# KEY

# Math 334 Midterm II Winter 2012 section 002 Instructor: Scott Glasgow

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Honor Code: After I have learned of the contents of this exam by any means, I will not disclose to anyone any of these contents by any means until after the exam has closed. My signature below indicates I accept this obligation.

Signature:

1. Determine the largest possible 'a priori' lower bound for the radius of convergence of the power series representation of the general solution of the following differential equation about the point  $x_0 = -2$  (i.e., determine the largest one knowable without solving the equation):

$$x((x+1)^{2}+1)y'' + xy' + y = 0.$$
 (1.1)

# **5** points

#### **Solution**

Equation (1.1) can be reduced to the equation

$$y'' + \frac{1}{(x+1)^2 + 1}y' + \frac{1}{x((x+1)^2 + 1)}y = 0$$
(1.2)

which has 'singularities' at the singularities of the coefficients of y' and y, which are at  $x = -1 \pm i$  and x = 0. In the complex plain the distance of the singularities to the expansion point  $x_0 = -2 = -2 + 0i$  is  $\sqrt{(-2+1)^2 + (0\mp 1)^2} = \sqrt{1+1} = \sqrt{2}$  and  $|-2-0| = 2 = \sqrt{4}$ , the smaller of which being  $\sqrt{2}$ . Thus, even along the real axis, we cannot guarantee a radius of convergence beyond  $\sqrt{2}$  without more information.

2. Find a (particular) solution of the following differential equation by the method of undetermined coefficients:

$$L(y) \coloneqq y'' - 2y' + 0y = 20t + 12.$$
(1.3)

Before 'walking away' from this problem, CHECK YOUR (LASER BEAM) ANSWER—SEE IF YOUR PROPOSED SOLUTION ACTUALLY DOES IN FACT SOLVE THE ODE!!!! If it doesn't, fix it. At least comment on what may have gone wrong.

#### 7 points

#### **Solution**

The usual explanation of the ansatz for developing a particular solution to a linear constant coefficient differential equation (with a RHS that is in the null space of a linear constant coefficient differential operator) is to first find a basis for the span of the RHS together with all its derivatives. Then, barring the phenomena of *resonance* (which is that one or more elements of such a basis are in the null space of the specific differential

operator in question, here *L*), one then forms a general element of the space spanned by the basis, which general element constitutes the "method of undetermined coefficients ansatz" for a solution of the equation in question. For the problem at hand, and since the RHS of (1.3) is spanned by the set of functions  $\{t,1\}$ , whose first derivatives are both spanned by the set of functions  $\{1\} \subset \{t,1\}$ , so that all subsequent/higher derivatives are spanned by  $\{t,1\}$ , the relevant basis for a particular solution of (1.3) is, barring resonance,  $\{t,1\}$ . One soon finds though that there is resonance in the case of (1.3), specifically that 1 is in this nullspace, and so, while the dimensionality of the space spanned by  $\{t,1\}$  must, by theory, be enough, nevertheless the 1 function must be multiplied by the minimum power of *t* able to "boost it out of the null space" of *L*. One power of *t* suffices, but then the span of the set  $\{t,1\cdot t^1\} = \{t,t\} = \{t\}$  is clearly only one dimensional, not two as is guaranteed to be sufficient, so that *t* in the original  $\{t,1\}$  must itself be boosted just enough to maintain a two dimensional span: the final set  $\{t\cdot t^1, 1\cdot t^1\} = \{t^2, t\}$  is now guaranteed to span a vector space in which a solution of (1.3) exists. Thus a solution of (1.3), together with relevant derivatives, is guaranteed to be of the form

$$y = At^{2} + Bt + 0$$
  

$$y' = 0t^{2} + 2At + B$$
  

$$y'' = 0t^{2} + 0t + 2A$$
(1.4)

Weighted appropriate for the equation (1.3), the equations (1.4) are

$$0y = 0t^{2} + 0t + 0$$
  
-2y' = 0t<sup>2</sup> - 4At - 2B  
y" = 0t<sup>2</sup> + 0t + 2A (1.5)

which sum to

$$0y = 0t^{2} + 0t + 0$$
  

$$-2y' = 0t^{2} - 4At - 2B$$
  

$$y'' = 0t^{2} + 0t + 2A$$
  

$$20t + 12 = 0t^{2} - 4At + 2A - 2B \Leftrightarrow \frac{-4A = 20}{2A - 2B = 12} \Leftrightarrow \frac{A = -5}{B = -11}.$$
(1.6)

Thus the following expresses a solution of (1.3):

$$y = At^{2} + Bt = -5t^{2} - 11t.$$
 (1.7)

3. A 125 kilogram mass stretches a spring 1/5 meter. If the mass is set in motion from the equilibrium position at 105 meters per second upward (yeah, that's pretty fast—need a detonating device), and there is no damping, determine the displacement u(t) of the mass above the equilibrium position at any subsequent time t. Use that the acceleration of gravity is 49/5 meters per second per second.

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# 9 points

#### **Solution**

The relevant version of Newton's second law is

$$0 = mu'' + ku = 125 \text{kg}u'' + ku . \tag{1.8}$$

Here we may determine the spring constant k from

$$k = F / s = ma / s = 125 \text{kg} 49 / 5 \text{meter} / \sec^2 / (1 / 5 \text{meter}) = 125 \cdot 7^2 \text{kg} / \sec^2$$
, (1.9)

so that (1.8) is

$$0 = 125 \text{kg}u'' + 125 \cdot 7^2 \text{kg} / \sec^2 u \iff 0 = u'' + 7^2 / \sec^2 u .$$
(1.10)

Rendering (1.10) unit-less, by measuring time in seconds, this is

$$0 = u'' + 7^2 u , (1.11)$$

the general solution to which being

$$u = A\cos(7t) + B\sin(7t)$$
. (1.12)

The initial data specifies that

$$u(0) = 0 = A, u'(0) = 105 = 7B$$

$$\Leftrightarrow \qquad (1.13)$$

$$A = 0, B = 15.$$

so that the required solution to the initial value problem is

$$u = A\cos(7t) + B\sin(7t) = 15\sin(7t).$$
(1.14)

4. Find the general solution of the following Euler equation, one that is valid for x > 0:

$$x^{2}y'' - 5xy' + 13y = 0.$$
 (1.15)

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## 11 points

## **Solution**

The differential equation (1.15) defines a linear differential operator  $L_x$ , in terms of which (1.15) can be written  $L_x[y] = 0$ . On a function  $y_r = x^r$  one finds that

$$L_{x}[y_{r}] = (r(r-1) - 5r + 13)x^{r} = (r^{2} - 6r + 13)x^{r} = ((r-3)^{2} + 2^{2})x^{r}, \qquad (1.16)$$

so that complex solutions of (1.15) are clearly then  $y_{3+2i} = x^{3+2i} = x^3 e^{2i\ln x} = x^3 (\cos(2\ln x) + i\sin(2\ln x))$  and  $y_{3-2i} = x^{3-2i} = x^3 e^{-2i\ln x} = x^3 (\cos(2\ln x) - i\sin(2\ln x))$ . Independent complex linear combinations of these linearly independent complex valued solutions gives the following real-representation of the general solution:

$$y = x^{3} \left( A \cos(2\ln x) + B \sin(2\ln x) \right).$$
(1.17)

5. Solve the following initial value problem:

$$y'' - 6y' + 13y = 0; y(0) = 1, y'(0) = 3.$$
 (1.18)

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## 12 points

# **Solution**

This linear homogeneous differential equation is associated with the following characteristic (polynomial) and characteristic exponents *r* :

$$0 = r^{2} - 6r + 13 = r^{2} - 6r + 9 + 4 = (r - 3)^{2} - (2i)^{2}$$
  

$$\Leftrightarrow \qquad (1.19)$$
  

$$r = 3 \pm 2i.$$

According to the usual theory, a real-representation of the general solution, and its corresponding first derivative, are

$$y = e^{3t} (C_1 \cos 2t + C_2 \sin 2t)$$
  
and  
$$y' = e^{3t} ((3C_1 + 2C_2) \cos 2t + (-2C_1 + 3C_2) \sin 2t).$$
 (1.20)

Inserting t = 0 into (1.20), and using the initial data given in (1.18), one obtains

$$y(0) = C_1 = 1$$
  
and (1.21)  
 $y'(0) = 3C_1 + 2C_2 = 3,$ 

the solution to which being  $C_1 = 1$  and  $C_2 = 0$ . Thus the solution to the initial value problem is then

$$y = e^{3t} \cos 2t$$
. (1.22)

6. Given that  $y_1 = t^3$  is a solution of

$$t^2 y'' - 5ty' + 9y = 0 \tag{1.23}$$

for t > 0, find a second, linearly independent solution  $y_2$  of (1.23) for t > 0 by making the D'Alembert ansatz  $y_2 = vy_1 = vt^3$ .

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## 14 points

## **Solution**

Using D'Alembert's ansatz in (1.23) gives

$$0 = t^{2} y_{2}'' - 5t y_{2}' + 9 y_{2} = t^{2} (vt^{3})'' - 5t (vt^{3})' + 9vt^{3} = t^{2} (v''t^{3} + 6v't^{2} + 6tv) - 5t (v't^{3} + 3t^{2}v) + 9vt^{3}$$
  
=  $t^{5} v'' + (6t^{4} - 5t^{4})v' + (6t^{3} - 15t^{3} + 9t^{3})v$   
=  $t^{5} v'' + t^{4} v' = t^{5} u' + t^{4} u = t^{4} (tu' + u),$   
(1.24)

and where we defined u = v'. By any one of a number of standard techniques, one finds that the first order homogeneous equation in (1.24) has a nontrivial solution  $u = t^{-1} = v' \leftarrow v = \ln t$ . Thus a second, linearly independent solution is

$$y_2 = vt^3 = t^3 \ln t.$$
(1.25)

7. Find the general solution of the following Euler equation, one that is valid for x > 0:

$$x^2 y'' - 7xy' + 16y = 0. (1.26)$$

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## 16 points

#### **Solution**

The differential equation (1.26) defines a linear differential operator  $L_x$ , in terms of which (1.26) can be written  $L_x[y] = 0$ . On a function  $y_r = x^r$  one finds that

$$L_{x}[y_{r}] = (r(r-1) - 7r + 16)x^{r} = (r^{2} - 8r + 16)x^{r} = (r-4)^{2}x^{r}, \qquad (1.27)$$

so that a solution of (1.26) is clearly then  $y_4 = x^4$ . To find the general solution to this second order differential equation we need to find a second, linearly independent solution. Since the ansatz  $y_r = x^r$  only produces solutions dependent upon  $y_4 = x^4$ , we

must use another ansatz. Fortunately the structure of (1.27), together with the fact that the differential operators  $\frac{d}{dr}$  and  $L_x$  commute, suggest such an alternative ansatz: applying  $\frac{d}{dr}$  to both sides of (1.27), and using the indicated commutivity, one obtains

$$L_{x}\left[\frac{d}{dr}y_{r}\right] = \left(r-4\right)^{2}x^{r}\ln x + 2\left(r-4\right)^{1}x^{r}, \qquad (1.28)$$

so that  $\frac{d}{dr} y_r \Big|_{r=r} = x^r \ln x \Big|_{r=4} = x^4 \ln x$  is clearly a second, linearly independent solution of (1.26). Thus the general solution to this linear homogeneous equation is

$$y = (A + B \ln x) x^4$$
. (1.29)

8. Find the first *two nonzero* terms (if there are that many) in the series representation of any *one* of 2 linearly independent solutions of the equation

$$x^{2}(1+4x)y'' + 4x(1+3x)y' + 2y = 0,$$
(1.30)

about the point  $x_0 = 0$ . Realize (1.30) can in fact be rewritten as

$$(x2 + 4x3)y'' + (4x + 12x2)y' + 2y = 0.$$
 (1.31)

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## 18 points

#### **Solution**

The point  $x_0 = 0$  is a singular point, so that the required series solution is not quite a Taylor series: insert  $y = \sum_n a_n x^{n+r}$  (with the assumption that  $a_n = 0$  for n < 0, and that the sum is over the integers, and that  $a_0 \neq 0$ )in (1.30) to obtain

$$\begin{aligned} 0 &= \sum_{n} (x^{2} + 4x^{3})(n+r)(n+r-1)a_{n}x^{n+r-2} + (4x + 12x^{2})(n+r)a_{n}x^{n+r-1} + 2a_{n}x^{n+r} \\ &= \sum_{n} \begin{cases} (n+r)(n+r-1)a_{n}x^{n+r} + 4(n+r)(n+r-1)a_{n}x^{n+r+1} \\ + 4(n+r)a_{n}x^{n+r} + 12(n+r)a_{n}x^{n+r} + 2a_{n}x^{n+r} \end{cases} \\ &= \sum_{n} \begin{cases} (n+r)(n+r-1)a_{n}x^{n+r} + 4(n+r-1)(n+r-2)a_{n-1}x^{n+r} \\ + 4(n+r)a_{n}x^{n+r} + 12(n+r-1)a_{n-1}x^{n+r} + 2a_{n}x^{n+r} \end{cases} \\ &= \sum_{n} \{ [(n+r)(n+r-1) + 4(n+r) + 2]a_{n} + [4(n+r-1)(n+r-2) + 12(n+r-1)]a_{n-1} \} x^{n+r} \\ &= \sum_{n=0}^{\infty} \{ [(n+r)(n+r-1) + 4(n+r) + 2]a_{n} + 4(n+r-1)(n+r+1)a_{n-1} \} x^{n+r} \\ &= [r(r-1) + 4r + 2]a_{0}x^{r} + \sum_{n=1}^{\infty} \left\{ [(n+r)(n+r-1) + 4(n+r) + 2]a_{n} + (n+r+1)a_{n-1} \\ 4(n+r-1)(n+r+1)a_{n-1} \end{array} \right\} x^{n+r} \\ &= (1.32) \end{aligned}$$

Evidently we require

$$0 = r(r-1) + 4r + 2 = r^{2} + 3r + 2 = (r+2)(r+1) \Leftrightarrow r = -2, -1..$$
(1.33)

Since these roots differ by an integer, according to the general theory we are only guaranteed that the larger of the two roots r gives a solution of the form

$$y = \sum_{n} a_{n} x^{n+r} = \sum_{n=0}^{\infty} a_{n} x^{n+r} = \sum_{n=0}^{\infty} a_{n} x^{n-1}.$$
 (1.34)

From the last expression in (1.32) with r = -1 we get that the coefficients  $a_n$  are determined by

$$\begin{bmatrix} (n-1)(n-1-1) + 4(n-1) + 2 \end{bmatrix} a_n + 4(n-1-1)(n-1+1)a_{n-1} = 0, \quad n = 1, 2, 3, \dots$$

$$\Leftrightarrow$$

$$n(n+1)a_n + 4(n-2)na_{n-1} = 0, \quad n = 1, 2, 3, \dots$$

$$\Leftrightarrow$$

$$a_n = -4\frac{n-2}{n+1}a_{n-1}, \quad n = 1, 2, 3, \dots$$
(1.35)

Thus we have

$$a_{1} = -4 \frac{1-2}{1+1} a_{1-1} = 2a_{0},$$

$$a_{2} = -4 \frac{2-2}{2+1} a_{2-1} = 0,$$

$$a_{3} = -4 \frac{3-2}{3+1} a_{3-1} = -a_{2} = -0 = 0,$$

$$\vdots$$

$$a_{n\geq 2} = 0.$$
(1.36)

Thus the "infinite" series terminates, and we get in (1.34) that

$$y = \sum_{n=0}^{\infty} a_n x^{n-1} = a_0 x^{0-1} + a_1 x^{1-1} = a_0 x^{-1} + a_1 = a_0 x^{-1} + 2a_0 = a_0 \left( x^{-1} + 2 \right)$$

$$= a_0 \left( \frac{1}{x} + 2 \right).$$
(1.37)

which gives a (closed form) solution of (1.30).

9. Find the general solution of the following linear but non-homogeneous differential equation by the method of variation of parameters. Do not use the (memorized) formula/theorem (involving a Wronskian), rather generate the relevant version of the formula afresh by using the "D'Alembert-like" ansatz that leads to that formula.

$$y'' - 3y' + 2y = e^{4t} \tag{1.38}$$

Before 'walking away' from this problem, CHECK YOUR (LASER BEAM) ANSWER—SEE IF YOUR PROPOSED SOLUTION ACTUALLY DOES IN FACT SOLVE THE ODE!!!! If it doesn't, fix it. At least comment on what may have gone wrong.

# 22 points

#### **Solution**

The characteristic equation of the homogeneous version of the constant coefficient differential equation (1.38) is

$$0 = r^{2} - 3r + 2 = (r - 1)(r - 2)$$
(1.39)

so that the general solution of the corresponding homogeneous equation is

$$y = Ae^{t} + Be^{2t}, (1.40)$$

where A and B are independent of t. But allowing the parameters A and B to vary with t in (1.40), we have also an ansatz there for the solution of the non-homogeneous equation (1.38): with such an ansatz one immediately has

$$y' = Ae^{t} + 2Be^{2t} + \left(e^{t}A' + e^{2t}B'\right).$$
(1.41)

But this ansatz is "initially consistent with A and B independent of t" if we choose here that

$$e^{t}A' + e^{2t}B' = 0, (1.42)$$

so that then (1.41) becomes

$$y' = Ae^t + 2Be^{2t}.$$
 (1.43)

Differentiating (1.43) gives

$$y'' = Ae^{t} + 4Be^{2t} + \left(e^{t}A' + 2e^{2t}B'\right).$$
(1.44)

Combining these derivatives with the appropriate weights (dictated by the differential equation) we get the ledger

$$2y = 2Ae^{t} + 2Be^{2t}$$
  
-3y' = -3Ae^{t} - 6Be^{2t} (1.45)  
+1y'' = Ae^{t} + 4Be^{2t} + (e^{t}A' + 2e^{2t}B').

and from which it is clear that the differential equation demands that

$$e^{t}A' + 2e^{2t}B' = e^{4t}.$$
 (1.46)

Combining this with the "consistency ansatz" (1.42) we get the (rather trivial) system of first order ODE's

$$\begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 0 \\ e^{4t} \end{bmatrix},$$
 (1.47)

which implies that

$$A' = \frac{\begin{vmatrix} 0 & e^{2t} \\ e^{4t} & 2e^{2t} \\ e^{t} & e^{2t} \\ e^{t} & 2e^{2t} \end{vmatrix}}{\begin{vmatrix} e^{t} & e^{2t} \\ e^{t} & 2e^{2t} \end{vmatrix}} = \frac{-e^{6t}}{e^{3t}} = -e^{3t}, \ B' = \frac{\begin{vmatrix} e^{t} & 0 \\ e^{t} & e^{4t} \end{vmatrix}}{\begin{vmatrix} e^{t} & e^{2t} \\ e^{t} & 2e^{2t} \end{vmatrix}} = \frac{e^{5t}}{e^{3t}} = e^{2t}.$$
 (1.48)

Solutions to (1.48) include the pair  $A = -\frac{e^{3t}}{3}$ ,  $B = \frac{e^{2t}}{2}$ , so that a solution to (1.38) is, according to (1.40),

$$y = Ae^{t} + Be^{2t} = -\frac{e^{3t}}{3}e^{t} + \frac{e^{2t}}{2}e^{2t} = \frac{1}{6}e^{4t}, \qquad (1.49)$$

and the general solution to (1.38) can be expressed as

$$y = c_1 e^t + c_2 e^{2t} + \frac{1}{6} e^{4t}, \qquad (1.50)$$

where  $c_1$  and  $c_2$  are (truly) constants now.

10. Solve the initial value problem obtained from combining the differential equation of problem 9 with the initial data y(0) = 0, y'(0) = 0. In order that errors don't "cascade", I will tell you that  $y = Ae^t + Be^{2t} + \frac{1}{6}e^{4t}$  (*A* and *B* truly constant) is the general solution of the differential equation of problem 9. (So now if you just write down this solution to 9 without very convincing work, you will get 0 points on problem 9.) Thus, I am only testing if you understand the correct principles needed to construct the solution to the initial value problem given the general solution to the associated differential equation.

## 20 points

#### Solution

From the information given we have

$$y(0) = 0 = A + B + \frac{1}{6}$$
  
y'(0) = 0 = A + 2B +  $\frac{2}{3}$ , (1.51)

or, equivalently, the following augmented matrix for the column vector (A, B), which, together with row reduction, is

$$\begin{bmatrix} 6 & 6 & -1 \\ 3 & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 3 & 6 & -2 \\ 6 & 6 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & 6 & -2 \\ 0 & -6 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 1 \\ 0 & -6 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}.$$
 (1.52)

From (1.52) one has that

$$A = 1/3, B = -1/2. \tag{1.53}$$

and the solution sought is

$$y = Ae^{t} + Be^{2t} + \frac{1}{6}e^{4t} = \frac{1}{3}e^{t} - \frac{1}{2}e^{2t} + \frac{1}{6}e^{4t}.$$
 (1.54)