KEY

Math 334 Midterm I
Winter 2012
section 002
Instructor: Scott Glasgow

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1. Solve the following initial value problem:

$$\frac{dy}{dt} = 1 + t + y + ty,\ y(2) = 0\,. \tag{1.1}$$

Also, what is the value of this solution at $t = -4$? (I.e., what is $y(-4)$?)

10 points

Solution

The equation separates to

$$\frac{dy}{1+y} = \left(1+t\right)dt, \tag{1.2}$$

which, with the initial data in (1.1), yields the integral statement

$$\int_0^y \frac{dy'}{1+y'} = \int_2^t \left(1+t'\right)dt'\,, \tag{1.3}$$

which, after some work, gives

$$\begin{aligned}
\log\left(1+y\right) &= \log\left(1+y\right) - \log\left(1+0\right) = \log\left(1+y'\right)\Big|_0^y = \\
&\int_0^y \frac{dy'}{1+y'} = \int_2^t \left(1+t'\right)dt' \\
&= t' + \frac{1}{2}t'^2\Big|_2^t = t + \frac{1}{2}t^2 - \left(2 + \frac{1}{2}2^2\right) = t + \frac{1}{2}t^2 - 4 \\
&= \frac{1}{2}\left(t^2 + 2t - 8\right) = \frac{1}{2}\left(t-2\right)\left(t+4\right),
\end{aligned} \tag{1.4}$$

i.e.,

$$y(t) = \exp\left(\frac{1}{2}\left(t-2\right)\left(t+4\right)\right) - 1. \tag{1.5}$$

Thus $y(-4) = \exp\left(\frac{1}{2}\left(-4-2\right)\left(-4+4\right)\right) - 1 = 1 - 1 = 0 = y(2)$.

2. Prove that the following differential equation is exact and then find an expression for its general solution.

$$\left(2xy^3+3x^2y^4+y^6\right)dx+\left(3x^2y^2+4x^3y^3+5y^4+6xy^5\right)dy=0\,. \tag{1.6}$$

14 points

Solution

The equation (1.6) is, by definition, *exact* if the left-hand side is the differential of a (continuously differentiable) function (of two variables x and y, in some simply-connected region of the x-y plane, etc., etc.), i.e. if there is a function $\psi(x,y)$ such that

$$d\psi(x,y)=\left(2xy^3+3x^2y^4+y^6\right)dx+\left(3x^2y^2+4x^3y^3+5y^4+6xy^5\right)dy\,. \tag{1.7}$$

But we have, by definition,

$$d\psi(x,y)=\psi_x(x,y)dx+\psi_y(x,y)dy, \tag{1.8}$$

so that the equation (1.7) is the (potentially) over-determined system of PDE's

$$\psi_x(x,y)=2xy^3+3x^2y^4+y^6,\ \text{and}\ \psi_y(x,y)=3x^2y^2+4x^3y^3+5y^4+6xy^5. \tag{1.9}$$

This over-determined pair of equations is consistent (or *integrable*) iff $\left(\psi_x\right)_y=\left(\psi_y\right)_x$, i.e. iff

$$\left(2xy^3+3x^2y^4+y^6\right)_y=\left(3x^2y^2+4x^3y^3+5y^4+6xy^5\right)_x. \tag{1.10}$$

(1.10) holds true, so that the equation (1.6) is indeed exact, because either side of (1.10) is $6xy^2+12x^2y^3+6y^5$.

As for developing the function $\psi(x,y)$, and then (an expression for) the general solution of(1.6), one notes that the equations (1.9) demand, respectively, that

$$\begin{aligned}\psi(x,y)&=\int\left(2xy^3+3x^2y^4+y^6\right)dx=x^2y^3+x^3y^4+xy^6+f(y),\\ \text{and}\ \psi(x,y)&=\int\left(3x^2y^2+4x^3y^3+5y^4+6xy^5\right)dy=x^2y^3+x^3y^4+y^5+xy^6+g(x),\end{aligned} \tag{1.11}$$

for some initially rather arbitrary functions $f(y)$ and $g(x)$. The two statements (1.11) are compatible iff
$x^2y^3+x^3y^4+xy^6+f(y)=x^2y^3+x^3y^4+y^5+xy^6+g(x)\Leftrightarrow f(y)-y^5=g(x)-0$, which

implies both sides of the last equation are independent of both x and y. As far as finding the general solution of (1.6) is concerned, without loss of generality we can choose $f(y) - y^5 = g(x) - 0 = 0$ so that (1.11) becomes ("in either case")

$$\psi(x,y) = x^2y^3 + x^3y^4 + xy^6 + y^5 = y^3\left(1+xy\right)\left(x^2+y^2\right). \tag{1.12}$$

(1.12) is NOT the general solution to the (exact) differential equation (1.6). It is not even a specific solution. Rather (1.12) defines a "potential (function) for the solution." Using it one notes that (1.6) can be written as

$$d\psi(x,y) = d\left(y^3\left(1+xy\right)\left(x^2+y^2\right)\right) = 0, \tag{1.13}$$

the general solution to which is clearly

$$y^3\left(1+xy\right)\left(x^2+y^2\right) = C. \tag{1.14}$$

3. By using one of the estimates from Picard's proof of the Fundamental Theorem of First Order ODE's, show that there exists a solution $y = \phi(t)$ to the IVP

$$\frac{dy}{dt} = \frac{3t^2}{2}\left(1+y^2\right),\ y(0) = 0, \tag{1.15}$$

at least throughout the interval

$$t \in \left[-h,h\right] = \left[-1,1\right].^{1} \tag{1.16}$$

Hint: If you do not remember the estimate, do the following to jog your memory. Instead of (1.15), and following Picard, write (for some $h > 0$ to be determined) that

$$\phi(t) = \frac{3}{2}\int_0^t s^2\left(1+\phi^2(s)\right)ds,\ \ |t| \le h, \tag{1.17}$$

and, so, deduce that

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[1] Note that the solution of (1.15) actually persists throughout the interval $t \in \left(-\left(\pi\right)^{1/3}, \left(\pi\right)^{1/3}\right) \doteq \left(-1.46, 1.46\right)$, since the solution has the formula $\tan\left(t^3/2\right)$, and since the domain of (the relevant instance of) the tangent function is $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$|t| \le h \Rightarrow |\phi(t)| \le \frac{3}{2}\int_0^{|t|}\left|s^2\left(1+\phi^2(s)\right)\right|ds = \frac{3}{2}\int_0^{|t|}s^2\left(1+\phi^2(s)\right)ds \le \frac{3}{2}\int_0^{h}s^2\left(1+\phi^2(s)\right)ds \,. \quad (1.18)$$

Now demand that h is the biggest number that is still small enough so that, for some $Y>0$, imposing $|\phi(s)| \le Y$ (for each $s \in [-h,h]$) on the right hand side of (1.18) certainly ensures that $|\phi(t)| \le Y$ (for each $t \in [-h,h]$) on the left of (1.18). By doing this you will get an h that depends on Y, i.e. you will get an $h(Y)$. Now find

$$h := \max_{Y>0} h(Y)\,. \quad (1.19)$$

This should be the number indicated in (1.16), i.e., the result of (1.19) should be the number 1.

10 points

Solution

Demanding $|\phi(s)| \le Y$ (for each $s \in [-h,h]$) on the right hand side of (1.18) gives there that

$$|t| \le h \Rightarrow |\phi(t)| \le \frac{3}{2}\int_0^{h}s^2\left(1+\phi^2(s)\right)ds \le \frac{3}{2}\int_0^{h}s^2\left(1+Y^2\right)ds = \frac{h^3}{2}\left(1+Y^2\right). \quad (1.20)$$

So now we certainly get $|\phi(t)| \le Y$ (for each $t \in [-h,h]$) provided we choose h small enough so that $\frac{h^3}{2}\left(1+Y^2\right) \le Y$, the largest such h accomplishing this being

$$h(Y) = \left(\frac{2Y}{1+Y^2}\right)^{1/3}. \quad (1.21)$$

So then we get the h indicated in (1.16) by noting that

$$h := \max_{Y>0} h(Y) = \max_{Y>0}\left(\frac{2Y}{1+Y^2}\right)^{1/3} = \left(\frac{2\cdot 1}{1+1^2}\right)^{1/3} = \left(\frac{2}{2}\right)^{1/3} = 1. \quad (1.22)$$

We could get the maximum indicated in (1.22) by using the relevant tools from calculus.

4. Suppose we have 2 continuous solutions $\phi(t)$ and $\psi(t)$, $t \in [-h,h] = [-1,1]$, to the integral equation indicated in (1.17), i.e. suppose that both

$$\begin{aligned} \phi(t) &= \frac{3}{2}\int_0^t s^2\left(1+\phi^2(s)\right)ds, \text{ and} \\ \psi(t) &= \frac{3}{2}\int_0^t s^2\left(1+\psi^2(s)\right)ds \end{aligned} \tag{1.23}$$

for each $t \in [-1,1]$. Then we could note that the difference $\phi(t)-\psi(t)$ of these two solutions obeys

$$\begin{aligned} \phi(t)-\psi(t) &= \frac{3}{2}\int_0^t \left[s^2\left(1+\phi^2(s)\right)-s^2\left(1+\psi^2(s)\right)\right]ds = \frac{3}{2}\int_0^t s^2\left(\phi^2(s)-\psi^2(s)\right)ds \\ &= \frac{3}{2}\int_0^t s^2\left(\phi(s)+\psi(s)\right)\left(\phi(s)-\psi(s)\right)ds, \end{aligned} \tag{1.24}$$

and, for $t \in [-1,1]$, we could get the estimate

$$\begin{aligned} \left|\phi(t)-\psi(t)\right| &= \left|\frac{3}{2}\int_0^t s^2\left(\phi(s)+\psi(s)\right)\left(\phi(s)-\psi(s)\right)ds\right| \\ &= \frac{3}{2}\int_0^{|t|} s^2\left|\phi(s)+\psi(s)\right|\left|\phi(s)-\psi(s)\right|ds \\ &\le \frac{3}{2}\int_0^{|t|}\left\{\max_{\tau\in[0,1]}\tau^2\left|\phi(\tau)+\psi(\tau)\right|\right\}\left|\phi(s)-\psi(s)\right|ds \\ &=: \frac{3}{2}\int_0^{|t|} K\left|\phi(s)-\psi(s)\right|ds = \frac{3}{2}K\int_0^{|t|}\left|\phi(s)-\psi(s)\right|ds \end{aligned} \tag{1.25}$$

where evidently

$$0 \le K := \max_{\tau\in[0,1]}\tau^2\left|\phi(\tau)+\psi(\tau)\right| < \infty, \tag{1.26}$$

the last inequality in (1.26) holding because we have a continuous function on the bounded interval $\tau \in [0,1]$. So now define the new function

$$U(t) := \int_0^t |\phi(s) - \psi(s)| \, ds \tag{1.27}$$

for each $t \in [-1,1]$, and note that

$$\begin{aligned} &U(t) \geq 0, t \in [0,1], \\ &U(0) = 0, \end{aligned} \tag{1.28}$$

and also then note that (1.25) can then be written as

$$U'(t) = |\phi(t) - \psi(t)| \leq \frac{3}{2} K \int_0^{|t|} |\phi(s) - \psi(s)| \, ds = \frac{3}{2} K U(t) \tag{1.29}$$

for $t \in [0,1]$, i.e., we get

$$U'(t) \leq \frac{3}{2} K U(t), t \in [0,1]. \tag{1.30}$$

Use (1.28) and (1.30) to show that

$$U(t) = 0\,, \tag{1.31}$$

for $t \in [0,1]$ and, so, deduce that $\phi(t) = \psi(t), t \in [0,1]$, i.e., deduce that there is at most one continuous solution to the integral equation (1.17) for $t \in [0,1]$. (You could also show $U(t) = 0$ for $t \in [-1,0]$ by a related but different argument.)

15 points

Solution

From (1.30) we have that, for each $t \in [0,1]$,

$$U'(t) - \frac{3}{2} K U(t) \leq 0, \tag{1.32}$$

and then that

$$e^{-3/2Kt} U'(t) - \frac{3}{2} e^{-3/2Kt} K U(t) \leq 0 \tag{1.33}$$

for $t \in [0,1]$. But then since

$$e^{-3/2Kt}U'(t) - \frac{3}{2}e^{-3/2Kt}KU(t) = \frac{d}{dt}\left(e^{-3/2Kt}U(t)\right), \tag{1.34}$$

(1.33) is

$$\frac{d}{dt}\left(e^{-3/2Kt}U(t)\right) \le 0, \tag{1.35}$$

again for $t \in [0,1]$. Together with $U(0) = 0$ (see (1.28)), (1.35) gives, for each $t \in [0,1]$,

$$e^{-3/2Kt}U(t) = e^{-3/2Kt}U(t) - 0 = e^{-3/2Kt}U(t) - e^{-3/2K\cdot 0}U(0) = e^{-3/2Ks}U(s)\Big|_{s=0}^{s=t} =$$
$$\int_0^t \frac{d\left(e^{-3/2Ks}U(s)\right)}{ds}ds \le \int_0^t 0ds \tag{1.36}$$
$$= 0,$$

i.e. (1.35) gives

$$e^{-3/2Kt}U(t) \le 0, \tag{1.37}$$

or, equivalently,

$$U(t) \le 0 \tag{1.38}$$

for each $t \in [0,1]$. Between (1.38) and $U(t) \ge 0$ (from (1.28)), we must have $U(t) = 0$. But then, using (1.27), we have

$$0 = \frac{d}{dt}0 = \frac{d}{dt}U(t) = \frac{d}{dt}\int_0^t |\phi(s) - \psi(s)|ds = |\phi(t) - \psi(t)| \tag{1.39}$$

for $t \in [0,1]$, which then gives $\phi(t) = \psi(t)$ for $t \in [0,1]$.

5. After a time $t = 0$, a solution of constant concentration of 1gram solute per liter solvent enters a (perfect) stirring tank at a constant rate of 6liters per minute. The well-stirred mixture exits the tank at a constant rate of 4 liters per minute. Suppose the solute takes no volume in solution. If the tank contains 10 liters of fluid at a time $t = 0$, write down a (self-contained) differential equation for the time evolution of the grams of solute $Q(t)$ accumulated in the tank at time t, one that is valid for as long as the tank is not overflowing. Then, assuming there are

10 grams of solute in the tank at $t=0$, give an expression for the grams $Q(t)$ of solute accumulated in the tank at time t (by solving the relevant IVP).

15 points

Solution

By stoichiometric / "unit-canceling" / "chain-rule" reasoning, one has

$$\begin{aligned}\frac{dQ}{dt} &= \left(\frac{dQ}{dt}\right)_{total} = \left(\frac{dQ}{dt}\right)_{in} - \left(\frac{dQ}{dt}\right)_{out} = \left(\frac{dQ}{dV}\right)_{in}\left(\frac{dV}{dt}\right)_{in} - \left(\frac{dQ}{dV}\right)_{out}\left(\frac{dV}{dt}\right)_{out} \\ &= C_{in}R_{in} - C_{out}R_{out} \\ &= R_{in}C_{in} - R_{out}\frac{Q}{V} = 6\cdot 1 - 4\frac{Q}{V} = 6 - 4\frac{Q}{V},\end{aligned} \tag{1.40}$$

where the fluid tank volume $V = V(t)$ is specified by

$$\frac{dV}{dt} = R_{in} - R_{out} = 6-4=2,\ V(0)=V_0=10, \tag{1.41}$$

the latter (trivial) initial value problem having the unique solution

$$V = V_0 + t(R_{in} - R_{out}) = 10 + t\cdot 2 = 10+2t\,. \tag{1.42}$$

Thus the required, "self-contained" differential equation is

$$\begin{aligned}\frac{dQ}{dt} &= R_{in}C_{in} - R_{out}\frac{Q}{V_0 + t(R_{in}-R_{out})} \\ &= 6 - \frac{4}{10+2t}Q \\ &\Leftrightarrow \\ \frac{dQ}{dt} &+ \frac{2}{5+t}Q = 6.\end{aligned} \tag{1.43}$$

We solve the initial value problem which is ODE (1.43) together with initial data

$$Q(0) = 10\,. \tag{1.44}$$

An integrating factor for the ODE (1.43) is, according to the standard theory,

$$\mu = \exp\int \frac{2}{5+t}dt = \exp\left(2\log\left(5+t\right)\right)$$
$$= \left(5+t\right)^2 . \quad (1.45)$$

Use of the integrating factor (1.45) in (1.43) gives

$$\frac{d\left(\left(5+t\right)^2 Q\right)}{dt} = \left(5+t\right)^2 \frac{dQ}{dt} + \left(5+t\right)^2 \frac{2}{5+t} Q = 6\left(5+t\right)^2 = \frac{d\,2\left(5+t\right)^3}{dt} . \quad (1.46)$$

Integration of (1.46) using relevant limits (and dummy variables) gives

$$\begin{aligned}
&\left(5+t\right)^2 Q(t) - 2\cdot 5^3 = \left(5+t\right)^2 Q(t) - 5^2\cdot 10 = \\
&\left(5+t\right)^2 Q(t) - \left(5+0\right)^2 Q(0) = \left(5+s\right)^2 Q(s)\Big|_0^t = \\
&\int_0^t d\left(\left(5+s\right)^2 Q(s)\right) = \int_0^t d\left(2\left(5+s\right)^3\right) \\
&= 2\left(5+s\right)^3\Big|_0^t = 2\left(5+t\right)^3 - 2\left(5+0\right)^3 = 2\left(5+t\right)^3 - 2\cdot 5^3 ,
\end{aligned} \quad (1.47)$$

or, equivalently,

$$Q(t) = 2\left(5+t\right) = 10 + 2t . \quad (1.48)$$

6. Show that the following differential equation, equation(1.49), is not exact, but can be rendered exact by multiplication by an integrating factor that is only a function of x or only a function of y . Find an expression of the general solution of the differential equation.

$$\left(2xy + 3x^2y^2 + y^4\right)dx + \left(3x^2 + 4x^3y + 5y^2 + 6xy^3\right)dy = 0 . \quad (1.49)$$

15 points

Solution

(1.49) is not exact since

$$\begin{aligned}
\psi_{xy} &= \left(\psi_x\right)_y := \left(2xy + 3x^2y^2 + y^4\right)_y = 2x + 6x^2y + 4y^3 \\
&\neq 6x + 12x^2y + 6y^3 = \left(3x^2 + 4x^3y + 5y^2 + 6xy^3\right)_x =: \left(\psi_y\right)_x = \psi_{yx}
\end{aligned} . \quad (1.50)$$

By theorem we know that, with an integrating factor μ , (1.49) can be made exact. We note from (1.50) that

$$\begin{aligned}\left(2xy+3x^2y^2+y^4\right)_y-\left(3x^2+4x^3y+5y^2+6xy^3\right)_x &= 2x+6x^2y+4y^3-\left(6x+12x^2y+6y^3\right)\\ &= -4x-6x^2y-2y^3\\ &= -2\left(2x+3x^2y+y^3\right)\end{aligned} \quad ,(1.51)$$

which divides the first term in (1.49), (which is $y\left(2x+3x^2y+y^3\right)$), the remaining factor ($-y/2$) being only a function of y . Thus we suspect the existence of an integrating factor only depending on y . At any rate, with the use of such a factor, ODE (1.49) becomes

$$\left(2xy+3x^2y^2+y^4\right)\mu(y)dx+\left(3x^2+4x^3y+5y^2+6xy^3\right)\mu(y)dy=0\,, \qquad (1.52)$$

and exactness demands that

$$\begin{aligned}0&=\left(\left(2xy+3x^2y^2+y^4\right)\mu(y)\right)_y-\left(\left(3x^2+4x^3y+5y^2+6xy^3\right)\mu(y)\right)_x\\ &=\left(2x+6x^2y+4y^3\right)\mu(y)+\left(2xy+3x^2y^2+y^4\right)\mu'(y)-\left(6x+12x^2y+6y^3\right)\mu(y)\\ &=-2\left(2x+3x^2y+y^3\right)\mu(y)+y\left(2x+3x^2y+y^3\right)\mu'(y)\\ &=\left(2x+3x^2y+y^3\right)\left(-2\mu(y)+y\mu'(y)\right)\\ &\Leftarrow\\ y\mu'(y)&=2\mu(y).\end{aligned} \qquad (1.53)$$

Thus, as suspected, there is an integrating factor depending only on y . A solution of the last differential equation in (1.53) is given by

$$\mu(y)=y^2\,, \qquad (1.54)$$

in which case (1.52) becomes

$$\begin{aligned}0&=\left(2xy+3x^2y^2+y^4\right)y^2dx+\left(3x^2+4x^3y+5y^2+6xy^3\right)y^2dy\\ &=\left(2xy^3+3x^2y^4+y^6\right)dx+\left(3x^2y^2+4x^3y^3+5y^4+6xy^5\right)dy,\end{aligned} \qquad (1.55)$$

which is the exact equation considered in problem 2. Thus the solution is the same as in problem 2, namely

$$x^2y^3 + x^3y^4 + xy^6 + y^5 = y^3(1+xy)(x^2+y^2) = C\,. \tag{1.56}$$

7. Find a linear, first order, ordinary differential equation with the property that *every* solution $y(t)$ of it approaches the function $f(t) = 1+t^2$ arbitrarily closely as $t \to +\infty$. Note that the (too) simple equation

$$y'(t) = f'(t) = (1+t^2)' = 2t \tag{1.57}$$

does not work since the general solution of (1.57) is

$$y(t) = \int 2t\,dt = C + t^2\,, \tag{1.58}$$

giving

$$\lim_{t\to+\infty}\left(y(t) - f(t)\right) = \lim_{t\to+\infty}\left(C + t^2 - (1+t^2)\right) = C - 1\,, \tag{1.59}$$

which is not zero for every possible choice of C.

10 points

Solution

Introduce a general solution of the form

$$y(t) = 1 + t^2 + Ce^{-at} \tag{1.60}$$

with $a > 0$ to get

$$\lim_{t\to+\infty}\left(y(t) - f(t)\right) = \lim_{t\to+\infty}\left(1 + t^2 + Ce^{-at} - (1+t^2)\right) = \lim_{t\to+\infty} Ce^{-at} = 0 \tag{1.61}$$

for every choice of C, as demanded by the problem. Thus, to get a first order ODE with the required property we differentiate (1.60) with respect to t and eliminate C between (1.60) and this new result. Differentiating (1.60) gives

$$y'(t) = 2t - aCe^{-at}\,, \tag{1.62}$$

and elimination of C between (1.60) and (1.62) gives the required first order ODE, namely

$$y'(t) = 2t - aCe^{-at} = 2t - a\left(y(t) - \left(1+t^2\right)\right)$$
$$= -ay(t) + a + 2t + at^2. \tag{1.63}$$

8. Solve the following initial value problem. State the properties of the solution as $t \to +\infty$ for all possible choices of the initial value y_0.

$$y'(t) = -y(t) + \left(1+t\right)^2,\ y(0) = y_0\,. \tag{1.64}$$

15 points

Solution

The ODE in (1.64) can be written as

$$y'(t) + 1y(t) = \left(1+t\right)^2. \tag{1.65}$$

(1.65) suggests the integrating factor

$$\mu(t) = \exp\int 1dt = e^t\,, \tag{1.66}$$

which renders the ODE (1.65) as

$$\begin{aligned}
&\frac{d}{dt}e^t y(t) = \\
&e^t y'(t) + e^t y(t) = \left(1+t\right)^2 e^t = \frac{d}{dt}\left(\left(1+t\right)^2 e^t\right) - e^t\frac{d}{dt}\left(1+t\right)^2 \\
&= \frac{d}{dt}\left(\left(1+t\right)^2 e^t\right) - 2e^t\left(1+t\right) \\
&= \frac{d}{dt}\left(\left(1+t\right)^2 e^t - 2e^t\left(1+t\right)\right) + 2e^t\frac{d}{dt}\left(1+t\right) \\
&= \frac{d}{dt}\left(\left(1+t\right)^2 e^t - 2e^t\left(1+t\right)\right) + 2e^t \\
&= \frac{d}{dt}\left(\left(1+t\right)^2 e^t - 2e^t\left(1+t\right) + 2e^t\right) \\
&= \frac{d}{dt}\left(\left(1+t^2\right)e^t\right)
\end{aligned} \tag{1.67}$$

which, with the initial data specified in (1.64), integrates to

$$e^t y(t) - y_0 = e^t y(t) - 1\cdot y_0 = e^t y(t) - e^0 y(0) = e^s y(s)\Big|_0^t =$$
$$\int_0^t de^s y(s) = \int_0^t d\left(1+s^2\right)e^s \tag{1.68}$$
$$= \left(1+s^2\right)e^s\Big|_0^t = \left(1+t^2\right)e^t - \left(1+0^2\right)e^0 = \left(1+t^2\right)e^t - 1,$$

or, equivalently,

$$y(t) = 1 + t^2 + e^{-t}\left(y_0 - 1\right). \tag{1.69}$$

Note then the differential equation in (1.64) gives a solution to problem 7, which indicates the desired properties.

9. Carefully state the (nonlinear) existence and uniqueness theorem for a single first order ODE.

15 points

Solution

Consider the initial value problem

$$y'(t) = f(t, y), \qquad y(t_0) = y_0. \tag{1.70}$$

Suppose $f(t, y)$ and $f_y(t, y)$ are both continuous in an open rectangle $\left(t_{-1}, t_{+1}\right)\times\left(y_{-1}, y_{+1}\right)$ containing the point $\left(t_0, y_0\right)$. Then there exists an $h > 0$ such that (1.70) has a unique, continuously differentiable solution $y = \phi(t)$ persisting over the t interval $\left(t_0 - h, t_0 + h\right)$ (potentially much smaller than the interval $\left(t_{-1}, t_{+1}\right)$).

10. Carefully state the linear existence and uniqueness theorem for a single first order ODE. Explain in general terms how it is proven.

15 points

Solution

Consider the initial value problem

$$y'(t) = p(t)y + q(t), y(t_0) = y_0. \tag{1.71}$$

Suppose $p(t)$ and $q(t)$ are both continuous in an open interval $\left(t_{-1}, t_{+1}\right)$ containing the point t_0. Then (1.71) has a unique, continuously differentiable solution $y = \phi(t)$ persisting over the t interval $\left(t_{-1}, t_{+1}\right)$.

The theorem is proven by explicitly integrating (1.71), using an integrating factor, the various theorems of calculus, including that the integral of a continuous function exists, etc.