) \qquad (1.22)$$

,

but that the following quotient is a pure function of y:

$$\frac{12x^2y+12x^3-\left(6x^2y+4x^3\right)}{3x^2y^2+4x^3y}=\frac{6x^2y+8x^3}{x^2y\left(3y+4x\right)}=\frac{2x^2\left(3y+4x\right)}{x^2y\left(3y+4x\right)}=\frac{2}{y}. \qquad (1.23)$$

Thus we can find an integrating factor that is a function of y only. As per formulae already developed (or you can re-derive it) the integrating factor $\mu=\mu(y)$ satisfies the ODE

$$\frac{d\mu}{dy}=\frac{2}{y}\mu \Leftrightarrow \frac{d\mu}{\mu}=2\frac{dy}{y} \Leftarrow \ln\mu = 2\ln y \Leftarrow \mu = y^2. \qquad (1.24)$$

Use of this factor gives the exact equation

$$\left(3x^2y^4+4x^3y^3\right)dx+\left(4x^3y^3+3x^4y^2\right)dy=0 \qquad (1.25)$$

So the following integrations are compatible

$$\begin{aligned}
&\psi_x(x,y)=3x^2y^4+4x^3y^3\\
&\psi_y(x,y)=4x^3y^3+3x^4y^2\\
&\Leftrightarrow\\
&\psi(x,y)=x^3y^4+x^4y^3+f(y)\\
&\psi(x,y)=x^3y^4+x^4y^3+g(x)\\
&\Leftarrow f(y)=0, g(x)=0.
\end{aligned} \qquad (1.26)$$

So then we can take

$$\psi(x,y)=x^3y^4+x^4y^3, \qquad (1.27)$$

and the O.D.E. is integrated with the expression

$$x^3y^4+x^4y^3=C. \qquad (1.28)$$

16)

A tank holds initially a solution with A_0 (mass) units of a contaminant and V_0 (volume) units of (solvent/contaminant) solution. After this initial preparation, solution with the constant contamination concentration of C_{in} (mass) units per unit volume pours in to the tank at the constant rate of R_{in} (volume) units per unit time. The well-mixed solution pours out of the tank at the constant rate of R_{out} (volume) units per unit time. Set up the IVP whose solution gives $A=A(t)$, the amount of contaminant (in mass) within the tank at time t units after the initial preparation.

(10 pt.s)

Solution: We have

$$\frac{dV_{in}}{dt} = R_{in}, \frac{dV_{out}}{dt} = R_{out} \Rightarrow \frac{dV}{dt} = R_{in} - R_{out}$$
$$\frac{dA_{in}}{dt} = C_{in}\frac{dV_{in}}{dt} = C_{in}R_{in}; \frac{dA_{out}}{dt} = C_{out}\frac{dV_{out}}{dt} = C_{out}R_{out} \Rightarrow \tag{1.29}$$
$$\frac{dA}{dt} = \frac{dA_{in}}{dt} - \frac{dA_{out}}{dt} = C_{in}R_{in} - C_{out}R_{out}.$$

But for the well-stirred mixture

$$C_{out} = C_{out}(t) = \frac{A(t)}{V(t)}. \tag{1.30}$$

Using $V(0) = V_0$, and $\frac{dV}{dt} = R_{in} - R_{out} =$ constant, we get

$$V = V(t) = V_0 + (R_{in} - R_{out})t. \tag{1.31}$$

Thus

$$C_{out} = C_{out}(t) = \frac{A(t)}{V(t)} = \frac{A(t)}{V_0 + (R_{in} - R_{out})t}, \tag{1.32}$$

and the required IVP is

$$\frac{dA(t)}{dt} = C_{in}R_{in} - \frac{R_{out}}{V_0 + (R_{in} - R_{out})t}A(t); \qquad A(0) = A_0. \tag{1.33}$$

.

17)
The I.V.P.

$$\frac{dS}{dt} = rS - p, \ S(0) = S_0,$$

with r and p positive constants, has a solution S which depends on time t, the parameters r and p, and the initial data S_0, i.e. we can write that

$$S = S(t; r, p, S_0).$$

Given a specific time $T > 0$ such that

$$S(T; r, p, S_0) = 0,$$

find the initial data S_0. Note: evidently we will get that

$$S_0 = S_0(T; r, p).$$

So find this function $S_0(T; r, p)$ of three variables. (Recall that you have solved this "annuity" or (in reverse) "mortgage" problem as homework, except that there the three variables $T,\ r,$ and p where specified. The problem is actually much easier to solve leaving them indeterminate, so be happy.)
(16 pt.s)

Solution: Find the general solution to the ODE by separation or integrating factor. I choose the latter: the form

$$\frac{dS}{dt} - rS = -p, \tag{1.34}$$

suggests the integrating factor $\mu = e^{-rt}$, giving

$$\begin{aligned} &e^{-rt}\frac{dS}{dt} - re^{-rt}S = -pe^{-rt} \Leftrightarrow \frac{de^{-rt}S}{dt} = -pe^{-rt} \Leftrightarrow e^{-rt}S = \frac{p}{r}e^{-rt} + C \Leftrightarrow \\ &S = \frac{p}{r} + Ce^{rt}. \end{aligned} \tag{1.35}$$

The initial data then demands that

$$S_0 = \frac{p}{r} + C \Leftrightarrow C = S_0 - \frac{p}{r} \tag{1.36}$$

and the solution to the IVP is then

$$S = \frac{p}{r} + \left(S_0 - \frac{p}{r}\right)e^{rt}. \tag{1.37}$$

The "final data" gives

$$0 = \frac{p}{r} + \left(S_0 - \frac{p}{r}\right)e^{rT} \Leftrightarrow S_0 = \frac{p}{r} - \frac{pe^{-rT}}{r} = \frac{p(1-e^{-rT})}{r}. \tag{1.38}$$