KEY

Math 334 Midterm III
Winter 2012
section 002
Instructor: Scott Glasgow

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Signature:

1. Solve the following initial value problem in terms of the convolution integral:

$$y'' + 16y = g(t);\ y(0) = y'(0) = 0. \tag{1.1}$$

10 points

Solution

Laplace transformation of (1.1) gives

$$s^2 L[y] + 16L[y] = L[y''] + 16L[y] = L[g]$$

$$\Leftrightarrow$$

$$L[y] = \frac{1}{s^2+4^2} L[g] = \frac{1}{4}\frac{4}{s^2+4^2} L[g] = L\left(\frac{1}{4}\sin(4\cdot)\right) L[g] = L\left[\frac{1}{4}\sin(4\cdot) * g\right] \tag{1.2}$$

$$\Leftrightarrow$$

$$y(t) = \left(\frac{1}{4}\sin(4\cdot) * g\right)(t) = \int_0^t \frac{1}{4}\sin(4(t-\tau))\, g(\tau)\, d\tau.$$

2. Find the general solution of the following system:

$$\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \mathbf{x} \tag{1.3}$$

15 points

Solution

The general solution of (1.3) is

$$\mathbf{x} = \mathbf{x}(t) = c_1 \boldsymbol{\xi}_1 e^{\lambda_1 t} + c_2 \boldsymbol{\xi}_2 e^{\lambda_2 t}, \tag{1.4}$$

where the $\boldsymbol{\xi}$'s and λ's are independent eigenvectors and distinct eigenvalues of the matrix in (1.3):

$$\begin{bmatrix} 2-\lambda & 1 \\ 4 & 2-\lambda \end{bmatrix} \boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \boldsymbol{\xi} = \mathbf{0}$$

unless

$$0 = \det \begin{bmatrix} 2-\lambda & 1 \\ 4 & 2-\lambda \end{bmatrix} = (\lambda - 2)^2 - 2^2 \tag{1.5}$$

$$\Leftrightarrow$$

$$\lambda = 2 \pm 2 = 4, 0 =: \lambda_1, \lambda_2.$$

So

$$\begin{aligned} \mathbf{0} &= \begin{bmatrix} 2-4 & 1 \\ 4 & 2-4 \end{bmatrix} \boldsymbol{\xi}_1 = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \boldsymbol{\xi}_1 = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{\xi}_1 \Leftarrow \boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \\ \mathbf{0} &= \begin{bmatrix} 2-0 & 1 \\ 4 & 2-0 \end{bmatrix} \boldsymbol{\xi}_2 = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \boldsymbol{\xi}_2 = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{\xi}_2 \Leftarrow \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{aligned} \tag{1.6}$$

Thus, explicitly, (1.4) is

$$\mathbf{x} = \mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{0t} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \tag{1.7}$$

3. Solve the following IVP. Using the Laplace transform and knowing how to deal with piecewise defined functions in this transform should make things easier. What is the value of $y(10)$? Do not express this value in terms of an abstract formula, but rather as a concrete number. To compute this value concretely, you will need to know that the function $\sin(t)$ is 2π periodic.

$$y'' + 4y = f(t) = \begin{cases} 0, \ t < 0 \\ 3t, \ 0 \le t < \pi \\ 6\pi - 3t, \ \pi \le t < 2\pi \\ 0, t \ge 2\pi \end{cases}, \ \ y(0) = y'(0) = 0. \tag{1.8}$$

20 points

<u>Solution</u>

Taking the Laplace transform of (1.8) we get

$$\begin{gathered}\left(s^2+2^2\right)\mathcal{L}\left[y\right](s)=\mathcal{L}\left[y''+4y\right](s)=\mathcal{L}\left[f\right](s)\\ \Leftrightarrow \\ \mathcal{L}\left[y\right](s)=\frac{1}{s^2+2^2}\mathcal{L}\left[f\right](s).\end{gathered}, \tag{1.9}$$

To compute $\mathcal{L}\left[f\right](s)$ we rewrite f in (1.3) as:

$$\begin{aligned}f(t)&=3\Big(t\big(u_0(t)-u_\pi(t)\big)+\big(2\pi-t\big)\big(u_\pi(t)-u_{2\pi}(t)\big)\Big)\\ &=3\Big(t\ u_0(t)+\big(2\pi-2t\big)u_\pi(t)+\big(t-2\pi\big)u_{2\pi}(t)\Big)\\ &=3\Big(\big(t-0\big)\ u_0(t)-2\big(t-\pi\big)u_\pi(t)+\big(t-2\pi\big)u_{2\pi}(t)\Big).\end{aligned} \tag{1.10}$$

So, according to the relevant theorem, and given $\mathcal{L}\left[g\right](s)=\frac{1}{s^2}$ when $g(t)=t\ u_0(t)$ (as per the table provided you), then

$$\begin{aligned}L\left[f\right](s)&=3\Big(L\left[g\right](s)-2e^{-\pi s}L\left[g\right](s)+e^{-2\pi s}L\left[g\right](s)\Big)\\ &=3L\left[g\right](s)\left(1-2e^{-\pi s}+e^{-2\pi s}\right)=\frac{3}{s^2}\left(1-2e^{-\pi s}+e^{-2\pi s}\right).\end{aligned} \tag{1.11}$$

Thus, explicitly, (1.9) is

$$L\left[y\right](s)=\frac{1}{s^2+2^2}L\left[f\right](s)=\frac{1}{s^2+2^2}\frac{3}{s^2}\left(1-2e^{-\pi s}+e^{-2\pi s}\right). \tag{1.12}$$

and we already know then that the solution is of the form

$$y(t)=3\big(h(t)u_0(t)-2h(t-\pi)u_\pi(t)+h(t-2\pi)u_{2\pi}(t)\big), \tag{1.13}$$

where then we need only find $h(t)$, whose Laplace transform is

$$\begin{aligned}L\left[h\right](s)&=\frac{1}{s^2+2^2}\cdot\frac{1}{s^2}=\frac{1}{s^2+2^2}\cdot\frac{1}{-2^2}+\frac{1}{0+2^2}\cdot\frac{1}{s^2}\\ &=\frac{1}{4}\left(\frac{1}{s^2}-\frac{1}{2}\frac{2}{s^2+2^2}\right).\end{aligned} \tag{1.14}$$

Thus, according to the table, or memorized formulae, we have that the solution is given by (1.13) and by

$$h(t)=\frac{1}{4}\left(t-\frac{1}{2}\sin(2t)\right). \tag{1.15}$$

Note that for any $t \geq 2\pi$, including $t = 10$, we have

$$
\begin{aligned}
y(t) &= 3\left(h(t)u_0(t) - 2h(t-\pi)u_\pi(t) + h(t-2\pi)u_{2\pi}(t)\right), \\
&= 3\left(h(t) - 2h(t-\pi) + h(t-2\pi)\right) \\
&= \frac{3}{4}\left(t - \frac{1}{2}\sin(2t) - 2\left(t - \pi - \frac{1}{2}\sin\left(2\left(t-\pi\right)\right)\right) + t - 2\pi - \frac{1}{2}\sin\left(2\left(t-2\pi\right)\right)\right) \\
&= \frac{3}{4}\left(t - \frac{1}{2}\sin(2t) - 2\left(t - \pi - \frac{1}{2}\sin\left(2t\right)\right) + t - 2\pi - \frac{1}{2}\sin\left(2t\right)\right) \\
&= 0.
\end{aligned}
$$

4. Find a real-valued representation of the general solution of the following system:

$$
\mathbf{x}' = \begin{bmatrix} 3 & 5 \\ -1 & -1 \end{bmatrix} \mathbf{x} \tag{1.16}
$$

25 points

<u>Solution</u>

The general solution of (1.16) can be expressed as

$$
\mathbf{x} = \mathbf{x}(t) = c_1 \boldsymbol{\xi}_1 e^{\lambda_1 t} + c_2 \boldsymbol{\xi}_2 e^{\lambda_2 t}\ , \tag{1.17}
$$

where the $\boldsymbol{\xi}$'s and λ's are independent eigenvectors and distinct eigenvalues of the matrix in (1.16):

$$
\begin{bmatrix} 3-\lambda & 5 \\ -1 & -1-\lambda \end{bmatrix} \boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \boldsymbol{\xi} = \mathbf{0}
$$

unless

$$
0 = \det \begin{bmatrix} 3-\lambda & 5 \\ -1 & -1-\lambda \end{bmatrix} = \left(\lambda+1\right)\left(\lambda-3\right) + 5 = \lambda^2 - 2\lambda + 2 = \left(\lambda-1\right)^2 - i^2 \tag{1.18}
$$

$$\Leftrightarrow$$

$$
\lambda = 1 \pm i = 1+i, 1-i =: \lambda_1, \lambda_2.
$$

So

$$\mathbf{0} = \begin{bmatrix} 3-(1+i) & 5 \\ -1 & -1-(1+i) \end{bmatrix} \xi_1 = \begin{bmatrix} 2-i & 5 \\ -1 & -2-i \end{bmatrix} \xi_1 = \begin{bmatrix} 2-i & 5 \\ -2+i & -5 \end{bmatrix} \xi_1 = \begin{bmatrix} 2-i & 5 \\ 0 & 0 \end{bmatrix} \xi_1 \Leftarrow \xi_1 = \begin{bmatrix} 2+i \\ -1 \end{bmatrix},$$
$$\mathbf{0} = \begin{bmatrix} 3-(1-i) & 5 \\ -1 & -1-(1-i) \end{bmatrix} \xi_2 = \begin{bmatrix} 2+i & 5 \\ -1 & -2+i \end{bmatrix} \xi_2 = \begin{bmatrix} 2+i & 5 \\ -2-i & -5 \end{bmatrix} \xi_2 = \begin{bmatrix} 2+i & 5 \\ 0 & 0 \end{bmatrix} \xi_2 \Leftarrow \xi_2 = \overline{\xi_1} = \begin{bmatrix} 2-i \\ -1 \end{bmatrix}. \tag{1.19}$$

Thus, explicitly, (1.17) is

$$\mathbf{x} = \mathbf{x}(t) = c_1 \begin{bmatrix} 2+i \\ -1 \end{bmatrix} e^{(1+i)t} + c_2 \begin{bmatrix} 2-i \\ -1 \end{bmatrix} e^{(1-i)t}. \tag{1.20}$$

As per the usual theory, we can find a real-valued representation by finding the real and imaginary parts of either of the above complex-valued solutions:

$$\mathbf{x}_1(t) = \begin{bmatrix} 2+i \\ -1 \end{bmatrix} e^{(1+i)t} = \begin{bmatrix} 2+i \\ -1 \end{bmatrix} e^t \left(\cos t + i \sin t\right) = \begin{bmatrix} 2\cos t - \sin t \\ -\cos t \end{bmatrix} e^t + i \begin{bmatrix} \cos t + 2\sin t \\ -\sin t \end{bmatrix} e^t, \tag{1.21}$$

whence a real-valued representation of the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2\cos t - \sin t \\ -\cos t \end{bmatrix} e^t + c_2 \begin{bmatrix} \cos t + 2\sin t \\ -\sin t \end{bmatrix} e^t. \tag{1.22}$$

5. Find the fundamental matrix of solutions $\Phi = \Phi(t)$ to the above problem that has the property that $\Phi(0) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

10 points

<u>Solution</u>

A fundamental matrix of solutions $\Psi = \Psi(t)$, one not necessarily having the desired property, can be found from the above general solution (1.22):

$$\Psi(t) = e^t \begin{bmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ -\cos t & -\sin t \end{bmatrix}. \tag{1.23}$$

The desired fundamental matrix $\Phi = \Phi(t)$ can be obtained from $\Psi = \Psi(t)$ via

$$\Phi = \Phi(t) = \Psi(t)\Psi^{-1}(0) = e^t \begin{bmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ -\cos t & -\sin t \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$$
$$= e^t \begin{bmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ -\cos t & -\sin t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = e^t \begin{bmatrix} \cos t + 2\sin t & 5\sin t \\ -\sin t & \cos t - 2\sin t \end{bmatrix}. \tag{1.24}$$

6. Solve the initial value problem given by the system of problem 4 and the initial data

$$\mathbf{x}(0) = \begin{pmatrix} 5 \\ -2 \end{pmatrix}. \tag{1.25}$$

(Hint: rather than "reinventing the wheel", just use the fundamental matrix of problem 5.)

7 points

Solution

Using the fundamental matrix of problem 5 we have

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) = e^t \begin{bmatrix} \cos t + 2\sin t & 5\sin t \\ -\sin t & \cos t - 2\sin t \end{bmatrix} \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$
$$= e^t \begin{bmatrix} 5\cos t + 10\sin t - 10\sin t \\ -5\sin t - 2\cos t + 4\sin t \end{bmatrix} = e^t \begin{bmatrix} 5\cos t \\ -2\cos t - \sin t \end{bmatrix}. \tag{1.26}$$

7. Calculate

$$e^{\begin{bmatrix} 3 & 5 \\ -1 & -1 \end{bmatrix}\pi}. \tag{1.27}$$

(Hint: use the result of problem 5).

5 points

Solution

We have, from problem 5,

$$e^{\begin{bmatrix} 3 & 5 \\ -1 & -1 \end{bmatrix}\pi} = \Phi(\pi) = e^{\pi}\begin{bmatrix} \cos\pi + 2\sin\pi & 5\sin\pi \\ -\sin\pi & \cos\pi - 2\sin\pi \end{bmatrix} = e^{\pi}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -e^{\pi} & 0 \\ 0 & -e^{\pi} \end{bmatrix} \quad (1.28)$$
$$= -e^{\pi}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

8. Find a representation of the general solution of the system

$$\mathbf{x}' = \begin{bmatrix} 6 & 9 \\ -1 & 0 \end{bmatrix}\mathbf{x}\,. \quad (1.29)$$

20 points

Solution

The matrix in (1.29) has a repeated eigenvalue with only one eigenvector. Hence the general solution is of the form

$$\mathbf{x} = \mathbf{x}(t) = c_1\boldsymbol{\xi}e^{\lambda t} + c_2\left(\boldsymbol{\xi}t + \boldsymbol{\eta}\right)e^{\lambda t} \quad (1.30)$$

where $\boldsymbol{\xi}$ is an eigenvector and $\boldsymbol{\eta}$ is an associated pseudo eigenvector:

$$\begin{bmatrix} 6-\lambda & 9 \\ -1 & 0-\lambda \end{bmatrix}\boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \boldsymbol{\xi} = \mathbf{0}$$

unless

$$0 = \det\begin{bmatrix} 6-\lambda & 9 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 9 = \left(\lambda - 3\right)^2 \quad (1.31)$$
$$\Leftrightarrow$$
$$\lambda = 3, 3,$$

So

$$\mathbf{0} = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}\boldsymbol{\xi} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\boldsymbol{\xi} \Leftarrow \boldsymbol{\xi} = \begin{bmatrix} 3 \\ -1 \end{bmatrix},$$

and (1.32)

$$\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}\boldsymbol{\eta} = \boldsymbol{\xi} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\boldsymbol{\eta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Leftrightarrow \boldsymbol{\eta} = \begin{bmatrix} 1-3\eta_2 \\ \eta_2 \end{bmatrix}.$$

Thus, explicitly, (1.30) is

$$\mathbf{x} = \mathbf{x}(t) = c_1\begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{3t} + c_2\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1-3\eta_2 \\ \eta_2 \end{bmatrix}\right) e^{3t}. \tag{1.33}$$

9. Find the fundamental matrix of solutions $\Phi = \Phi(t)$ for the system of problem 8 that satisfies $\Phi(0) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

15 points

Solution

From (1.33) we have a fundamental matrix of solutions

$$\Psi(t) = e^{3t}\begin{bmatrix} 3 & 3t+1-3\eta_2 \\ -1 & -t+\eta_2 \end{bmatrix}, \tag{1.34}$$

whence the one desired is

$$\begin{aligned} \Phi(t) = \Psi(t)\Psi^{-1}(0) &= e^{3t}\begin{bmatrix} 3 & 3t+1-3\eta_2 \\ -1 & -t+\eta_2 \end{bmatrix}\begin{bmatrix} 3 & 1-3\eta_2 \\ -1 & \eta_2 \end{bmatrix}^{-1} \\ &= e^{3t}\begin{bmatrix} 3 & 3t+1-3\eta_2 \\ -1 & -t+\eta_2 \end{bmatrix}\begin{bmatrix} \eta_2 & -1+3\eta_2 \\ 1 & 3 \end{bmatrix} = e^{3t}\begin{bmatrix} 1+3t & 9t \\ -t & 1-3t \end{bmatrix}. \end{aligned} \tag{1.35}$$

10. Solve the initial value problem obtained from combining the differential equation of problem 8 with the initial data

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{1.36}$$

(Hint: do not "reinvent the wheel", but rather use the result from problem 9.)

6 points

Solution

From problem 9 we have

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) = e^{3t}\begin{bmatrix} 1+3t & 9t \\ -t & 1-3t \end{bmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = e^{3t}\begin{bmatrix} 1+21t \\ 2-7t \end{bmatrix}. \tag{1.37}$$