

KEY

**Math 334 Midterm III
Winter 2011
section 002
Instructor: Scott Glasgow**

Write your name very clearly on this exam. In this booklet, write your mathematics clearly, legibly, in big fonts, and, most important, “have a point”, i.e. make your work logically and even pedagogically acceptable. (Other human beings not already understanding 334 should be able to learn from your exam.) To avoid excessive erasing, first put your ideas together on scratch paper, then commit the logically acceptable fraction of your scratchings to this exam booklet. More is not necessarily better: say what you mean and mean what you say.

Honor Code: After I have learned of the contents of this exam by any means, I will not disclose to anyone any of these contents by any means until after the exam has closed. My signature below indicates I accept this obligation.

Signature:

(Exams without this signature will not be graded.)

1) Write the definition of the Laplace transform:

$$\mathcal{L}[f](s) := \quad (0.1)$$

10 points

Solution:

$$\mathcal{L}[f](s) := \int_0^\infty e^{-st} f(t) dt. \quad (0.2)$$

2) Recall $\cos bt = (e^{ibt} + e^{-ibt}) / 2$. By assuming the relevant elementary integration formulas extend to complex, find the following (when it is finite):

$$\int_0^\infty \exp(-st) e^{at} \cos bt dt \quad (0.3)$$

In (0.3), think of s, a, b as all real. For which real values of s is your formula relevant? What is the Laplace transform of f , $\mathcal{L}[f]$, when we choose $f(t) := e^{at} \cos bt$?

15 points

Solution: (0.3) converges for real s iff $s > a$, in which case by (0.2) we get

$$\begin{aligned} \mathcal{L}[f](s) &:= \int_0^\infty e^{-st} f(t) dt = \int_0^\infty \exp(-st) e^{at} \cos bt dt = \\ &= \frac{1}{2} \int_0^\infty \exp(-(s-a)t) (e^{ibt} + e^{-ibt}) dt \\ &= \frac{1}{2} \int_0^\infty (\exp(-(s-a-ib)t) + \exp(-(s-a+ib)t)) dt \\ &= \frac{1}{2} \left(\frac{\exp(-(s-a-ib)t)}{-(s-a-ib)} + \frac{\exp(-(s-a+ib)t)}{-(s-a+ib)} \right) \Bigg|_0^\infty \\ &= \frac{1}{2} \left[\left(\frac{0}{-(s-a-ib)} + \frac{0}{-(s-a+ib)} \right) - \left(\frac{1}{-(s-a-ib)} + \frac{1}{-(s-a+ib)} \right) \right] \\ &= \frac{1}{2} \left(\frac{1}{s-a-ib} + \frac{1}{s-a+ib} \right) = \frac{1}{2} \frac{s-a+ib+s-a-ib}{(s-a-ib)(s-a+ib)} = \frac{s-a}{(s-a)^2 + b^2}. \end{aligned} \quad (0.4)$$

This answers all the questions.

3) Find $\mathcal{L}[f'](s)$ in terms of $\mathcal{L}[f](s)$ (and $f(0)$) by integrating formula (0.1) by parts. (Of course we will assume as true whatever is needed to make integration by parts actually work.)

10 points

Solution:

By (0.2) and integration by parts we have

$$\begin{aligned}\mathcal{L}[f'](s) &:= \int_0^\infty e^{-st} f'(t) dt = \int_0^\infty e^{-st} df(t) =: \int_0^\infty u(t) dv(t) = u(t)v(t) \Big|_0^\infty - \int_0^\infty v(t) du(t) \\ &:= e^{-st} f(t) \Big|_0^\infty - \int_0^\infty f(t) d e^{-st} = 0 - e^0 f(0) + s \int_0^\infty f(t) e^{-st} dt \\ &= s \mathcal{L}[f](s) - f(0).\end{aligned}\tag{0.5}$$

Here we used, say, that $|f|$ is bounded by some Ke^{at} and considered only $s > a$.

- 4) By using the formula of the last problem twice, find $\mathcal{L}[f''](s) = \mathcal{L}\left[(f')'\right](s)$ in terms of $\mathcal{L}[f'](s)$ (and $f'(0)$), and then in terms of $\mathcal{L}[f](s)$ (and $f'(0)$ and $f(0)$).

15 points

Solution:

Since by (0.5) we have $\mathcal{L}[f'](s) = s \mathcal{L}[f](s) - f(0)$, then clearly

$$\begin{aligned}\mathcal{L}[f''](s) &= \mathcal{L}\left[(f')'\right](s) = s \mathcal{L}[f'](s) - f'(0) = s(s \mathcal{L}[f](s) - f(0)) - f'(0) \\ &= s^2 \mathcal{L}[f](s) - sf(0) - f'(0).\end{aligned}\tag{0.6}$$

- 5) Solve the following initial value problem by means of the Laplace transform.

$$y'' - 2y' + 2y = \cos t; \quad y(0) = 1, \quad y'(0) = 0.\tag{0.7}$$

20 points

Solution

Taking the Laplace transform of both sides of the differential equation in (0.7), and, in particular, using (0.4), we get

$$\begin{aligned}
((s-1)^2 + 1^2) \mathcal{L}[y](s) - s + 2 &= (s^2 - 2s + 2) \mathcal{L}[y](s) - sy(0) - y'(0) + 2y(0) \\
&= (s^2 \mathcal{L}[y](s) - sy(0) - y'(0)) - 2(s \mathcal{L}[y](s) - y(0)) + 2 \mathcal{L}[y](s) \\
&= \mathcal{L}[y''] (s) - 2 \mathcal{L}[y'] (s) + 2 \mathcal{L}[y] (s) = \mathcal{L}[y'' - 2y' + 2y] (s) \\
&= \mathcal{L}[\cos] (s) = \frac{s}{s^2 + 1^2}.
\end{aligned}
\tag{0.8}$$

Here we also used formula (0.6) and (0.5), and the data in (0.7), as well as linearity of the transform. (0.8) holds iff

$$\begin{aligned}
\mathcal{L}[y](s) &= \frac{\frac{s}{s^2 + 1^2} + s - 2}{(s-1)^2 + 1^2} = \frac{s + (s-2)(s^2 + 1^2)}{(s^2 + 1^2)((s-1)^2 + 1^2)} = \frac{\frac{s + (s-2)(s^2 + 1^2)}{(s^2 + 1^2)(s - (1-i))}}{s - (1+i)} \Bigg]_{s=1+i} + \\
&\quad \frac{\frac{s + (s-2)(s^2 + 1^2)}{(s^2 + 1^2)(s - (1+i))}}{s - (1-i)} \Bigg]_{s=1-i} + \frac{\frac{s + (s-2)(s^2 + 1^2)}{(s+i)((s-1)^2 + 1^2)}}{s - i} \Bigg]_{s=i} + \\
&\quad \frac{\frac{s + (s-2)(s^2 + 1^2)}{(s-i)((s-1)^2 + 1^2)}}{s + i} \Bigg]_{s=-i} \\
&= \frac{1+i + (1+i-2)((1+i)^2 + 1^2)}{((1+i)^2 + 1^2)((1+i) - (1-i))} + \frac{1-i + (1-i-2)((1-i)^2 + 1^2)}{((1-i)^2 + 1^2)((1-i) - (1+i))} + \\
&\quad \frac{i + (i-2)(i^2 + 1^2)}{(i+i)((i-1)^2 + 1^2)} + \frac{-i + (-i-2)((-i)^2 + 1^2)}{(-i-i)((-i-1)^2 + 1^2)} \\
&\quad \frac{1+i + (-1+i)(1+2i)}{(1+2i)(2i)} + \frac{1-i + (-1-i)(1-2i)}{(1-2i)(-2i)} + \frac{i}{(2i)(1-2i)} + \frac{-i}{(-2i)(1+2i)} \tag{0.9} \\
&= \frac{1}{5} \left(\frac{2+i}{s-(1+i)} + \frac{2-i}{s-(1-i)} \right) + \frac{1}{10} \left(\frac{1+2i}{s-i} + \frac{1-2i}{s+i} \right),
\end{aligned}$$

so evidently

$$\begin{aligned}
y(t) &= 2 \operatorname{Re} \left(\frac{1}{5} (2+i) \exp((1+i)t) + \frac{1}{10} (1+2i) \exp(it) \right) \\
&= 2 \operatorname{Re} \left(\frac{e^t}{5} (2+i) (\cos t + i \sin t) + \frac{1}{10} (1+2i) (\cos t + i \sin t) \right) \\
&= 2 \left(\frac{e^t}{5} (2 \cos t - \sin t) + \frac{1}{10} (\cos t - 2 \sin t) \right) \\
&= \frac{1}{5} (\cos t - 2 \sin t) + \frac{e^t}{5} (4 \cos t - 2 \sin t).
\end{aligned} \tag{0.10}$$

6) Solve the following initial value problem (I.V.P.) by means of the Laplace transform, and by use of unit step functions.

$$y'' + 4y = \begin{cases} \sin t, & 0 \leq t \leq 2\pi \\ 0, & t \geq 2\pi \end{cases}, \quad y(0) = 0 = y'(0). \tag{0.11}$$

What is $y(7)$?

15 points

Solution

(0.11) is equivalent to

$$\begin{aligned}
y'' + 4y &= (u_0(t) - u_{2\pi}(t)) \sin t = u_0(t) \sin t - u_{2\pi}(t) \sin(t - 2\pi), \\
y(0) &= 0 = y'(0),
\end{aligned} \tag{0.12}$$

where we used \sin is 2π periodic. Taking transforms, and using the quiescent data, we get

$$(s^2 + 2^2) \mathcal{L}[y](s) = \frac{1}{s^2 + 1^2} - \frac{e^{-2\pi s}}{s^2 + 1^2} = (1 - e^{-2\pi s}) \frac{1}{s^2 + 1^2} \tag{0.13}$$

i.e.,

$$\mathcal{L}[y](s) = (1 - e^{-2\pi s}) \frac{1}{(s^2 + 1^2)(s^2 + 2^2)} =: (1 - e^{-2\pi s}) \mathcal{L}[h](s) \tag{0.14}$$

whence

$$y(t) = u_0(t)h(t) - u_{2\pi}(t)h(t - 2\pi) = u_0(t)h(t) - u_{2\pi}(t)h(t) = (u_0(t) - u_{2\pi}(t))h(t) \tag{0.15}$$

where we used that $h(t)$ satisfying $\mathcal{L}[h](s) = 1/(s^2 + 1^2)(s^2 + 2^2)$ will clearly be 2π periodic. In fact since

$$\begin{aligned}\mathcal{L}[h](s) &= \frac{1}{(s^2+1^2)(s^2+2^2)} = \frac{1/(s^2+2^2)}{s^2+1^2} + \frac{1/(s^2+1^2)}{s^2+2^2} = \frac{1}{3} \left(\frac{1}{s^2+1^2} - \frac{1}{s^2+2^2} \right) \\ &= \frac{1}{3} \left(\frac{1}{s^2+1^2} - \frac{1}{2} \frac{2}{s^2+2^2} \right) = \frac{1}{6} \left(2 \frac{1}{s^2+1^2} - \frac{2}{s^2+2^2} \right),\end{aligned}\quad (0.16)$$

in (0.15) we have

$$h(t) = \frac{1}{6} (2 \sin t - \sin 2t). \quad (0.17)$$

Since $7 > 2\pi$, from (0.15) clearly $y(7) = 0$.

- 7) Solve the following I.V.P. by means of the Laplace transform, and by use of unit step functions.

$$y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0 = y'(0) \quad (0.18)$$

What is $y(7)$?

10 points

Solution:

The transform gives:

$$\begin{aligned}(s^2 + 2^2) \mathcal{L}[y](s) &= e^{-\pi s} - e^{-2\pi s} \\ \Leftrightarrow \\ \mathcal{L}[y](s) &= (e^{-\pi s} - e^{-2\pi s}) \frac{1}{s^2 + 2^2} = \frac{1}{2} (e^{-\pi s} - e^{-2\pi s}) \frac{2}{s^2 + 2^2} \\ &=: \frac{1}{2} (e^{-\pi s} - e^{-2\pi s}) \mathcal{L}[h](s)\end{aligned}\quad (0.19)$$

whence

$$\begin{aligned}y(t) &= \frac{1}{2} (u_\pi(t)h(t - \pi) - u_{2\pi}(t)h(t - 2\pi)) = \frac{1}{2} (u_\pi(t)h(t) - u_{2\pi}(t)h(t)) \\ &= \frac{1}{2} (u_\pi(t) - u_{2\pi}(t))h(t)\end{aligned}\quad (0.20)$$

since $h(t)$ will have the symmetry $h(t - \pi) = h(t) \Rightarrow h(t - 2\pi) = h(t)$. In fact, clearly $h(t) = \sin(2t)$. Note then $y(7) = 0$ since $7 > 2\pi > \pi$.

- 8) Express the solution of the following IVP in terms of a convolution integral.

$$y'' + 4y' + 4y = g(t), \quad y(0) = 0 = y'(0) \quad (0.21)$$

10 points

Solution

Transformation gives

$$\begin{aligned}
 (s^2 + 4s + 4)\mathcal{L}[y](s) &= \mathcal{L}[g](s) \\
 &\Leftrightarrow \\
 \mathcal{L}[y](s) &= \frac{1}{s^2 + 4s + 4} \mathcal{L}[g](s) = \frac{1!}{(s+2)^2} \mathcal{L}[g](s) \\
 &=: \mathcal{L}[h](s)\mathcal{L}[g](s) = \mathcal{L}[h * g](s)
 \end{aligned} \tag{0.22}$$

whence, for $t \geq 0$,

$$y(t) = (h * g)(t) = \int_0^t h(t - \tau)g(\tau)d\tau, \tag{0.23}$$

where

$$\mathcal{L}[h](s) := \frac{1!}{(s+2)^2} \Leftrightarrow h(t) = t^1 e^{-2t} = t e^{-2t}. \tag{0.24}$$

9) Suppose that both

$$\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ t \end{bmatrix} \text{ and } \mathbf{x}_2(t) = \begin{bmatrix} -t \\ 1 \end{bmatrix} \tag{0.25}$$

solve the system

$$\mathbf{x}' = \frac{1}{1+t^2} \begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix} \mathbf{x} \tag{0.26}$$

and that

$$\mathbf{x}_p(t) = \begin{bmatrix} 1+t \\ 1-t \end{bmatrix} \tag{0.27}$$

solves the system

$$\mathbf{x}' = \frac{1}{1+t^2} \begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix} \mathbf{x} - \frac{2}{1+t^2} \begin{bmatrix} t-1 \\ t+1 \end{bmatrix}. \tag{0.28}$$

Then what is the solution of the IVP

$$\mathbf{x}' = \frac{1}{1+t^2} \begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix} \mathbf{x} - \frac{2}{1+t^2} \begin{bmatrix} t-1 \\ t+1 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} ? \tag{0.29}$$

This problem tests your general understanding of the structure of solutions of linear systems without requiring you to regurgitate a theorem (with all of the intricacies of its articulation). Note the point value—this problem is straightforward if you recall the theory.

7 points

Solution

According to the information given, the general solution of (0.29) can be expressed as

$$\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} \mathbf{c} + \begin{bmatrix} 1+t \\ 1-t \end{bmatrix} \quad (0.30)$$

Choosing $t = 0$ and using the initial data in (0.29), (0.30) becomes

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \mathbf{x}(0) = \begin{bmatrix} 1 & -0 \\ 0 & 1 \end{bmatrix} \mathbf{c} + \begin{bmatrix} 1+0 \\ 1-0 \end{bmatrix} = \mathbf{c} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftrightarrow \mathbf{c} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}. \quad (0.31)$$

So (0.30) becomes

$$\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1+t \\ 1-t \end{bmatrix} = \begin{bmatrix} -2t \\ 2 \end{bmatrix} + \begin{bmatrix} 1+t \\ 1-t \end{bmatrix} = \begin{bmatrix} 1-t \\ 3-t \end{bmatrix}. \quad (0.32)$$

10) Find the fundamental matrix of solutions $\Phi = \Phi(t)$ to the system

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \mathbf{x}, \quad (0.33)$$

the one that has the property that $\Phi(0) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

21 points

Solution

First we find a representation of the general solution of the system(0.33). This can be expressed as

$$\mathbf{x} = \mathbf{x}(t) = c_1 \xi_+ e^{\lambda_+ t} + c_2 \xi_- e^{\lambda_- t} \quad (0.34)$$

provided the ξ 's are independent eigenvectors associated with the eigenvalues λ_+ and λ_- of the matrix in (0.33). To determine this we note

$$\begin{bmatrix} 0-\lambda & 1 \\ -2 & 2-\lambda \end{bmatrix} \xi = \mathbf{0} \Leftrightarrow \xi = \mathbf{0}$$

unless

$$0 = \det \begin{bmatrix} 0-\lambda & 1 \\ -2 & 2-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1^2 \quad (0.35)$$

\Leftrightarrow

$$\lambda = 1 \pm i = 1 + i, 1 - i =: \lambda_+, \lambda_-.$$

So

$$\mathbf{0} = \begin{bmatrix} 0-(1 \pm i) & 1 \\ -2 & 2-(1 \pm i) \end{bmatrix} \xi_{\pm} = \begin{bmatrix} -1 \mp i & 1 \\ -2 & 1 \mp i \end{bmatrix} \xi_{\pm} \Leftarrow \xi_{\pm} = \begin{bmatrix} 1 \\ 1 \pm i \end{bmatrix}. \quad (0.36)$$

Thus, explicitly, (0.34) is

$$\mathbf{x} = \mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{(1+i)t} + c_2 \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(1-i)t}. \quad (0.37)$$

As per the usual theory, we can find a real-valued representation by finding the real and imaginary parts of either of the above complex-valued solutions:

$$\mathbf{x}_1(t) := \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{(1+i)t} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^t (\cos t + i \sin t) = \begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} e^t + i \begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix} e^t, \quad (0.38)$$

whence a real-valued representation of the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} e^t + c_2 \begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix} e^t. \quad (0.39)$$

A fundamental matrix of solutions $\Psi = \Psi(t)$, one not necessarily having the desired property, can be found from the above general solution (0.39) as in

$$\Psi(t) = e^t \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix} \quad (0.40)$$

The desired fundamental matrix $\Phi = \Phi(t)$ can be obtained from $\Psi = \Psi(t)$ via

$$\begin{aligned} \Phi = \Phi(t) &= \Psi(t) \Psi^{-1}(0) = e^t \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \\ &= e^t \begin{bmatrix} \cos t & \sin t \\ \cos t - \sin t & \cos t + \sin t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\ &= e^t \begin{bmatrix} \cos t - \sin t & \sin t \\ -2 \sin t & \cos t + \sin t \end{bmatrix}. \end{aligned} \quad (0.41)$$