KEY

Math 334 Midterm I
Winter 2011
section 002
Instructor: Scott Glasgow

Write your name very clearly on this exam. In this booklet, write your mathematics clearly, legibly, in big fonts, and, most important, "have a point", i.e. make your work logically and even pedagogically acceptable. (Other human beings not already understanding 334 should be able to learn from your exam.) To avoid excessive erasing, first put your ideas together on scratch paper, then commit the logically acceptable fraction of your scratchings to this exam booklet. More is not necessarily better: say what you mean and mean what you say.

Honor Code: After I have learned of the contents of this exam by any means, I will not disclose to anyone any of these contents by any means until after the exam has closed. My signature below indicates I accept this obligation.

Signature:

(*Exams without this signature will not be graded.*)

1) Solve the I.V.P.

$$\frac{dy}{dt} = -\frac{1}{1-t}y + 1 - t,\ y(0) = y_0. \tag{0.1}$$

15 points

Solution: We first put the equation in the standard form for calculating an integrating factor:

$$\frac{dy}{dt} + \frac{1}{1-t}y = 1 - t. \tag{0.2}$$

According to the relevant theory, an integrating factor is

$$e^{\int \frac{1}{1-t}dt} = e^{-\ln(1-t)} = e^{\ln(1-t)^{-1}} = \frac{1}{1-t} \tag{0.3}$$

use of which giving

$$\frac{1}{1-t}\frac{dy}{dt} + \frac{1}{(1-t)^2}y = 1 \Leftrightarrow \frac{d}{dt}\frac{y}{1-t} = 1 \Leftrightarrow \frac{y}{1-t} = t + c. \tag{0.4}$$

The initial data specifies that

$$\frac{y_0}{1-0} = 0 + c \Leftrightarrow c = y_0 \tag{0.5}$$

so that the solution to the I.V.P. is

$$\frac{y}{1-t} = t + y_0 \Leftrightarrow y = (1-t)(t + y_0). \tag{0.6}$$

2) Show that the following differential equation is exact and then find an expression for its general solution.

$$\left(2x + 3x^2y^2\right)dx + \left(2x^3y + 3y^2\right)dy = 0. \tag{0.7}$$

12 points

Solution

The equation (0.7) is, by definition, *exact* if the left-hand side is the differential of a (continuously differentiable) function (of two variables x and y, in some simply-connected region of the x-y plane, etc., etc.), i.e. if there is a function $\psi(x,y)$ such that

$$d\psi(x,y) = \left(2x+3x^2y^2\right)dx + \left(2x^3y+3y^2\right)dy\,. \tag{0.8}$$

But we have, by definition,

$$d\psi(x,y) = \psi_x(x,y)dx + \psi_y(x,y)dy, \tag{0.9}$$

so that the equation (0.8) is the (potentially) over-determined system of equations

$$\psi_x(x,y) = 2x+3x^2y^2, \text{ and } \psi_y(x,y) = 2x^3y+3y^2\,. \tag{0.10}$$

This over-determined pair of equations is consistent (or *integrable*) iff $\left(\psi_x\right)_y = \left(\psi_y\right)_x$, i.e. iff

$$\left(2x+3x^2y^2\right)_y = \left(2x^3y+3y^2\right)_x\,. \tag{0.11}$$

(0.11) holds true, so that the equation (0.7) is indeed exact, because either side of (0.11) is $6x^2y$.

As for developing the function $\psi(x,y)$, and then (an expression for) the general solution of(0.7), one notes that the equations (0.10) demand, respectively, that

$$\begin{aligned} \psi(x,y) &= \int\left(2x+3x^2y^2\right)dx = x^2+x^3y^2+f(y), \\ \text{and } \psi(x,y) &= \int\left(2x^3y+3y^2\right)dy = x^3y^2+y^3+g(x), \end{aligned} \tag{0.12}$$

for some initially rather arbitrary functions $f(y)$ and $g(x)$. The two statements (0.12) are not contradictory iff $x^2+x^3y^2+f(y) = x^3y^2+y^3+g(x) \Leftrightarrow f(y)-y^3 = g(x)-x^2$, which implies both sides of this last equation are independent of both x and y. As far as finding the general solution of (0.7) is concerned, without loss of generality we can choose $f(y)-y^3 = g(x)-x^2 = 0$ so that (0.12) becomes ("in either case")

$$\psi(x,y) = x^2+x^3y^2+y^3\,. \tag{0.13}$$

(0.13) is NOT the general solution to the (exact) differential equation (0.7). It is not even a specific solution. Rather (0.13) defines a "potential (function) for the solution." Using it one notes that (0.7) can be written as

$$d\psi(x,y) = d\left(x^2 + x^3y^2 + y^3\right) = 0\,, \tag{0.14}$$

the general solution to which clearly being

$$x^2 + x^3y^2 + y^3 = C\,. \tag{0.15}$$

3) Without solving the following IVP, but rather just using the relevant Picard iteration-like theory on it, determine an interval $[0,h]$ over which the solution $\phi(t)$ of the initial value problem

$$\frac{dy}{dt} = 1 + y^2,\ y(0) = 0 \tag{0.16}$$

is guaranteed to exist. In fact, show that the solution is guaranteed to exist for t's in $[0,1/2]$.

Hint: Recall that solutions of IVP (0.16) are equivalent to (continuous) solutions of the integral equation

$$\phi(t) = \int_0^t \left(1+\phi^2(s)\right)ds, \tag{0.17}$$

and that since the function $f(t,y) := 1+y^2$ is continuous in any box of the form $(t,y)\in[-a,a]\times[-b,b]$, that it is relevant to use the fact that for $(t,\phi(t))$ in such a box, we have (for $[0,h]\in[-a,a]$) the estimate

$$\left|\phi(t)\right| = \left|\int_0^t \left(1+\phi^2(s)\right)ds\right| \le \int_0^h \left(1+\phi^2(s)\right)ds \le \int_0^h \left(1+b^2\right)ds = \left(1+b^2\right)h, \tag{0.18}$$

i.e. we have the estimate

$$\left|\phi(t)\right| \le \left(1+b^2\right)h. \tag{0.19}$$

But now note that (0.19) does not guarantee by itself that $\phi(t)$ is in the box, i.e. (0.19) does not by itself guarantee that

$$\left|\phi(t)\right| \le b, \tag{0.20}$$

and this eventhough (0.20) was used in getting the estimate (0.18)/(0.19). Hmmm…. Well, for any fixed $b>0$ is there a way to choose $h>0$ (small enough) so that (0.19) *implies* (0.20)? (Recall (0.20) certainly implies (0.19).) Well, for h near zero, (0.19) implies that $\left|\phi(t)\right|$ is really small, which then implies (0.20). What is the biggest h that could be used in (0.19) to imply (0.20) for any given b? What then is the best choice of b to make $h = h(b)$ as large as possible? Answers these questions and you will arrive at $h = 1/2$.

10 points

Solution

Essentially all the hard work has already been done in the hint. One simply notes that a) the largest $h > 0$ in (0.19) that still allows (0.19) to imply (0.20) for any given $b > 0$ is clearly given by the equation

$$\left(1+b^2\right)h = b \Leftrightarrow h = \frac{b}{1+b^2}, \qquad (0.21)$$

and that b) for all $b > 0$ we have

$$h = \frac{b}{1+b^2} \le \frac{1}{1+1^2} = \frac{1}{2}, \qquad (0.22)$$

with equality, as indicated, at $b = 1$. ((0.22) holds because, say,

$$\frac{b}{1+b^2} \le \frac{1}{2} \Leftrightarrow 2b \le 1+b^2 \Leftrightarrow 0 \le 1-2b+b^2 \Leftrightarrow 0 \le \left(b-1\right)^2, \qquad (0.23)$$

the latter clearly true for all b, positive, negative or zero. One could also do the calculus, and rely on theorems from that field, but here we have proved our own theorem about a maximum without using any calculus.)

4) Solve the I.V.P.

$$\frac{dy}{dt} = ay+b, y(t_0) = y_0 \qquad (0.24)$$

in terms of y_0 ($\ne b/a, a \ne 0$) without using an integrating factor.

12 points

Solution:

Separation is about the only recourse:

$$\begin{aligned}
&\frac{dy}{dt} = ay+b = a\left(y+b/a\right) \Leftrightarrow \\
&\frac{dy}{y+b/a} = adt \Leftrightarrow \\
&\ln\left|y+b/a\right| = at + C \Leftrightarrow \\
&y + b/a = C'e^{at} \Leftrightarrow \\
&y = -b/a + C'e^{at}.
\end{aligned}$$

So then

$$\begin{aligned}
&y_0 = y(t_0) = -b/a + C'e^{at_0} \Leftrightarrow \\
&C' = \left(y_0 + b/a\right)e^{-at_0},
\end{aligned}$$

and the solution to the I.V.P. is

$$
\begin{aligned}
y &= -b/a + C'e^{at} \\
&= -b/a + \left(y_0 + b/a\right)e^{-at_0}e^{at} \\
&= -b/a + \left(y_0 + b/a\right)e^{a(t-t_0)} \\
&= y_0 e^{a(t-t_0)} + b\frac{e^{a(t-t_0)} - 1}{a}.
\end{aligned}
$$

5) After a time $t = 0$, a solution of constant concentration of 1 gram solute per liter solvent enters a (perfect) stirring tank at a constant rate of 3 liters per minute. The perfectly-stirred mixture exits the tank at a constant rate of 2 liters per minute. Suppose the solute takes no volume in solution. If the tank contains 10 liters of fluid at a time $t = 0$, write down a (self-contained) differential equation for the time evolution of the grams of solute $Q(t)$ accumulated in the tank at time t, one that is valid for as long as the tank is not overflowing. Then, assuming there are 10 grams of solute in the tank at $t = 0$, give an expression for the grams $Q(t)$ of solute accumulated in the tank at time t (by solving the relevant IVP).

18 points

<u>Solution</u>

By stoichiometric / "unit-canceling" / "chain-rule" reasoning, one has

$$
\begin{aligned}
\frac{dQ}{dt} &= \left(\frac{dQ}{dt}\right)_{total} = \left(\frac{dQ}{dt}\right)_{in} - \left(\frac{dQ}{dt}\right)_{out} = \left(\frac{dQ}{dV}\right)_{in}\left(\frac{dV}{dt}\right)_{in} - \left(\frac{dQ}{dV}\right)_{out}\left(\frac{dV}{dt}\right)_{out} \\
&= C_{in}R_{in} - C_{out}R_{out} \qquad (0.25) \\
&= R_{in}C_{in} - R_{out}\frac{Q}{V} = 3 \cdot 1 - 2\frac{Q}{V} = 3 - 2\frac{Q}{V},
\end{aligned}
$$

where the fluid tank volume $V = V(t)$ is specified by

$$
\frac{dV}{dt} = R_{in} - R_{out} = 3 - 2 = 1,\ V(0) = V_0 = 10, \qquad (0.26)
$$

the latter (trivial) initial value problem having the unique solution

$$
V = V_0 + t(R_{in} - R_{out}) = 10 + t \cdot 1 = 10 + t\ . \qquad (0.27)
$$

Thus the required, "self-contained" differential equation is

$$
\begin{aligned}
\frac{dQ}{dt} &= R_{in}C_{in} - R_{out}\frac{Q}{V_0 + t(R_{in} - R_{out})} \\
&= 3 - \frac{2}{10+t}Q \\
&\Leftrightarrow \\
\frac{dQ}{dt} &+ \frac{2}{10+t}Q = 3.
\end{aligned}
\tag{0.28}
$$

We solve the initial value problem which is ODE (0.28) together with initial data

$$
Q(0) = 10\,. \tag{0.29}
$$

An integrating factor for the ODE (0.28) is, according to the standard theory,

$$
\begin{aligned}
\mu &= \exp\int \frac{2}{10+t}dt = \exp\big(2\log(10+t)\big) \\
&= (10+t)^2\,.
\end{aligned}
\tag{0.30}
$$

Use of the integrating factor (0.30) in (0.28) gives

$$
\frac{d\left((10+t)^2 Q\right)}{dt} = (10+t)^2\frac{dQ}{dt} + (10+t)^2\frac{2}{10+t}Q = 3(10+t)^2 = \frac{d\,(10+t)^3}{dt}\,. \tag{0.31}
$$

Integration of (0.31) using relevant limits (and dummy variables) gives

$$
\begin{aligned}
&(10+t)^2 Q(t) - 10^3 = (10+t)^2 Q(t) - 10^2\cdot 10 = \\
&(10+t)^2 Q(t) - (10+0)^2 Q(0) = (10+s)^2 Q(s)\Big|_0^t = \\
&\int_0^t d\left((10+s)^2 Q(s)\right) = \int_0^t d\,(10+s)^3 \\
&= (10+s)^3\Big|_0^t = (10+t)^3 - (10+0)^3 = (10+t)^3 - 10^3,
\end{aligned}
\tag{0.32}
$$

or, equivalently,

$$
Q(t) = 10 + t\,. \tag{0.33}
$$

6) Show that the following differential equation, (0.34), is not exact, but can be rendered exact by multiplication by an integrating factor that is only a function of

x or only a function of y. Find an expression of the general solution of the differential equation.

$$\left(7x^2 + 5y^3\right)dx + 3xy^2 dy = 0. \tag{0.34}$$

15 points

<u>Solution</u>

(0.34) is not exact since

$$\psi_{xy} = \left(\psi_x\right)_y := \left(7x^2 + 5y^3\right)_y = 15y^2 \neq 3y^2 = \left(3xy^2\right)_x =: \left(\psi_y\right)_x = \psi_{yx}. \tag{0.35}$$

By theorem we know that, with an integrating factor μ, (0.34) can be made exact. We note from (0.35) that

$$\left(7x^2 + 5y^3\right)_y - \left(3xy^2\right)_x = 15y^2 - 3y^2 = 12y^2, \tag{0.36}$$

which divides the second term in (0.34), the remaining factor ($x/4$) being only a function of x. Thus we suspect the existence of an integrating factor only depending on x. At any rate, with the use of such a factor, ODE (0.34) becomes

$$\left(7x^2 + 5y^3\right)\mu(x)dx + 3xy^2\mu(x)dy = 0, \tag{0.37}$$

and exactness demands that

$$\begin{aligned} 0 &= \left(\left(7x^2 + 5y^3\right)\mu(x)\right)_y - \left(3xy^2\mu(x)\right)_x = \\ &\quad 15y^2\mu(x) - \left(3y^2\mu(x) + 3xy^2\mu'(x)\right) \\ &= -3xy^2\mu'(x) + 12y^2\mu(x) \\ &= -3y^2\left(x\mu'(x) - 4\mu(x)\right) \\ &\Leftarrow \\ x\mu'(x) &= 4\mu(x). \end{aligned} \tag{0.38}$$

Thus, as suspected, there is an integrating factor depending only on x. A solution of the last differential equation in (0.38) is given by

$$\mu(x) = x^4, \tag{0.39}$$

in which case (0.37) becomes

$$0=\left(7x^2+5y^3\right)x^4dx+3xy^2x^4dy=\left(7x^6+5x^4y^3\right)dx+3x^5y^2dy\,. \tag{0.40}$$

Writing

$$\begin{aligned}\psi_x &= 7x^6+5x^4y^3,\\ \psi_y &= 3x^5y^2,\end{aligned} \tag{0.41}$$

gives

$$\begin{aligned}\psi &= x^7+x^5y^3+f(y),\\ \psi &= x^5y^3+g(x),\end{aligned} \tag{0.42}$$

which can be reconciled by $f(y)=g(x)-x^7=0$. Thus (0.40) can be written as

$$d\left(x^7+x^5y^3\right)=0\,, \tag{0.43}$$

an expression of the general solution to which clearly being

$$x^7+x^5y^3=C\,. \tag{0.44}$$

7) Find a linear, first order, ordinary differential equation with the property that *every* solution $y(t)$ of it approaches the function $f(t)=\sin(t)$ arbitrarily closely as $t\to+\infty$. Note that the (too) simple equation

$$y'(t)=f'(t)=\sin'(t)=\cos(t) \tag{0.45}$$

does not work since the general solution of (0.45) is

$$y(t)=\int\cos(t)dt=\sin(t)+C\,, \tag{0.46}$$

giving

$$\lim_{t\to+\infty}\left(y(t)-f(t)\right)=\lim_{t\to+\infty}\left(\sin(t)+C-\sin(t)\right)=C\,, \tag{0.47}$$

which is not zero for most choices of C.

10 points

Solution

Introduce a general solution of the form

$$y(t) = \sin(t) + Ce^{-at} \tag{0.48}$$

with $a > 0$ to get

$$\lim_{t\to+\infty}\left(y(t) - f(t)\right) = \lim_{t\to+\infty}\left(\sin(t) + Ce^{-at} - \sin(t)\right) = \lim_{t\to+\infty} Ce^{-at} = 0 \tag{0.49}$$

for every choice of C, as demanded by the problem. Thus, to get a first order ODE with the required property we differentiate (0.48) with respect to t and eliminate C between (0.48) and this new equation (acquired from differentiating the old one). Differentiating (0.48) gives

$$y'(t) = \cos(t) - aCe^{-at}, \tag{0.50}$$

and elimination of C between (0.48) and (0.50) gives the required first order ODE:

$$\begin{aligned} y'(t) &= \cos(t) - aCe^{-at} = \cos(t) - a\left(y(t) - \sin(t)\right) \\ &= -ay(t) + \cos(t) + a\sin(t). \end{aligned} \tag{0.51}$$

Choosing, say, $a = 1$ we get the example ODE

$$y'(t) = -y(t) + \cos(t) + \sin(t). \tag{0.52}$$

8) Solve the following initial value problem. State the properties of the solution as $t \to \infty$ for all possible choices of the initial value y_0.

$$y'(t) + y(t) = \cos(t) + \sin(t),\ y(0) = y_0. \tag{0.53}$$

12 points

<u>Solution</u>

(0.53) suggests the integrating factor

$$\mu(t) = \exp\int 1dt = e^t, \tag{0.54}$$

which renders the ODE (0.53) as

$$\frac{d}{dt}e^t y(t) = e^t y'(t) + e^t y(t) = \left(\cos(t) + \sin(t)\right)e^t = \frac{d}{dt}\sin(t)e^t \tag{0.55}$$

which, with the initial data specified in (0.53), integrates to

$$\begin{aligned} & e^t y(t) - y_0 = e^t y(t) - 1 \cdot y_0 = e^t y(t) - e^0 y(0) = e^s y(s)\Big|_0^t = \\ & \int_0^t de^s y(s) = \int_0^t d\sin(s)e^s = \sin(s)e^s\Big|_0^t = \sin(t)e^t - \sin(0)e^0 = \sin(t)e^t, \end{aligned} \tag{0.56}$$

or, equivalently,

$$y(t) = e^{-t}y_0 + \sin(t). \tag{0.57}$$

Note then the differential equation in (0.53) gives a solution to problem 7, which indicates the desired properties.

9) Carefully state the (nonlinear) existence and uniqueness theorem for a single first order ODE.

15 points

Solution

Consider the initial value problem

$$y'(t) = f(t, y), y(t_0) = y_0. \tag{0.58}$$

Suppose $f(t, y)$ and $f_y(t, y)$ are both continuous in an open rectangle $\left(t_{-1}, t_{+1}\right) \times \left(y_{-1}, y_{+1}\right)$ containing the point $\left(t_0, y_0\right)$. Then there exists an $h > 0$ such that (0.58) has a unique, continuously differentiable solution $y = \phi(t)$ persisting over the t interval $\left(t_0 - h, t_0 + h\right)$ (potentially much smaller than the interval $\left(t_{-1}, t_{+1}\right)$).

10) Carefully state the linear existence and uniqueness theorem for a single first order ODE. Explain in general terms how it is proven.

15 points

Solution

Consider the initial value problem

$$y'(t) = p(t)y + q(t), y(t_0) = y_0. \tag{0.59}$$

Suppose $p(t)$ and $q(t)$ are both continuous in an open interval (t_{-1}, t_{+1}) containing the point t_0. Then (0.59) has a unique, continuously differentiable solution $y = \phi(t)$ persisting over the t interval (t_{-1}, t_{+1}).

The theorem is proven by explicitly integrating (0.59), using the various theorems of calculus, including that the integral of a continuous function exists, etc.