343 Final Fall 2003

1. a) Find polynomials of zeroth, first, and second degree that are orthogonal with respect to the (real) innerproduct

$$\langle f,g\rangle := \int_0^1 f(t)g(t)t(1-t)dt. \tag{0.1}$$

b) Check that the polynomials f_0, f_1, and f_2 that you generated in a) are, in fact, orthogonal with respect to the (unusual) innerproduct (0.1), i.e. check that $\langle f_0,f_1\rangle = \langle f_0,f_2\rangle = \langle f_1,f_2\rangle = 0$: Yes, you will be graded on this checking (this part b)) as well as part a).

P.S. Make life easy on the grader (me) by "normalizing" as follows: i) make all polynomials f_0, f_1, and f_2 have integer coefficients, ii) make the leading order coefficient positive, and iii) eliminate common integer factors from all coefficients. For example (and in that same order) $2/7-4t/3-6t^2 \overset{i)}{\rightarrow} 6-28t-126t^2 \overset{ii)}{\rightarrow} -6+28t+126t^2 \overset{iii)}{\rightarrow} -3+14t+63t^2$.

Note that for each nonnegative integer n, $\int_0^1 t^n dt = \dfrac{1}{n+1}$.

<u>Solution:</u> Clearly the set of polynomials $\{1,t,t^2\}$ span the same subspace as the polynomials desired and have, in order, the required degrees. Thus we can perform Gram-Schmidt on them (in the usual order) to produce the desired orthogonal set: Label these original (non-orthogonal) polynomials via

$$g_0(t)=1,\ g_1(t)=t,\ g_2(t)=t^2.$$

Then, for all nonzero scalars α and β, the following polynomials have the required degrees and are orthogonal w.r.t. the innerproduct (0.1):

$$f_0 := g_0,\ f_1 := \alpha\left(g_1 - \frac{\langle g_1,f_0\rangle}{\langle f_0,f_0\rangle}f_0\right),\ f_2 := \beta\left(g_2 - \frac{\langle g_2,f_0\rangle}{\langle f_0,f_0\rangle}f_0 - \frac{\langle g_2,f_1\rangle}{\langle f_1,f_1\rangle}f_1\right). \tag{0.2}$$

Evidently we must calculate several innerproducts. The first ones are

$$\langle g_1, f_0\rangle = \langle g_1, g_0\rangle = \int_0^1 g_1(t)g_0(t)t(1-t)dt = \int_0^1 t\cdot 1\cdot t(1-t)dt$$
$$= \int_0^1 (t^2 - t^3)dt = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}, \text{ and}$$
$$\langle f_0, f_0\rangle = \langle g_0, g_0\rangle = \int_0^1 g_0(t)g_0(t)t(1-t)dt = \int_0^1 1\cdot 1\cdot t(1-t)dt$$
$$= \int_0^1 (t - t^2)dt = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Since $\langle g_1, f_0\rangle / \langle f_0, f_0\rangle = 1/2$, we fulfill the "normalization" criteria when we pick α to be 2:

$$f_1(t) := 2\left(g_1(t) - \frac{\langle g_1, f_0\rangle}{\langle f_0, f_0\rangle} g_0(t) \right) = 2\left(g_1(t) - \frac{1}{2} g_0(t) \right) = 2g_1(t) - g_0(t) = 2t - 1. \ (0.3)$$

The second set of inner products are then

$$\langle g_2, f_0\rangle = \langle g_2, g_0\rangle = \int_0^1 g_2(t)g_0(t)t(1-t)dt = \int_0^1 t^2\cdot 1\cdot t(1-t)dt$$
$$= \int_0^1 (t^3 - t^4)dt = \frac{1}{4} - \frac{1}{5} = \frac{1}{20},$$
$$\langle g_2, f_1\rangle = \langle g_2, 2g_1 - g_0\rangle = 2\langle g_2, g_1\rangle - \langle g_2, g_0\rangle = 2\int_0^1 g_2(t)g_1(t)t(1-t)dt - \frac{1}{20}$$
$$= 2\int_0^1 t^2\cdot t\cdot t(1-t)dt - \frac{1}{20} = 2\int_0^1 (t^4 - t^5)dt - \frac{1}{20} = 2\left(\frac{1}{5} - \frac{1}{6}\right) - \frac{1}{20}$$
$$= 2\frac{1}{30} - \frac{1}{20} = \frac{1}{60}, \text{ and}$$
$$\langle f_1, f_1\rangle = \langle 2g_1 - g_0, 2g_1 - g_0\rangle = 4\langle g_1, g_1\rangle - 4\langle g_1, g_0\rangle + \langle g_0, g_0\rangle = 4\langle g_1, g_1\rangle - 4\frac{1}{12} + \frac{1}{6}$$
$$= 4\langle g_1 t, g_1/t\rangle - \frac{1}{3} + \frac{1}{6} = 4\langle g_2, g_0\rangle - \frac{1}{3} + \frac{1}{6} = 4\frac{1}{20} - \frac{1}{6} = \frac{1}{5} - \frac{1}{6} = \frac{1}{30}.$$

Thus the last polynomial is

$$f_2 := \beta\left(g_2 - \frac{\langle g_2, f_0\rangle}{\langle f_0, f_0\rangle} f_0 - \frac{\langle g_2, f_1\rangle}{\langle f_1, f_1\rangle} f_1 \right) = \beta\left(g_2 - \frac{\frac{1}{20}}{\frac{1}{6}} g_0 - \frac{\frac{1}{60}}{\frac{1}{30}}(2g_1 - g_0) \right)$$
$$= \beta\left(g_2 - \frac{3}{10} g_0 - \frac{1}{2}(2g_1 - g_0) \right) = \beta\left(g_2 - \frac{3}{10} g_0 - g_1 + \frac{1}{2} g_0 \right)$$
$$= \beta\left(g_2 - g_1 + \frac{1}{5} g_0 \right) = \frac{\beta}{5}(5g_2 - 5g_1 + g_0),$$

and we fulfill the "normalization" criteria by picking $\beta = 5$: in summary

$$f_0(t) = g_0(t) = 1,$$
$$f_1(t) = 2g_1(t) - g_0(t) = 2t - 1, \text{ and}$$
$$f_2(t) = 5g_2(t) - 5g_1(t) + g_0(t) = 5t^2 - 5t + 1.$$

Check:

$$\langle f_0, f_1 \rangle = \int_0^1 f_0(t) f_1(t) t(1-t)dt = \int_0^1 1(2t-1)t(1-t)dt = \int_0^1 (2t-1)(t-t^2)dt$$
$$= \int_0^1 (2t^2 - t - 2t^3 + t^2)dt = \int_0^1 (-2t^3 + 3t^2 - t)dt = -2\frac{1}{4} + 3\frac{1}{3} - \frac{1}{2} = -\frac{1}{2} + 1 - \frac{1}{2} = 0.$$
$$\langle f_0, f_2 \rangle = \int_0^1 f_0(t) f_2(t) t(1-t)dt = \int_0^1 1(5t^2 - 5t + 1)t(1-t)dt = \int_0^1 (5t^2 - 5t + 1)(t - t^2)dt$$
$$= \int_0^1 (5t^3 - 5t^2 + t - 5t^4 + 5t^3 - t^2)dt = \int_0^1 (-5t^4 + 10t^3 - 6t^2 + t)dt = -5\frac{1}{5} + 10\frac{1}{4} - 6\frac{1}{3} + \frac{1}{2}$$
$$= -1 + \frac{5}{2} - 2 + \frac{1}{2} = \frac{6}{2} - 3 = 0, \text{ and}$$
$$\langle f_1, f_2 \rangle = \int_0^1 f_1(t) f_2(t) t(1-t)dt = \int_0^1 (2t-1)(5t^2 - 5t + 1)t(1-t)dt$$
$$= \int_0^1 (10t^3 - 10t^2 + 2t - 5t^2 + 5t - 1)(t - t^2)dt = \int_0^1 (10t^3 - 15t^2 + 7t - 1)(t - t^2)dt$$
$$= \int_0^1 (10t^4 - 15t^3 + 7t^2 - t - 10t^5 + 15t^4 - 7t^3 + t^2)dt = \int_0^1 (-10t^5 + 25t^4 - 22t^3 + 8t^2 - t)dt$$
$$= -10\frac{1}{6} + 25\frac{1}{5} - 22\frac{1}{4} + 8\frac{1}{3} - \frac{1}{2} = -\frac{5}{3} + 5 - \frac{11}{2} + \frac{8}{3} - \frac{1}{2} = 1 + 5 - 6 = 0.$$

2. a) Find (real) polynomials of zeroth, first, and second degree that minimize their distance to $F(t) = t^2$ on the interval $[0,1]$. Define the (square of the) distance between two (real-valued) functions via

$$d^2(f,g) := \|f - g\|^2 := \langle f - g, f - g \rangle := \int_0^1 [f(t) - g(t)]^2 \, t(1-t)dt, \qquad (0.4)$$

i.e. via the "natural" notion of distance dictated by (unusual) innerproduct (0.1). In order to make your life much easier, I will tell you the results of the following integrations (some of which may be irrelevant):

$$\int_0^1 t\cdot 1\cdot t(1-t)dt=\frac{1}{12}=\frac{1}{2^2\cdot 3},\ \int_0^1 t^2\cdot 1\cdot t(1-t)dt=\frac{1}{20}=\frac{1}{2^2\cdot 5},$$

$$\int_0^1 t\cdot t^2\cdot t(1-t)dt=\frac{1}{30}=\frac{1}{2\cdot 3\cdot 5},\ \int_0^1 t^2\cdot t^2\cdot t(1-t)dt=\frac{1}{42}=\frac{1}{2\cdot 3\cdot 7},$$

$$\int_0^1 1\cdot 1\cdot t(1-t)dt=\frac{1}{6}=\frac{1}{2\cdot 3},\ \int_0^1 t^2(2t-1)t(1-t)dt=\frac{1}{60}=\frac{1}{2^2\cdot 3\cdot 5},\tag{0.5}$$

$$\int_0^1 (2t-1)^2 t(1-t)dt=\frac{1}{2\cdot 3\cdot 5},\ \int_0^1 t^2(5t^2-5t+1)t(1-t)dt=\frac{1}{420}=\frac{1}{2^2\cdot 3\cdot 5\cdot 7},$$

$$\int_0^1 (5t^2-5t+1)^2 t(1-t)dt=\frac{1}{84}=\frac{1}{2^2\cdot 3\cdot 7}.$$

This table will also be useful to you in part b):

b) Check that the polynomials h_0, h_1, and h_2 that you generated in a) are in fact correct by solving the following ("$A^T A\mathbf{x}=A^T\mathbf{b}$-like") systems of equations: write

$$h_0(t)=\alpha_0,\quad h_1(t)=\alpha_1+\beta_1 t,\ \text{and}\quad h_2(t)=\alpha_2+\beta_2 t+\gamma_2 t^2,$$

and then solve the following three systems of equations for the 6 coefficients α_0, α_1, β_1, α_2, β_2, γ_2:

System I: $\langle 1,\alpha_0\rangle=\langle 1,t^2\rangle$

System II:
$$\langle 1,\alpha_1+\beta_1 t\rangle=\langle 1,t^2\rangle$$
$$\langle t,\alpha_1+\beta_1 t\rangle=\langle t,t^2\rangle$$

System III:
$$\langle 1,\alpha_2+\beta_2 t+\gamma_2 t^2\rangle=\langle 1,t^2\rangle$$
$$\langle t,\alpha_2+\beta_2 t+\gamma_2 t^2\rangle=\langle t,t^2\rangle\ .$$
$$\langle t^2,\alpha_2+\beta_2 t+\gamma_2 t^2\rangle=\langle t^2,t^2\rangle$$

As in problem 1) you will be graded on your checking (this part b)) as well as part a). Obviously your answers to a) and b) should match exactly.

Solution: From the results of problem 1) (and a "huge" theorem about orthogonal projections—the generalized "Fourier" theorem) we have that the distances are minimized when we choose

$$h_0 = \frac{\langle F, f_0 \rangle}{\langle f_0, f_0 \rangle} f_0 = \frac{\int_0^1 F(t) f_0(t) t(1-t)dt}{\int_0^1 f_0(t) f_0(t) t(1-t)dt} f_0 = \frac{\int_0^1 t^2 \cdot 1 \cdot t(1-t)dt}{\int_0^1 1 \cdot 1 \cdot t(1-t)dt} f_0 = \frac{\frac{1}{2^2 \cdot 5}}{\frac{1}{2 \cdot 3}} f_0 = \frac{2 \cdot 3}{2^2 \cdot 5} f = \frac{3}{2 \cdot 5} f_0$$

$$= \frac{3}{10} f_0,$$

$$h_1 - h_0 = \frac{\langle F, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1 = \frac{\int_0^1 F(t) f_1(t) t(1-t)dt}{\int_0^1 f_1(t) f_1(t) t(1-t)dt} f_1 = \frac{\int_0^1 t^2 (2t-1) t(1-t)dt}{\int_0^1 (2t-1)^2 t(1-t)dt} = \frac{\frac{1}{2^2 \cdot 3 \cdot 5}}{\frac{1}{2 \cdot 3 \cdot 5}} f_1 = \frac{2 \cdot 3 \cdot 5}{2^2 \cdot 3 \cdot 5} f_1$$

$$= \frac{1}{2} f_1, \text{ and}$$

$$h_2 - h_1 = \frac{\langle F, f_2 \rangle}{\langle f_2, f_2 \rangle} f_2 = \frac{\int_0^1 F(t) f_2(t) t(1-t)dt}{\int_0^1 f_2(t) f_2(t) t(1-t)dt} f_2 = \frac{\int_0^1 t^2 (5t^2 - 5t + 1) t(1-t)dt}{\int_0^1 (5t^2 - 5t + 1)^2 t(1-t)dt} = \frac{\frac{1}{2^2 \cdot 3 \cdot 5 \cdot 7}}{\frac{1}{2^2 \cdot 3 \cdot 7}} f_2$$

$$= \frac{2^2 \cdot 3 \cdot 7}{2^2 \cdot 3 \cdot 5 \cdot 7} f_2 = \frac{1}{5} f_2.$$

Here we used table (0.5). Hence the "Fourier" approximations to F (of "degree" 0, 1, and 2) are

$$h_0 = \frac{3}{10} f_0, \; h_1 = \frac{3}{10} f_0 + \frac{1}{2} f_1, \; h_2 = \frac{3}{10} f_0 + \frac{1}{2} f_1 + \frac{1}{5} f_2.$$

In terms of the original monomials $g_0(t) = 1, \; g_1(t) = t, \; g_2(t) = t^2$ we have

$$h_0 = \frac{3}{10} f_0 = \frac{3}{10} g_0,$$

$$h_1 = \frac{3}{10} g_0 + \frac{1}{2} f_1 = \frac{3}{10} g_0 + \frac{1}{2}(2g_1 - g_0) = g_1 - \frac{1}{5} g_0$$

$$h_2 = g_1 - \frac{1}{5} g_0 + \frac{1}{5} f_2 = g_1 - \frac{1}{5} g_0 + \frac{1}{5}(5g_2 - 5g_1 + g_0) = g_2,$$

the last result as expected.

Check: First, table (0.5) (and some simple manipulations) dictates that

$$\langle 1,1 \rangle = \frac{1}{2 \cdot 3}, \langle 1,t \rangle = \langle t,1 \rangle = \frac{1}{2^2 \cdot 3}, \; \langle 1,t^2 \rangle = \langle t^2,1 \rangle = \frac{1}{2^2 \cdot 5}, \; \langle t,t \rangle = \langle 1,t^2 \rangle = \frac{1}{2^2 \cdot 5},$$

$$\langle t,t^2 \rangle = \langle t^2,t \rangle = \frac{1}{2 \cdot 3 \cdot 5}, \langle t^2,t^2 \rangle = \frac{1}{2 \cdot 3 \cdot 7}.$$

Inserting these into systems I, II, and III one gets

System I: $\langle 1,\alpha_0\rangle=\langle 1,t^2\rangle \Leftrightarrow \alpha_0\langle 1,1\rangle=\langle 1,t^2\rangle \Leftrightarrow \alpha_0=\dfrac{\langle 1,t^2\rangle}{\langle 1,1\rangle}=\dfrac{\frac{1}{2^2\cdot 5}}{\frac{1}{2\cdot 3}}=\dfrac{2\cdot 3}{2^2\cdot 5}=\dfrac{3}{2\cdot 5}=\dfrac{3}{10}.$

System II: $\begin{matrix}\langle 1,\alpha_1+\beta_1 t\rangle=\langle 1,t^2\rangle\\ \langle t,\alpha_1+\beta_1 t\rangle=\langle t,t^2\rangle\end{matrix} \Leftrightarrow \begin{matrix}\langle 1,1\rangle\alpha_1+\langle 1,t\rangle\beta_1=\dfrac{1}{2^2\cdot 5}\\ \langle t,1\rangle\alpha_1+\langle t,t\rangle\beta_1=\dfrac{1}{2\cdot 3\cdot 5}\end{matrix} \Leftrightarrow$

$$\begin{matrix}\dfrac{1}{2\cdot 3}\alpha_1+\dfrac{1}{2^2\cdot 3}\beta_1=\dfrac{1}{2^2\cdot 5}\\ \dfrac{1}{2^2\cdot 3}\alpha_1+\dfrac{1}{2^2\cdot 5}\beta_1=\dfrac{1}{2\cdot 3\cdot 5}\end{matrix} \Leftrightarrow \begin{matrix}\dfrac{2^2\cdot 3\cdot 5}{2\cdot 3}\alpha_1+\dfrac{2^2\cdot 3\cdot 5}{2^2\cdot 3}\beta_1=\dfrac{2^2\cdot 3\cdot 5}{2^2\cdot 5}\\ \dfrac{2^2\cdot 3\cdot 5}{2^2\cdot 3}\alpha_1+\dfrac{2^2\cdot 3\cdot 5}{2^2\cdot 5}\beta_1=\dfrac{2^2\cdot 3\cdot 5}{2\cdot 3\cdot 5}\end{matrix} \Leftrightarrow \begin{matrix}2\cdot 5\alpha_1+5\beta_1=3\\ 5\alpha_1+3\beta_1=2\end{matrix}$$

$$\Leftrightarrow \begin{matrix}(2-2)5\alpha_1+(5-2\cdot 3)\beta_1=3-2\cdot 2\\ 5\alpha_1+3\beta_1=2\end{matrix} \Leftrightarrow \begin{matrix}\beta_1=1\\ 5\alpha_1+3\beta_1=2\end{matrix} \Leftrightarrow \begin{matrix}\beta_1=1\\ \alpha_1=\dfrac{-1}{5}\end{matrix}.$$

System III: $\begin{matrix}\langle 1,\alpha_2+\beta_2 t+\gamma_2 t^2\rangle=\langle 1,t^2\rangle\\ \langle t,\alpha_2+\beta_2 t+\gamma_2 t^2\rangle=\langle t,t^2\rangle\\ \langle t^2,\alpha_2+\beta_2 t+\gamma_2 t^2\rangle=\langle t^2,t^2\rangle\end{matrix} \Leftrightarrow \begin{matrix}\langle 1,1\rangle\alpha_2+\langle 1,t\rangle\beta_2+\langle 1,t^2\rangle\gamma_2=\dfrac{1}{2^2\cdot 5}\\ \langle t,1\rangle\alpha_2+\langle t,t\rangle\beta_2+\langle t,t^2\rangle\gamma_2=\dfrac{1}{2\cdot 3\cdot 5}\\ \langle t^2,1\rangle\alpha_2+\langle t^2,t\rangle\beta_2+\langle t^2,t^2\rangle\gamma_2=\dfrac{1}{2\cdot 3\cdot 7}\end{matrix}$

$$\begin{matrix}\dfrac{1}{2\cdot 3}\alpha_2+\dfrac{1}{2^2\cdot 3}\beta_2+\dfrac{1}{2^2\cdot 5}\gamma_2=\dfrac{1}{2^2\cdot 5}\\ \dfrac{1}{2^2\cdot 3}\alpha_2+\dfrac{1}{2^2\cdot 5}\beta_2+\dfrac{1}{2\cdot 3\cdot 5}\gamma_2=\dfrac{1}{2\cdot 3\cdot 5}\\ \dfrac{1}{2^2\cdot 5}\alpha_2+\dfrac{1}{2\cdot 3\cdot 5}\beta_2+\dfrac{1}{2\cdot 3\cdot 7}\gamma_2=\dfrac{1}{2\cdot 3\cdot 7}\end{matrix} \Leftrightarrow \begin{matrix}\dfrac{2^2\cdot 3\cdot 5}{2\cdot 3}\alpha_2+\dfrac{2^2\cdot 3\cdot 5}{2^2\cdot 3}\beta_2+\dfrac{2^2\cdot 3\cdot 5}{2^2\cdot 5}\gamma_2=\dfrac{2^2\cdot 3\cdot 5}{2^2\cdot 5}\\ \dfrac{2^2\cdot 3\cdot 5}{2^2\cdot 3}\alpha_2+\dfrac{2^2\cdot 3\cdot 5}{2^2\cdot 5}\beta_2+\dfrac{2^2\cdot 3\cdot 5}{2\cdot 3\cdot 5}\gamma_2=\dfrac{2^2\cdot 3\cdot 5}{2\cdot 3\cdot 5}\\ \dfrac{2^2\cdot 3\cdot 5\cdot 7}{2^2\cdot 5}\alpha_2+\dfrac{2^2\cdot 3\cdot 5\cdot 7}{2\cdot 3\cdot 5}\beta_2+\dfrac{2^2\cdot 3\cdot 5\cdot 7}{2\cdot 3\cdot 7}\gamma_2=\dfrac{2^2\cdot 3\cdot 5\cdot 7}{2\cdot 3\cdot 7}\end{matrix}$$

$$\begin{array}{lll} 2\cdot 5\alpha_2+5\beta_2+3\gamma_2=3 & & (2-2)\cdot 5\alpha_2+(5-2\cdot 3)\beta_2+(3-2\cdot 2)\gamma_2=3-2\cdot 2 \\ \Leftrightarrow \quad 5\alpha_2+3\beta_2+2\gamma_2=2 & \Leftrightarrow & 5\alpha_2+3\beta_2+2\gamma_2=2 \\ 3\cdot 7\alpha_2+2\cdot 7\beta_2+2\cdot 5\gamma_2=2\cdot 5 & & (3\cdot 7-5\cdot 5)\alpha_2+(2\cdot 7-5\cdot 3)\beta_2+(2\cdot 5-5\cdot 2)\gamma_2=2\cdot 5-5\cdot 2 \end{array}$$

$$\begin{array}{lllll} -\beta_2-\gamma_2=-1 & -\beta_2-\gamma_2=-1 & -\beta_2-\gamma_2=-1 & & \\ \Leftrightarrow 5\alpha_2+3\beta_2+2\gamma_2=2 \Leftrightarrow & (5-4)\alpha_2+(3-1)\beta_2+2\gamma_2=2 \Leftrightarrow & \alpha_2+2\beta_2+2\gamma_2=2 & & \\ -4\alpha_2-\beta_2=0 & -4\alpha_2-\beta_2=0 & \beta_2=-4\alpha_2 & & \\ 4\alpha_2-\gamma_2=-1 & \gamma_2=4\alpha_2+1 & \gamma_2=4\alpha_2+1 & \gamma_2=4\alpha_2+1 & \gamma_2=1 \\ \Leftrightarrow \alpha_2-8\alpha_2+2\gamma_2=2 \Leftrightarrow & -7\alpha_2+2\gamma_2=2 \Leftrightarrow & -7\alpha_2+8\alpha_2+2=2 \Leftrightarrow & \alpha_2=0 & \Leftrightarrow \alpha_2=0. \\ \beta_2=-4\alpha_2 & \beta_2=-4\alpha_2 & \beta_2=-4\alpha_2 & \beta_2=-4\alpha_2 & \beta_2=0 \end{array}$$

Thus, via this method, our "closest" polynomials are

$$h_0(t)=\alpha_0=\frac{3}{10}, \quad h_1(t)=\frac{-1}{5}+1t=t-\frac{1}{5}, \text{ and}$$
$$h_2(t)=\alpha_2+\beta_2 t+\gamma_2 t^2=0+0t+1t^2=t^2,$$

just as calculated in part a).

3. a) Calculate the adjoint of the matrix

$$S=\begin{bmatrix} 18 & -8 & 6 \\ -8 & 6 & -2 \\ 6 & -2 & 4 \end{bmatrix}.$$

b) Calculate the determinant of S, by (an appropriate use of) row reduction. c) Using a) and b) calculate S^{-1}. d) Check your answer to c) by calculating the matrix product SS^{-1}.

Solution: a) Since S is symmetric, its adjoint will be also. Thus we need only calculate

$$(\operatorname{adj}S)_{11}=(-1)^{1+1}S_{11}=+\det\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}=4\det\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}=4(6-1)=20$$

$$(\operatorname{adj}S)_{12}=(-1)^{1+2}S_{21}=-\det\begin{bmatrix} -8 & 6 \\ -2 & 4 \end{bmatrix}=-4\det\begin{bmatrix} 4 & -3 \\ 1 & -2 \end{bmatrix}=-4(-8+3)=20$$

$$(\operatorname{adj}S)_{13}=(-1)^{1+3}S_{31}=+\det\begin{bmatrix} -8 & 6 \\ 6 & -2 \end{bmatrix}=4\det\begin{bmatrix} 4 & -3 \\ -3 & 1 \end{bmatrix}=4(4-9)=-20$$

$$(\operatorname{adj}S)_{22}=(-1)^{2+2}S_{22}=+\det\begin{bmatrix} 18 & 6 \\ 6 & 4 \end{bmatrix}=12\det\begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}=12(6-3)=36$$

$$(\operatorname{adj}S)_{23}=(-1)^{2+3}S_{32}=-\det\begin{bmatrix} 18 & 6 \\ -8 & -2 \end{bmatrix}=12\det\begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}=12(3-4)=-12$$

$$(\operatorname{adj}S)_{33}=(-1)^{3+3}S_{32}=+\det\begin{bmatrix} 18 & -8 \\ -8 & 6 \end{bmatrix}=4\det\begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}=4(27-16)=44.$$

Thus the adjoint is

$$\operatorname{adj}S = \begin{bmatrix} 20 & 20 & -20 \\ 20 & 36 & -12 \\ -20 & -12 & 44 \end{bmatrix} = 4\begin{bmatrix} 5 & 5 & -5 \\ 5 & 9 & -3 \\ -5 & -3 & 11 \end{bmatrix}.$$

b) We'll calculate the determinant by row reduction:

$$\det S = \det\begin{bmatrix} 18 & -8 & 6 \\ -8 & 6 & -2 \\ 6 & -2 & 4 \end{bmatrix} = 8\det\begin{bmatrix} 9 & -4 & 3 \\ -4 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix} = -8\det\begin{bmatrix} 9 & -4 & 3 \\ 4 & -3 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

$$= -8\det\begin{bmatrix} 0 & -1 & -3 \\ 1 & -2 & -1 \\ 3 & -1 & 2 \end{bmatrix} = 8\det\begin{bmatrix} 0 & 1 & 3 \\ 1 & -2 & -1 \\ 0 & 5 & 5 \end{bmatrix} = 40\det\begin{bmatrix} 0 & 1 & 3 \\ 1 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= 40\det\begin{bmatrix} 0 & 1 & 3 \\ 1 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix} = 40\det\begin{bmatrix} 0 & 0 & 2 \\ 1 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix} = 80\det\begin{bmatrix} 0 & 0 & 1 \\ 1 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= -80\det\begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 80\det\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 80.$$

c) By theorem we have (since $\det S \neq 0$)

$$S^{-1} = \frac{\operatorname{adj}S}{\det S} = \frac{1}{80}4\begin{bmatrix} 5 & 5 & -5 \\ 5 & 9 & -3 \\ -5 & -3 & 11 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 5 & 5 & -5 \\ 5 & 9 & -3 \\ -5 & -3 & 11 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 & -1/4 \\ 1/4 & 9/20 & -3/20 \\ -1/4 & -3/20 & 11/20 \end{bmatrix}.$$

d) We haven't made a mistake:

$$S^{-1}S = \frac{1}{20}\begin{bmatrix} 5 & 5 & -5 \\ 5 & 9 & -3 \\ -5 & -3 & 11 \end{bmatrix}\begin{bmatrix} 18 & -8 & 6 \\ -8 & 6 & -2 \\ 6 & -2 & 4 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 5(18-8-6) & 5(-8+6+2) & 5(6-2-4) \\ 5\cdot 18-9\cdot 8-3\cdot 6 & -5\cdot 8+9\cdot 6+3\cdot 2 & 5\cdot 6-9\cdot 2-3\cdot 4 \\ -5\cdot 18+3\cdot 8+11\cdot 6 & 5\cdot 8-3\cdot 6-11\cdot 2 & -5\cdot 6+3\cdot 2+11\cdot 4 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

as it should be.

4. Find a parabola $y = ax^2 + bx + c$ that comes as close as possible to describing the data $\{(x,y)\} = \{(-2,3),(-1,1),(0,0),(1,3)\}$. As it is the quickest way to proceed, you may use the " $A^T A\mathbf{x} = A^T\mathbf{b}$ " method, which I have justified previously in the course.

Solution: It would be wonderful if the system

$$\begin{matrix} ax^2+bx+c=y\big|_{(x,y)\to(-2,3)} \\ ax^2+bx+c=y\big|_{(x,y)\to(-1,1)} \\ ax^2+bx+c=y\big|_{(x,y)\to(0,0)} \\ ax^2+bx+c=y\big|_{(x,y)\to(1,3)} \end{matrix} \Leftrightarrow \begin{matrix} 4a-2b+1c=3 \\ 1a-1b+1c=1 \\ 0a+0b+1c=0 \\ 1a+1b+1c=3 \end{matrix}$$

actually had a solution $(a,b,c) \in \mathbb{R}^3$, but this in unlikely since there are four (not three) data points. In fact we have

$$\begin{matrix} 4a-2b+1c=3 \\ 1a-1b+1c=1 \\ 0a+0b+1c=0 \\ 1a+1b+1c=3 \end{matrix} \Leftrightarrow \begin{matrix} 4a-2b=3 \\ a-b=1 \\ c=0 \\ a+b=3 \end{matrix} \Leftrightarrow \begin{matrix} 4a-2b=3 \\ (a-b)+(a+b)=1+3 \\ c=0 \\ (a-b)-(a+b)=1-3 \end{matrix} \Leftrightarrow \begin{matrix} 4a-2b=3 \\ 2a=4 \\ c=0 \\ -2b=-2 \end{matrix} \Leftrightarrow \begin{matrix} 8-2=3 \\ a=2 \\ c=0 \\ b=1 \end{matrix}$$

$$\Leftrightarrow \begin{matrix} 6=3 \\ a=2 \\ c=0 \\ b=1 \end{matrix},$$

the (first of the) latter being nonsense. So we write our system as

$$\begin{matrix} 4a-2b+1c=3 \\ 1a-1b+1c=1 \\ 0a+0b+1c=0 \\ 1a+1b+1c=3 \end{matrix} \leftrightarrow \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 3 \end{bmatrix} \leftrightarrow A\mathbf{z} = \mathbf{r}.$$

Again, this has no solution. But we know we minimize the (Euclidean) distance between $A\mathbf{z}$ and $\mathbf{r}$ at a $\mathbf{z}_0$ satisfying $A^T A\mathbf{z}_0 = A^T\mathbf{r}$. Evidently we must first make the following calculations

$$A^TA=\begin{bmatrix}4&1&0&1\\-2&-1&0&1\\1&1&1&1\end{bmatrix}\begin{bmatrix}4&-2&1\\1&-1&1\\0&0&1\\1&1&1\end{bmatrix}=\begin{bmatrix}18&-8&6\\-8&6&-2\\6&-2&4\end{bmatrix}$$

$$A^T\mathbf{r}=\begin{bmatrix}4&1&0&1\\-2&-1&0&1\\1&1&1&1\end{bmatrix}\begin{bmatrix}3\\2\\0\\3\end{bmatrix}=\begin{bmatrix}17\\-5\\8\end{bmatrix}$$

Thus we want to solve

$$A^TA\begin{bmatrix}a\\b\\c\end{bmatrix}=A^T\mathbf{r}=\begin{bmatrix}17\\-5\\8\end{bmatrix}\Leftrightarrow\begin{bmatrix}a\\b\\c\end{bmatrix}=\left(A^TA\right)^{-1}A^T\mathbf{r}=S^{-1}A^T\mathbf{r}$$

$$=\frac{1}{20}\begin{bmatrix}5&5&-5\\5&9&-3\\-5&-3&11\end{bmatrix}\begin{bmatrix}17\\-5\\8\end{bmatrix}=\frac{1}{20}\begin{bmatrix}5(17-5-8)\\5\cdot17-9\cdot5-3\cdot8\\-5\cdot17+3\cdot5+11\cdot8\end{bmatrix}$$

$$=\frac{1}{20}\begin{bmatrix}20\\16\\18\end{bmatrix}=\begin{bmatrix}1\\4/5\\9/10\end{bmatrix}.$$

where we used the inverse calculated in problem 3). Thus the best parabolic fit to the data points $\{(x,y)\}=\{(-2,3),(-1,1),(0,0),(1,3)\}$ is $y=x^2+\frac{4}{5}x+\frac{9}{10}=\frac{10x^2+8x+9}{10}$. Note that this curve goes through the points $\{(-2,3.1),(-1,1.1),(0,.9),(1,2.7)\}$: not bad.

5. a) Consider the quadratic form

$$Q(x,y,z)=1x^2+2y^2+3z^2. \tag{0.6}$$

Insert the change of variable

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{6}}&-\frac{2}{\sqrt{6}}&\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{2}}&0&-\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}$$

into (0.6) to develop a new quadratic form $\tilde{Q}$ in the new variables (u,v,w). Make sure and expand the form out, i.e. write the new form as

$$\tilde{Q}(u,v,w)=au^2+bv^2+cw^2+duv+euw+fvw$$

for some new (but very specific) constants a, b, c, d, e, and f. (That is, what are a, b, c, d, e, and f?)

b) Write $\tilde{Q}(u,v,w)$ as

$$\tilde{Q}(u,v,w)=\begin{bmatrix} u & v & w \end{bmatrix} A \begin{bmatrix} u \\ v \\ w \end{bmatrix} =: \mathbf{f}^T A \mathbf{f}$$

for some symmetric matrix A. (That is, what is this symmetric matrix? It will be easier to proceed with the last part of this problem, part d), if you write $A=\frac{1}{6}B$: the "large" denominator 6 appears in the calculations).

c) Calculate the (factored version of the) characteristic polynomial of the symmetric matrix A. I'm expecting you to have some insight here, i.e. to notice a connection between the various parts of this problem: if you calculate the characteristic polynomial via the definition, you probably won't finish this exam.

d) Use the results of c) to calculate eigenvectors of A. Where have you seen these before? Note it is possible to avoid fractions in this calculation, despite the "fractional" form of A.

Solution: a) Evidently we get

$$\tilde{Q}(u,v,w)=1x^2+2y^2+3z^2=1\left(\frac{1u+1v+1w}{\sqrt{3}}\right)^2+2\left(\frac{1u-2v+1w}{\sqrt{6}}\right)^2+3\left(\frac{1u+0v-1w}{\sqrt{2}}\right)^2$$

$$=\frac{1}{3}(u+v+w)^2+\frac{1}{3}(u-2v+w)^2+\frac{3}{2}(u-w)^2$$

$$=\frac{1}{6}\left[2(u+v+w)^2+2(u-2v+w)^2+9(u-w)^2\right]$$

$$=\frac{1}{6}\Big[2\left[(u+v)^2+2(u+v)w+w^2\right]+2\left[(u-2v)^2+2(u-2v)w+w^2\right]+$$
$$9\left(u^2-2uw+w^2\right)\Big]$$

$$=\frac{1}{6}\Big[2\left(u^2+2uv+v^2+2uw+2vw+w^2\right)+2\left(u^2-4uv+4v^2+2uw-4vw+w^2\right)+$$
$$9u^2-18uw+9w^2\Big]$$

$$=\frac{1}{6}\Big[2u^2+4uv+2v^2+4uw+4vw+2w^2+2u^2-8uv+8v^2+4uw-8vw+$$
$$2w^2+9u^2-18uw+9w^2\Big]$$

$$=\frac{1}{6}\left(13u^2+10v^2+13w^2-4uv-10uw-4vw\right).$$

b) From a), evidently the matrix is

$$A=\frac{1}{6}\begin{bmatrix}13 & -2 & -5\\ -2 & 10 & -2\\ -5 & -2 & 13\end{bmatrix}.$$

c) The characteristic polynomial of a matrix A is $P(\lambda):=\det[\lambda I-A]$. Note that for a 3X3 matrix, this gives a polynomial of degree 3, with leading order term λ^3. Also note that the zeroes of $P(\lambda)$ are the eigenvalues A, and that such appear in obvious places in the diagonalized form of $\tilde{Q}$, the latter being the original Q written in (0.6)! Hence these eigenvalues are 1, 2, and 3, so that $P(\lambda)$ must be

$$P(\lambda)=(\lambda-1)(\lambda-2)(\lambda-3).$$

d) We have

$$E_A(1)=\text{Nul}[A-1I]=\text{Nul}[6A-6I]=\text{Nul}\begin{bmatrix}13-6 & -2 & -5\\ -2 & 10-6 & -2\\ -5 & -2 & 13-6\end{bmatrix}$$

$$=\text{Nul}\begin{bmatrix}7 & -2 & -5\\ -2 & 4 & -2\\ -5 & -2 & 7\end{bmatrix}=\text{Nul}\begin{bmatrix}7 & -2 & -5\\ 1 & -2 & 1\\ -5 & -2 & 7\end{bmatrix}=\text{Nul}\begin{bmatrix}7-7(1) & -2-7(-2) & -5-7(1)\\ 1 & -2 & 1\\ -5+5(1) & -2+5(-2) & 7+5(1)\end{bmatrix}$$

$$=\text{Nul}\begin{bmatrix}0 & 12 & -12\\ 1 & -2 & 1\\ 0 & -12 & 12\end{bmatrix}=\text{Nul}\begin{bmatrix}0 & 1 & -1\\ 1 & -2 & 1\\ 0 & 0 & 0\end{bmatrix}=\text{Nul}\begin{bmatrix}1 & -2 & 1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}=\text{Nul}\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$$

$$=Span\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}=Span\left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}\right\},$$

$$E_A(2)=\text{Nul}[A-2I]=\text{Nul}[6A-12I]=\text{Nul}\begin{bmatrix}13-12 & -2 & -5\\ -2 & 10-12 & -2\\ -5 & -2 & 13-12\end{bmatrix}$$

$$=\text{Nul}\begin{bmatrix}1 & -2 & -5\\ -2 & -2 & -2\\ -5 & -2 & 1\end{bmatrix}=\text{Nul}\begin{bmatrix}1 & -2 & -5\\ -2+2(1) & -2+2(-2) & -2+2(-5)\\ -5+5(1) & -2+5(-2) & 1+5(-5)\end{bmatrix}$$

$$=\text{Nul}\begin{bmatrix}1 & -2 & -5\\ 0 & -6 & -12\\ 0 & -12 & -24\end{bmatrix}=\text{Nul}\begin{bmatrix}1 & -2 & -5\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix}=\text{Nul}\begin{bmatrix}1 & -2+2(1) & -5+2(2)\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix}$$

$$=\text{Nul}\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix}=Span\left\{\begin{bmatrix}1\\ -2\\ 1\end{bmatrix}\right\}=Span\left\{\frac{1}{\sqrt{6}}\begin{bmatrix}1\\ -2\\ 1\end{bmatrix}\right\},\text{ and}$$

$$E_A(3)=\text{Nul}[A-3I]=\text{Nul}[6A-18I]=\text{Nul}\begin{bmatrix}13-18 & -2 & -5\\ -2 & 10-18 & -2\\ -5 & -2 & 13-18\end{bmatrix}$$

$$=\text{Nul}\begin{bmatrix}-5 & -2 & -5\\ -2 & -8 & -2\\ -5 & -2 & -5\end{bmatrix}=\text{Nul}\begin{bmatrix}5 & 2 & 5\\ 1 & 4 & 1\\ 0 & 0 & 0\end{bmatrix}=\text{Nul}\begin{bmatrix}5-5(1) & 2-5(4) & 5-5(1)\\ 1 & 4 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$$=\text{Nul}\begin{bmatrix}0 & -18 & 0\\ 1 & 4 & 1\\ 0 & 0 & 0\end{bmatrix}=\text{Nul}\begin{bmatrix}0 & 1 & 0\\ 1 & 4 & 1\\ 0 & 0 & 0\end{bmatrix}=\text{Nul}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}=Span\left\{\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}\right\}$$

$$=Span\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}\right\}.$$

Of course we've seen these in the (orthogonal!) transformation generating the nondiagonal quadratic form $\tilde{Q}(u,v,w)$ from the diagonal one $Q(x,y,z)$.