Problem:
Let **u** = (u1, u2) and **v** = (v1, v2) be vectors in R^2. Verify that the weighted Euclidean inner product <**u, v** > = 4u1v1 + 3u2v2 is an inner product by showing that it satisfies the inner product axioms.

Solution:
If **u** and **v** are interchanged, the right side remains the same. < **u, v** > = <**v, u**> This satisfies the first axiom.

For the second axiom use the vector **x** = (x1, x2) and place it into the second axiom.

<**u** + **v**, **x**> = 4(u1+v1) x1 + 3(u2+v2)x2 = 4u1x1 + 4v1x1 + 3u2x2 + 3v2x2
combine into the proper order = (4u1x1 + 3u2x2) + (4v1x1 + 3v2x2)
= < **u, z** > + < **v,z** >

For the 3rd axiom.
<*k***u,v**> = 4(*k*u1)v1 + 3(*k*u2)v2 = *k*4u1v1 + *k*3u2v2 = *k*(4u1v1 + 3u2v2) = *k*<**u,v**>

And the 4th axiom.

<**v,v**> = 4v1v1 + 3v2v2 = 4v1^2 + 3v2^2

It is easy to see <**v,v**> = 4v1^2 + 3v2^2 > = 0. Also <**v,v**> = 4v1^2 + 3v2^2 = 0 if and only if v1 = v2 = 0.

Problem: Good problem
Find the orthogonal projection of **u** onto the subspace of R^4 spanned by the vectors **v**1, **v**2,**v**3.

u = (2,2,1,5); **v**1 = (2,1,1,1); **v**2 = (1,0,1,1); **v**3 = (-2,-1,0,-1)

Solution:
The subspace spanned by the vectors is the column space of

$$\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

So we can find the orthogonal projection by finding the least squares solution and calculating the projection from that.

$$A^T * A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}$$

A^T * **u** = [2 1 1 1] [2] [12]
[1 0 1 1] * [2] = [8]
[-2 -1 0 1] [1] [-11]
[5]

The normal system A^T*A**x** = A^T***u** is now
[7 4 -6] [x1] [12]
[4 3 -3] * [x2] = [8]
[-6 -3 6] [x3] [-11]

Solving the system
[7 4 6 : 12] [1 1 0 : 1] [1 1 0 : 1]
[4 3 -3: 8] ~ [0 -1 -3: 4] ~ [0 -1 -3: 4]
[-6 -3 6 : -11][0 3 6 :-5] [0 0 -3: 7]

[1 1 0: 1] [1 0 0: -2]
[0 -1 0: -3] ~ [0 1 0: 3]
0 0 -3: 7] [0 0 1: -7/3]

so now Projection of **u** on W = A**x** = [2 1 -2] [-2] [11/3]
[1 0 -1] * [3] = [1/3]
[1 1 0] [-7/3] [1]
[1 1 -1] [10/3]

Problem:
Find the characteristic equation, eigenvalues, and eigenspaces of the following matrix.

[10 -9]
[4 -2]

Solution:
I will use L for lambda
To find the characteristic equation: det(LI- A)= det [L -10 9]
[-4 L+2]

= (L-10)(L +2) + 36 = L^2 -8L-20+36 = L^2 -8L+16=0
the characteristic equation is L^2 -8L+16=0

The eigenvalues are simply the roots of this quadratic equation. They are both 4.

To solve for the eigenspaces, you replace lambas with the eigenvalues in the matrix that you took the determinant of. and then row reduce.
[4-10 9] = [-6 9] ~ [-2 3] ~ [0 0]
[-4 4+2] [-4 6] [2 -3] [1 -3/2]

$x_1 = 3/2 x_2 \quad x_2 = x_2 \quad x = \begin{bmatrix} 3/2 x_2 \\ x_2 \end{bmatrix} = x_2 * \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$ the basis vector is $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$

this vector is the basis for the eigenspace corresponding to lamba = 4

Problem:
Find a basis for the orthogonal complement of the subspace of R^4 spanned by the vectors $\mathbf{v}_1 = (1,2,4,-3)$, $\mathbf{v}_2 = (-5,-7,-8,6)$, $\mathbf{v}_3 = (2,0,0,2)$.

write $(\text{Row}A)^\perp = \text{Nul}A$ so we know why you're doing this.

Solution:
Let $A = \begin{bmatrix} 1 & 2 & 4 & -3 \\ -5 & -7 & -8 & 6 \\ 2 & 0 & 0 & 2 \end{bmatrix}$ and then find the basis.

$$\text{Nul}\begin{bmatrix} 1 & 2 & 4 & -3 \\ -5 & -7 & -8 & 6 \\ 2 & 0 & 0 & 2 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 2 & 4 & -3 \\ 0 & 3 & 12 & -9 \\ 0 & -4 & -8 & 8 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 2 & 4 & -3 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 8 & -4 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1/2 \end{bmatrix} =$$

— Can you find equation editor?

$$\text{Nul}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1/2 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 1/2 \end{bmatrix}\right\}$$

since the orthogonal complement of the row space of A is the nullspace of A.

$[\,1\,] \in \mathbb{R}^4$ not $\mathbb{R}^3$.

O.K. but late

Problem:
Find the transition matrix from $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ to $B' = \{\mathbf{u}_1', \mathbf{u}_2'\}$, where

$\mathbf{u}_1 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{u}_1' = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $\mathbf{u}_2' = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

Solution:
$v =$ "$[\mathbf{u}_1|\mathbf{u}_2](\mathbf{v})_B = [\mathbf{u}_1'|\mathbf{u}_2'](\mathbf{v})_{B'}$"

$$\Rightarrow P_{B'B} = [\mathbf{u}_1'|\mathbf{u}_2']^{-1}[\mathbf{u}_1|\mathbf{u}_2] = \begin{bmatrix} 0 & 5 \\ 3 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 4 & 2 \\ 6 & -1 \end{bmatrix} = (-1/15)\begin{bmatrix} 4 & -5 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 6 & -1 \end{bmatrix}$$

$$= (-1/15)\begin{bmatrix} 14 & 13 \\ -12 & -6 \end{bmatrix} = \begin{bmatrix} -14/15 & -13/15 \\ 4/5 & 2/5 \end{bmatrix}$$

Problem:
Find a matrix P that diagonalizes

$$A = \begin{bmatrix} -5 & 3 \\ 0 & 1 \end{bmatrix}.$$

Solution :
λ is an eigenvalue $\Leftrightarrow 0 = \det[\lambda I - A]$

$$= \det\begin{vmatrix} \lambda+5 & 3 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda+5)(\lambda-1) = 0 \Rightarrow \lambda = -5, 1.$$

When $\lambda = -5$,

$\text{Nul}[\lambda I - A] = \text{Nul}\begin{bmatrix} 0 & 3 \\ 0 & -6 \end{bmatrix} = \text{Nul}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \Rightarrow \mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

When $\lambda = 1$,

$\text{Nul}[\lambda I - A] = \text{Nul}\begin{bmatrix} 6 & 3 \\ 0 & 0 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & ½ \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\} \Rightarrow \mathbf{p}_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

So, $P = [\mathbf{p}_1 | \mathbf{p}_2] = \begin{bmatrix} 1 & ½ \\ 0 & 1 \end{bmatrix}$.

Checking the answer:

$P^{-1}AP = \begin{bmatrix} 1 & ½ \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -5 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & ½ \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 5/2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & ½ \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$

Thus, P diagonalizes A.

Problem:

Consider the following vectors:

$\mathbf{u}_1 = (0, 2, 0)$, $\mathbf{u}_2 = (2, 0, 2)$, $\mathbf{u}_3 = (2, 0, -2)$,

a.) Determine if the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is an orthogonal set ,
b.) If S is an orthogonal set, then determine its corresponding Orthonormal set.

Solution:

a.) S is an orthogonal set since $<\mathbf{u}_1, \mathbf{u}_2> = <\mathbf{u}_1, \mathbf{u}_3> = <\mathbf{u}_2, \mathbf{u}_3> = 0$
b.) The Euclidean norms of the vectors in part a.) are:

$\| \mathbf{u}_1 \| = \sqrt{(2)}$, $\| \mathbf{u}_2 \| = 2$, $\| \mathbf{u}_3 \| = 2$

Consequently, normalizing $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ yields

$\mathbf{v}_1 = \mathbf{u}_1 / \| \mathbf{u}_1 \| = (0, 2/\sqrt{(2)}, 0) = (0, \sqrt{(2)}/2, 0)$ (0, 1, 0)
$\mathbf{v}_2 = \mathbf{u}_2 / \| \mathbf{u}_2 \| = (1, 0, 1)$
$\mathbf{v}_3 = \mathbf{u}_3 / \| \mathbf{u}_3 \| = (1, 0, -1)$

Verify if the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthonormal by showing that $<\mathbf{v}_1, \mathbf{v}_2> = <\mathbf{v}_1, \mathbf{v}_3> = <\mathbf{v}_2, \mathbf{v}_3> = 0$ and $\| \mathbf{v}_1 \| = \| \mathbf{v}_2 \| = \| \mathbf{v}_3 \| = 1$ – it isn't!

Problem:

Determine if the given matrix A is orthogonal.

/ not a different problem fundamentally from the previous one & this approach is overkill.

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 3 & 1 & -1 \\ 0 & 2 & 3 \end{bmatrix}$$

Solution:

Since a square matrix A is orthogonal when $A^{-1} = A^T$ and when $AA^T = I$.
Calculate AA^T

$$AA^T = \begin{bmatrix} 1 & 1 & -2 \\ 3 & 1 & -1 \\ 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 2 \\ -2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 6 & -4 \\ 6 & 11 & -1 \\ -4 & -1 & 13 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, A is NOT orthogonal.

Problem: Good problem

Prove the following equivalent statements given what we have studied so far

If A is a symmetric matrix, then

~~(a) The eigenvalues of A are all real numbers.~~

(b) Eigenvectors from different eigenspaces are orthogonal.

(Hint: we have not learned all necessary methods to prove (a), and assume all entries contain real entries)

Solution:

Proof (b).

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2 of the matrix A. We want to show that $\langle \mathbf{v}_1$ ~~and~~, $\mathbf{v}_2 \rangle = 0$. The proof of this involves the trick of starting with the expression

$$A\mathbf{v}_1 * \mathbf{v}_2 = \mathbf{v}_1 * A^T\mathbf{v}_2 = \mathbf{v}_1 * A\mathbf{v}_2 \qquad (1)$$

But $\mathbf{v}_1$ is an eigenvector of A corresponding to λ_1, and $\mathbf{v}_2$ is an eigenvector of A corresponding to λ_2, so (1) yields the relationship

$$\lambda_1\mathbf{v}_1 * \mathbf{v}_2 = \mathbf{v}_1 * \lambda_2\mathbf{v}_2$$

Which can be rewritten as

$$(\lambda_1 - \lambda_2)\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0 \qquad (2)$$

But $\lambda_1 - \lambda_2 \neq 0$, since λ_1 and λ_2 were assumed distinct. Thus it follows from (2) that $\mathbf{v}_1 * \mathbf{v}_2 = 0$

Problem:

Three equivalent statements that we learned are:

1) matrix A is orthogonal
2) $\langle A\mathbf{x}, A\mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle$ for all values of x
3) $\langle A\mathbf{x}, A\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ for all values of x and y

Prove the validity of one of these statements given one of the others. (ex. If 3 is true, then 1 is true because…)

Solution:

1) If 1 is true then 2 is true because $\langle A\mathbf{x}, A\mathbf{x} \rangle = (A\mathbf{x})^T A\mathbf{x} = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T\mathbf{x} = \langle \mathbf{x}, \mathbf{x} \rangle$
2) If 2 is true then 3 is true because (let * represent "normal Euclidian" dot product, and //A// represent length of A) $\langle A\mathbf{x}, A\mathbf{y} \rangle = A\mathbf{x} * A\mathbf{y} = \frac{1}{4}[//A\mathbf{x}+A\mathbf{y}//^2 - //A\mathbf{x}-A\mathbf{y}//^2] = \frac{1}{4}[//A(\mathbf{x}+\mathbf{y})//^2 - //A(\mathbf{x}-\mathbf{y})//^2] = \frac{1}{4}[//(\mathbf{x}+\mathbf{y})//^2 - //(\mathbf{x}-\mathbf{y})//^2] = \mathbf{x}*\mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle$
3) If 3 is true then 1 is true because $\langle A\mathbf{x}, A\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle = (A\mathbf{x})^T A\mathbf{y} = \mathbf{x}^T\mathbf{y}$ which implies that $\mathbf{x}^T A^T A\mathbf{y} - \mathbf{x}^T\mathbf{y} = 0 \Rightarrow \mathbf{x}^T(A^TA - I)\mathbf{y} = 0 \Rightarrow \langle \mathbf{x}^T, (A^TA-I)\mathbf{y} \rangle = 0$ and since this must be true for all values of x and y we will choose x to be $(A^TA-I)\mathbf{y}$ so the last equation can be rewritten $//(A^TA-I)\mathbf{y}//^2 = 0 \Rightarrow (A^TA-I)\mathbf{y} = \mathbf{0}$ which means that ~~either $(A^TA-I) = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$~~ and since this must hold for all values of ~~x and~~ y, not just $\mathbf{y} = \mathbf{0}$ then (A^TA-I) must equal $\mathbf{0}$. If $(A^TA-I) = \mathbf{0}$ then $A^TA = I \Rightarrow A^T = A^{-1}$ which means that A is orthogonal

*Though not necessary to answer the problem, the fact that we have proven that 2 is true if 1 is, 3 is true if 2 is, 1 is true if 2 is shows that if any of the above statements are true then all of them are true.

not the correct logic – see book. – This is a good problem (3 is) but get the proof right.

Good problem.

Extra Credit Problem:
What is π thinking?

Solution:
That Elvis lives on Mars.

5 pts. 1. Given that <u,v> represents the Euclidean Inner Product and that **u** = (1,-5,-7) and **v** = (-3,18,1), calculate <5**u**,**v**>.

Solution:

$$< 5\mathbf{u},\mathbf{v} >= 5 < \mathbf{u},\mathbf{v} >= 5(\mathbf{u}_1\mathbf{v}_1 + \mathbf{u}_2\mathbf{v}_2 + \mathbf{u}_3\mathbf{v}_3) = 5(-3-80-1) = 5(-90) = -450$$

10 pts. 2. Let W be the subspace of R^4 spanned by vectors $\mathbf{w}_1 = (2,0,4,6)$, $\mathbf{w}_2 = (1,1,1,-4)$, $\mathbf{w}_3 = (2,-1,5,-5)$. Find a basis for the orthogonal complement of W.

Solution: The space W spanned by $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ is the same as the row space of

$$\begin{bmatrix} 2 & 0 & 4 & -6 \\ 1 & 1 & -1 & -4 \\ 2 & -1 & 7 & -5 \end{bmatrix}$$

. The nullspace is the orthogonal complement of W:

$$\left[\begin{array}{cccc|c} 2 & 0 & 4 & -6 & 0 \\ 1 & 1 & -1 & -4 & 0 \\ 2 & -1 & 7 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & 0 \\ 1 & 1 & -1 & -4 & 0 \\ 2 & -1 & 7 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & -1 & 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & -3 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 &= -2s + 3t \\ x_2 &= 3s + t \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

, so the orthogonal complement of W, is [spanned by] $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, [&] $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

15 pts. 3. Let $\{\mathbf{u}\} \in R^3$ with the Euclidean Inner Product. Transform the basis $\{\mathbf{u}\}$ into an orthonormal basis $\{\mathbf{w}\}$ when

$$\mathbf{u}_1 = (1,0,0)$$
$$\mathbf{u}_2 = (3,7,-2).$$
$$\mathbf{u}_3 = (0,4,1)$$

Solution: First find the orthogonal basis:

$$\mathbf{v}_1 = \mathbf{u}_1 = (1,0,0)$$

$$\mathbf{v}_2 = -\frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \mathbf{u}_2 = -\frac{3}{1}(1,0,0) + (3,7,-2) = (0,7,-2)$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 = (0,4,1) - 0 \cdot \mathbf{v}_1 - \frac{26}{53}(0,7,-2) = (0, \frac{30}{53}, \frac{105}{53})$$

where's the rest of this problem?

Good type of problem, but could be better engineered,

20 pts. 4. Find the least squares solution of the linear system **Ax** = **b**, and find the orthogonal projection of **b** onto the column space of A, where $A=\begin{bmatrix}1&0&2\\-2&1&1\\1&3&0\\0&-1&-1\end{bmatrix};b=\begin{bmatrix}2\\-2\\0\\-1\end{bmatrix}$.

Solution:

$$A^TA=\begin{bmatrix}1&-2&1&0\\0&1&3&-1\\2&1&0&-1\end{bmatrix}\begin{bmatrix}1&0&2\\-2&1&1\\1&3&0\\0&-1&-1\end{bmatrix}=\begin{bmatrix}6&1&0\\1&11&2\\0&2&6\end{bmatrix},$$

$$A^Tb=\begin{bmatrix}1&-2&1&0\\0&1&3&-1\\2&1&0&-1\end{bmatrix}\begin{bmatrix}2\\-2\\0\\-1\end{bmatrix}=\begin{bmatrix}6\\-1\\2\end{bmatrix},$$

$$\left[\begin{array}{ccc|c}6&1&0&6\\1&11&2&-1\\0&2&6&2\end{array}\right]\sim\left[\begin{array}{ccc|c}1&11&2&-1\\0&1&3&1\\0&-65&-12&12\end{array}\right]\sim\left[\begin{array}{ccc|c}1&11&2&-1\\0&1&3&1\\0&0&183&77\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&0&\frac{191}{183}\\0&1&0&-\frac{16}{61}\\0&0&1&\frac{77}{183}\end{array}\right]$$

$$x=\begin{bmatrix}\frac{191}{183}\\-\frac{16}{61}\\\frac{77}{183}\end{bmatrix},proj_Ab=Ax=\begin{bmatrix}1&0&2\\-2&1&1\\1&3&0\\0&-1&-1\end{bmatrix}\begin{bmatrix}\frac{191}{183}\\-\frac{16}{61}\\\frac{77}{183}\end{bmatrix}=\begin{bmatrix}\frac{115}{61}\\\frac{-353}{183}\\\frac{47}{183}\\\frac{-29}{183}\end{bmatrix}.$$

5. Consider the bases B = {$\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3$} and B' = {$\mathbf{u}'_1,\mathbf{u}'_2,\mathbf{u}'_3$} for $\mathbf{R}^3$, where $u_1=(1,0,0),u_2=(1,1,0),u_3=(1,1,1),u_1'=(1,2,1),u_2'=(0,2,3),u_3'=(0,0,4)$.

5 pts a) Find the transition matrix from B to B'.

5 pts b) Find the transition matrix from B' to B.

5 pts. c) Find $[\mathbf{v}]_B$ if $[\mathbf{v}]_{B'} = \begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix}$.

try my way – less insight needed

Solution: a) First, we must find coordinate vectors for $\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3$ relative to B'.

maybe – I can't remember this "trick" – my way involves no tricks, only basic principles.

$$u_1 = u_1' - u_2' + \frac{1}{2}u_3'$$

By inspection, $u_2 = u_1' - \frac{1}{2}u_2' + \frac{1}{8}u_3'$. Thus, $P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{8} & \frac{3}{8} \end{bmatrix}$.

$$u_3 = u_1' - \frac{1}{2}u_2' + \frac{3}{8}u_3'$$

b) First, we must find coordinate vectors for $\mathbf{u}_1',\mathbf{u}_2',\mathbf{u}_3'$ relative to B.

$$u_1' = u_3 + u_2 - u_1$$

By inspection, $u_2' = 3u_3 - u_2 - 2u_1$. Thus, $P = \begin{bmatrix} 1 & 3 & 4 \\ 1 & -1 & -4 \\ -1 & -2 & 0 \end{bmatrix}$.

$$u_3' = 4u_3 - 4u_2 + 0u_1$$

c) $\begin{bmatrix} 1 & 3 & 4 \\ 1 & -1 & -4 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 31 \\ -21 \\ -12 \end{bmatrix}$.

20 pts. 6. Prove the equivalence of the following:
a) A is orthogonal, b) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n, c) $A\mathbf{x}\cdot A\mathbf{y} = \mathbf{x}\cdot\mathbf{y}$ for all $\mathbf{x},\mathbf{y}$ in R^n.

a~b: Assume A is orthogonal. Then $A^T A = I$, and
$\|Ax\| = (Ax \cdot Ax)^{1/2} = (x \cdot A^T Ax)^{1/2} = (x \cdot x)^{1/2} = \|x\|$.

b~c: Assume b. Then
$$Ax \cdot Ay = \tfrac{1}{4}\|Ax + Ay\|^2 - \tfrac{1}{4}\|Ax - Ay\|^2 = \tfrac{1}{4}\|A(x+y)\|^2 - \tfrac{1}{4}\|A(x-y)\|^2 = \tfrac{1}{4}\|x+y\|^2 - \tfrac{1}{4}\|x-y\|^2 = x \cdot y$$

c~a: Assume c. Then
$x \cdot y = x \cdot A^T Ay \Rightarrow x \cdot (A^T Ay - y) = 0 \Rightarrow x \cdot (A^T A - I)y = 0$.
This holds if $x = (A^T A - I)y$, so
$(A^T A - I)y \cdot (A^T A - I)y = 0 \Rightarrow (A^T A - I)y = 0 \Rightarrow A^T A = I$, because the system is consistent for all y based on our assumption of c. – why? Book asks why and so do I.

7. Consider the matrix $A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

5 pts a) What are the eigenvalues of A?
5 pts b) What are the eigenvalues of A^2?
5 pts c) What are the eigenvalues of A^7? (Hint: there is an easy way to do this)
5 pts d) Which eigenvalue will A never have if A^{-1} exists? — general statement — this A is more general than this one (misleading in other order)

Solution:

a) A~ $\begin{bmatrix} -1 & +2 & +4 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. $\mathbf{det}(\lambda I - A) = (\lambda + 1)(\lambda - 1)(\lambda + 2)$

$\lambda = -2, -1, 1$

b) eigval(A^k) = [eigval(A)]^k. k = 2, so $\lambda = 4, 1, 1$.

c) k = 7, so $\lambda = -128, -1, 1$.

d) eigval(A) $\neq 0$.

10 pts. 8. Determine whether this matrix is diagonizable: $A = \begin{bmatrix} 6 & 5 & 0 & 3 \\ 0 & 3 & 10 & -7 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Solution: This is a 4x4 matrix, so if it has 4 distinct eigenvalues it is diagonizable (since an nxn diagonizable matrix always has n distinct eigenvalues). This matrix is also upper-triangular, meaning its eigenvalues are the entries on the main diagonal. These entries are distinct and there are 4, so the matrix is diagonizable.

true statement, but the other one isn't

not true – for example $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Good

15 pts 9. Prove that if A is orthogonal, then the row vectors of A form an orthonormal set in R^n with the Euclidean inner product.

Solution: AA^T can be expressed as $\begin{bmatrix} r_1 \cdot r_1 & r_1 \cdot r_2 & \cdots & r_1 \cdot r_n \\ r_2 \cdot r_1 & r_2 \cdot r_2 & \cdots & r_2 \cdot r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_n \cdot r_1 & r_n \cdot r_2 & \cdots & r_n \cdot r_n \end{bmatrix}$, where the row vectors of A are r sub 1 through n. Thus, $AA^T = I$ if and only if $r_1 \cdot r_1 = \ldots r_n \cdot r_n = 1$ and $r_i \cdot r_j = 0$ when i does not equal j, which are true only if the rows are an orthonormal set in R^n.

20 pts 10. Find a matrix P that orthogonally diagonalizes A, where

$$A = \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ 36 & 0 & -23 \end{bmatrix}, \text{ and determine } P^{-1}AP$$

<u>Solution:</u>

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda+2 & 0 & 36 \\ 0 & \lambda+3 & 0 \\ 36 & 0 & \lambda+23 \end{bmatrix} = (\lambda+2)(\lambda+3)(\lambda+23) - 1296(\lambda+3)$$

$$= (\lambda+3)(\lambda^2+25\lambda-1250) = (\lambda-25)(\lambda+50)(\lambda+3)$$

So, the eigenvalues of A are λ_1=3, λ_2=25, λ_3= -50. Therefore,

λ =3:

$$\begin{bmatrix} -1 & 0 & 36 \\ 0 & 0 & 0 \\ 36 & 0 & 20 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{matrix} x1 = r \\ x2 = s \\ x3 = t \end{matrix} \Rightarrow \mathbf{u1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

x1=0 x3=0 only this one.

λ =25:

$$\begin{bmatrix} 27 & 0 & 36 \\ 0 & 28 & 0 \\ 36 & 0 & 48 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x1 = -4/3 * s \\ x2 = r \\ x3 = s \end{matrix} \Rightarrow \mathbf{u4} = \begin{bmatrix} -4/3 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u5} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

x2=0 only this one

λ =-50:

$$\begin{bmatrix} -48 & 0 & 36 \\ 0 & -47 & 0 \\ 36 & 0 & -27 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -3/4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x1 = -3/4 * s \\ x2 = r \\ x3 = s \end{matrix} \Rightarrow \mathbf{u6} = \begin{bmatrix} -3/4 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u7} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

x2=0 only this one

And use Gram-Schmidt

"geometric multiplicity cannot exceed algebraic multiplicity"

λ =3: $\mathbf{v1} = \mathbf{u1} = (1,0,0)$ $\quad \mathbf{v2} = \mathbf{u2} - \frac{\langle \mathbf{u2}, \mathbf{v1} \rangle}{\|\mathbf{v1}\|^2}\mathbf{v1} = (0,1,0)$

$$\mathbf{v3} = \mathbf{u3} - \frac{\langle \mathbf{u3}, \mathbf{v1} \rangle}{\|\mathbf{v1}\|^2}\mathbf{v1} - \frac{\langle \mathbf{u3}, \mathbf{v2} \rangle}{\|\mathbf{v2}\|^2}\mathbf{v2} = (0,0,1)$$

← I don't know what you're going to do here, since you messed up on eigenvectors.

$$\mathbf{q1} = \frac{\mathbf{v1}}{\|\mathbf{v1}\|} = (1,0,0) \quad \mathbf{q2} = (0,1,0) \quad \mathbf{q3} = (0,0,1)$$

$\lambda = 25$: $\mathbf{v}4 = \mathbf{u}4 = (-4/3,0,1)$ $\quad \mathbf{v}5 = \mathbf{u}5 - \dfrac{\langle \mathbf{u}5, \mathbf{v}4 \rangle}{\|\mathbf{v}4\|^2}\mathbf{v}4 = (0,1,0)$

$\mathbf{q}4 = \dfrac{\mathbf{v}4}{5/3} = (-4/5,0,3/5)$ $\qquad \mathbf{q}5 = (0,1,0)$

$\lambda = 50$: $\mathbf{v}6 = (-3/4,0,1)$ $\qquad \mathbf{v}7 = (0,1,0)$

$\mathbf{q}6 = \dfrac{\mathbf{v}6}{5/4} = (3/5,0,4/5)$ $\qquad \mathbf{q}7 = (0,1,0)$

So, $P = [\mathbf{q}4, \mathbf{q}5, \mathbf{q}6] = \begin{bmatrix} -4/5 & 0 & 3/5 \\ 0 & 1 & 0 \\ 3/5 & 0 & 4/5 \end{bmatrix}$

and $P^{-1}AP = P^{T}AP$:

$$\begin{bmatrix} -4/5 & 0 & 3/5 \\ 0 & 1 & 0 \\ 3/5 & 0 & 4/5 \end{bmatrix}\begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ 36 & 0 & -23 \end{bmatrix}\begin{bmatrix} -4/5 & 0 & 3/5 \\ 0 & 1 & 0 \\ 3/5 & 0 & 4/5 \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -50 \end{bmatrix}$$

One only does G.S. once, not "n-times", for an nxn matrix.
— who knows
— maybe, maybe not since everything else is in error

1) Given A and B:

$$A = \begin{vmatrix} 4 & 0 \\ 3 & 8 \end{vmatrix} \qquad B = \begin{vmatrix} 2 & 6 \\ 3 & 4 \end{vmatrix}$$

a. Find <A, B> (the inner product). — *inner product must be defined – ambiguous question.*

b. Find ||A|| (the norm). *\ ditto*

c. Find ||B||. *\ ditto*

Solution:

maybe, maybe not. Hard to say.

a. The inner product is simply the sum of the product of the corresponding entries between matrix A and B. Here is the following work to determine the inner product:

(4)(2) + (0) + (3)(3) + (8)(4) = 49.

b. To find the norm of a matrix all the entries have to be squared, added together, and then the sum has to be taken as follows:

||A|| = (4^2 + 3^2 + 8^2) ^ (1/2) = 9.43.

c. Exact same process as in part a, only this uses matrix B as follows:

||B|| = (2^2 + 6^2 + 3^2 + 4^2) ^ (1/2) = 8.06. *– Excellent problem – well engineered.*

2) Find the least squares solution of the linear system, Ax = b, given by:

$x_1 - x_2 = 1$
$2x_1 + 2x_2 = 4$
$-x_1 + 3x_2 = 3$

Solution:

$$A = \begin{vmatrix} 1 & -1 \\ 2 & 2 \\ -1 & 3 \end{vmatrix} \qquad B = \begin{vmatrix} 1 \\ 4 \\ 3 \end{vmatrix}$$

Theorem 6.4.2 states: A^T*Ax = A^T*b so both sides need to be solved for to determine the matrix x:

$$A^T*A = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 2 & 3 \end{vmatrix} \begin{vmatrix} 1 & -1 \\ 2 & 2 \\ -1 & 3 \end{vmatrix} = \begin{vmatrix} 6 & 0 \\ 0 & 14 \end{vmatrix}$$

$$A^T*b = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 2 & 3 \end{vmatrix} \begin{vmatrix} 1 \\ 4 \\ 3 \end{vmatrix} = \begin{vmatrix} 6 \\ 16 \end{vmatrix}$$

Now solving for the matrix x we can set up the following:

$$\begin{vmatrix} 6 & 0 \\ 0 & 14 \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \begin{vmatrix} 6 \\ 16 \end{vmatrix}$$

Now the equation must be solved for as follows:

$$\left|\begin{matrix} 6 & 0 \\ 0 & 14 \end{matrix}\right| \left.\begin{matrix} 6 \\ 16 \end{matrix}\right|$$

After row reducing we arrive with:

$X_1 = 1$

$X_2 = 8/7$

3) Find the coordinate vector for v relative to $S = \{v_1, v_2, v_3\}$ for:

$v = 5 + 4x + 3x^2$
$v_1 = 1 + 2x + 3x^2$
$v_2 = 2 + 3x + x^2$
$v_3 = 9 + 6x + 4x^2$

Solution:

This is a rather simple problem if the question is understood. All that has to be done is to set up the system of equations as columns of a matrix and multiply the coefficients together including the solution matrix to determine the coordinate vector:

$$\begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} \quad \begin{vmatrix} 2 \\ 3 \\ 1 \end{vmatrix} \quad \begin{vmatrix} 9 \\ 6 \\ 4 \end{vmatrix} \quad \begin{vmatrix} 5 \\ 4 \\ 3 \end{vmatrix}$$

Column vectors

why? without this explanation this is "Alchemy"! See my key for last exam for an explanation of why this works. (Assuming it does - I can't tell if it does or doesn't w/o thinking of good explanation)

- What is this? This is "way off." Again see my key for last exam.

1 * 2 * 9 * 5 = 90
2 * 3 * 6 * 4 = 144
3 * 1 * 4 * 3 = 36

Therefore, the coordinate vector is (90, 144, 36).

various other problem) Prove that $\|u + v\| \leq \|u\| + \|v\|$, where u and v are vectors in an inner product space V.

Good problem.

Solution:

$$(\|u + v\|)^2 = \langle u + v, u + v \rangle$$

$$= \langle u, u \rangle + 2 \langle u, v \rangle + \langle v, v \rangle$$

$$\leq \langle u, u \rangle + 2 |\langle u, v \rangle| + \langle v, v \rangle \quad \text{(Absolute value property)}$$

$$\leq \langle u, u \rangle + 2 \|u\| \|v\| + \langle v, v \rangle \quad \text{(Cauchy-Schwarz Inequality)}$$

$$= \|u\|^2 + 2 \|u\| \|v\| + \|v\|^2$$

$$= (\|u\| + \|v\|)^2$$

And once the square roots of both sides are taken we arrive with the inequality of:

$\|u + v\| \leq \|u\| + \|v\|$.

, or spanned by? Probably the latter.

4) Find a basis for the orthogonal complement of W, where W contains the vectors:

$w_1 = (6, 6, -2, 0, 2)$ $w_2 = (-2, -1, 3, -3, 1)$ $w_3 = (1, 1, -1, 0, -1)$ and $w_4 = (0, 0, 1, 1, 2)$

Solution: The space W spanned by these four vectors is the same as the row space of the matrix:

'good'

$$A = \begin{bmatrix} 6 & 6 & -2 & 0 & 2 \\ -2 & -1 & 3 & -3 & 1 \\ 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$ and by theorem, the null space of A is the orthogonal complement of W. To find the null space, we reduce the matrix and then use algebra to solve for "the values:"

$$\begin{bmatrix} 6 & 6 & -2 & 0 & 2 \\ -2 & -1 & 3 & -3 & 1 \\ 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 6 & -2 & 0 & 2 \\ 0 & 3 & 7 & -9 & 5 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 6 & -2 & 0 & 2 \\ 0 & 3 & 7 & -9 & 5 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix}$$

better writing

We know that:
$x_4 = 0$
$x_5 = t$

So,
$3x_2 - 14t + 5t = 0$ (where does this come from?)
$x_2 = 3t$
$x_3 = -2t$

$6x_1 + 18t + 4t + 2t = 0$
$x_1 = -4t$

So, the null space is "equal to" $t\begin{bmatrix} -4 \\ 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$. ~~The~~ A basis for the orthogonal component, then, is the vector $v = (-4, 3, -2, 0, 1)$

5) Given the two matrices:

$$A = \begin{bmatrix} 3/7 & -6/7 & 2/7 \\ 2/7 & 3/7 & 6/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Are the following matrices orthogonal?

a) A
b) B
c) AB
d) What is the det(AB) and how is this value related to whether or not AB is orthogonal?

Solution:

overkill!

a) In order to determine whether the matrices are orthogonal, we take A^TA because by theorem, A is orthogonal if and only if either $AA^T = I$ or $A^TA = I$.

$$\begin{bmatrix} 3/7 & -6/7 & 2/7 \\ 2/7 & 3/7 & 6/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix} \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

, which tells us that A is orthogonal!!!

b) following the same process as in a, we get that the identity matrix times itself is still the identity matrix, which gives us that B is also orthogonal.
c) By theorem, a product of orthogonal matrices is orthogonal, so we know that AB will be orthogonal.
d) To find the determinant, we first multiply A by B, which gives us A because B is the identity matrix. Then, we use the formula for a 3x3 matrix that the

$$\det() = aei + bfg + cdh - ceg - afh - bdi$$

which gives us

(3/7)(3/7)(-3/7) + (-6/7)(6/7)(6/7) + (2/7)(2/7)(2/7) – (2/7)(3/7)(6/7) – (-6/7)(2/7)(-3/7) – (3/7)(6/7)(2/7) =

-27/343 – 216/343 + 8/343 – 36/343 – 36/343 – 36/343 = -343/343 = -1

The determinant of an orthogonal matrix must be 1 or -1. Because the determinant is equal to -1 in this case, it also helps to show that this matrix is orthogonal.

6) Find the eigenvalues of the following matrices:

a) $\begin{bmatrix} 1 & 22/3 & 19/7 & 8/3 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 0 & 9 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

c) And find the characteristic polynomial of the following matrix:

$$\begin{bmatrix} 5 & 2 & 1 \\ 0 & 6 & 8 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution:

a) By theorem, if A is an nxn triangular matrix, then the eigen values of A are the entries on the main diagonal of A. So, $\lambda = 1$, $\lambda = 5$, $\lambda = 6$, and $\lambda = 9$
b) By the same theorem, $\lambda = 1$
c) To find the characteristic polynomial, we must first find the characteristic equation, which is $\det(I\lambda - A) = 0$, and then expand it.

After subtraction, we have the equation:

$\det\left(\begin{bmatrix} \lambda - 5 & -2 & -1 \\ 0 & \lambda - 6 & -8 \\ 0 & -1 & \lambda \end{bmatrix}\right) = 0$. By using the equation to find 3x3 determinants, we get:

$(\lambda-5)(\lambda-6)(\lambda) + (-2)(-8)(0) + (-1)(0)(-1) - (-1)(\lambda-6)(0) - (-2)(0)(\lambda) - (\lambda-5)(-8)(-1)$
$= (\lambda-5)(\lambda-6)(\lambda) - 8(\lambda-5) = (\lambda^2 - 6\lambda)(\lambda-5) - 8\lambda + 40 = \lambda^3 - 5\lambda^2 - 6\lambda^2 + 30\lambda - 8\lambda + 40 = \lambda^3 - 11\lambda^2 + 22\lambda + 40$

So, our characteristic polynomial is:
$\lambda^3 - 11\lambda^2 + 22\lambda + 40$

7)
V is an inner product space with inner product $\langle \mathbf{X}, \mathbf{Y} \rangle = tr(\mathbf{X}^T\mathbf{Y}) = tr(\mathbf{Y}^T\mathbf{X})$. — irrelevant to this question.
If the set of matrices $S = \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}\}$ (where

$$\mathbf{A} = \begin{bmatrix} 33 & -27 \\ 21 & 21 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -6 & 2 \\ 8 & 4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} -27 & -19 \\ -21 & 37 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} -7 & -7 \\ -9 & 11 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 6 & 12 \\ -3 & 9 \end{bmatrix}$$)

span V, find a basis (S') for V.

Solution

The first step is to observe that a basis for M22 is going to contain 4 linearly independent matrices. We need then to find which of the five matrices is a "linearly dependent multiple" of the others.

It turns out that **C** is linearly dependent ($\mathbf{C} = \mathbf{B} + 3\mathbf{D}$) and so will not be part of our basis. Orthogonality for each vector is next tested using the given inner product formula and recalling that $\langle \mathbf{X}, \mathbf{Y} \rangle = 0$ (scalar) if **X** and **Y** are orthogonal. To help with calculations it is beneficial to simplify our inner product to the equation — why are you doing this?

$$\langle \mathbf{X}, \mathbf{Y} \rangle = x_{11}y_{11} + x_{21}y_{21} + x_{12}y_{12} + x_{22}y_{22}$$

It is now the trivial activity of multiplying indices and adding them.

8)
Normalize the basis found above. — inner product finally relevant.

To normalize each matrix we ~~multiply~~ divide each matrix by its respective norms, that is $\frac{1}{\|\mathbf{X}\|}\mathbf{X}$.

$$\|\mathbf{A}\| = (\langle \mathbf{A}, \mathbf{A} \rangle)^{1/2} = (33^2 + (-27)^2 + 21^2 + 21^2)^{1/2} = (2700)^{1/2} = 30\sqrt{3}$$

$$\frac{1}{\|\mathbf{A}\|}\mathbf{A} = \begin{bmatrix} \frac{11}{10\sqrt{3}} & \frac{-9}{10\sqrt{3}} \\ \frac{7}{10\sqrt{3}} & \frac{7}{10\sqrt{3}} \end{bmatrix} = \mathbf{A}'$$

Repeating this process for matrices **B** **D** and **E** we obtain

$$\mathbf{B}' = \begin{bmatrix} \frac{-3}{\sqrt{30}} & \frac{1}{\sqrt{30}} \\ \frac{4}{\sqrt{30}} & \frac{2}{\sqrt{30}} \end{bmatrix} \quad \mathbf{D}' = \begin{bmatrix} \frac{-7}{10\sqrt{3}} & \frac{-7}{10\sqrt{3}} \\ \frac{-9}{10\sqrt{3}} & \frac{11}{10\sqrt{3}} \end{bmatrix} \quad \mathbf{E}' = \begin{bmatrix} \frac{2}{\sqrt{30}} & \frac{4}{\sqrt{30}} \\ \frac{-1}{\sqrt{30}} & \frac{3}{\sqrt{30}} \end{bmatrix}$$

and $S' = \{ \mathbf{A}', \mathbf{B}', \mathbf{D}', \mathbf{E}' \}$.

9)

Suppose that the invertible matrix $\mathbf{A}$ is diagonalized by the matrix $\mathbf{B}$. Show that $\mathbf{A}^{-1}$ is diagonalized ~~by~~ $\mathbf{B}^{-1}$. not so. this means

$\mathbf{A} = \mathbf{PBP}^{-1}$ – irrelevant

$\mathbf{A}^{-1} = (\mathbf{PBP}^{-1})^{-1}$

$\mathbf{A}^{-1} = (\mathbf{P}^{-1})^{-1}\mathbf{B}^{-1}\mathbf{P}^{-1}$

$\mathbf{A}^{-1} = \mathbf{PB}^{-1}\mathbf{P}^{-1}$

Therefore $\mathbf{A}^{-1}$ is diagonalized by $\mathbf{B}^{-1}$.

$B^{-1} A B = D$ diagonal matrix

$\Rightarrow B^{-1} A^{-1} B = \text{"}D^{-1}\text{"}$ diagonal matrix "

so B diagonalizes A iff B " A^{-1}.

Suppose that λ is an eigenvalue of the matrix A with associated eigenvector $\mathbf{v}$. Show that λ^k is an eigenvalue of $\mathbf{A}^k$ with associated eigenvector $\mathbf{v}$ (where k is a positive integer).

Suppose — proof by induction.

$\mathbf{A}^n\mathbf{v} = \lambda^n\mathbf{v}$ $(n<k)$ is true. Now we will show that it is true for $\mathbf{A}^{n+1}\mathbf{v} = \lambda^{n+1}\mathbf{v}$

Multiplying both sides by $\mathbf{A}$ gives

$$\mathbf{AA}^n\mathbf{v} = \mathbf{A}(\lambda^n\,\mathbf{v}) \Rightarrow \mathbf{A}^{n+1}\mathbf{v} = \lambda^n\mathbf{Av}$$

and we know

$$\mathbf{Av} = \lambda\mathbf{v}$$

so we have

$$\mathbf{A}^{n+1}\mathbf{v} = \lambda^n\lambda\mathbf{v} \Rightarrow \mathbf{A}^{n+1}\mathbf{v} = \lambda^{n+1}\mathbf{v}$$

therefore by induction we have shown that $\mathbf{A}^k\mathbf{v} = \lambda^k\mathbf{v}$ and that λ^k is an eigenvalue of $\mathbf{A}^k$.

10)

Given the matrix **A** above is

$$\mathbf{A} = \begin{bmatrix} 9 & -8 \\ 6 & -5 \end{bmatrix}$$

find $\mathbf{A}^3$.

– Here P diagonalizes A, not B.

Recall that $\mathbf{A}^k = \mathbf{PB}^k\mathbf{P}^{-1}$

so the problem can be reduced to finding **P** and **B** then calculating $\mathbf{PB}^3\mathbf{P}^{-1}$.

The characteristic equation $|\lambda\mathbf{I} - \mathbf{A}| = \mathbf{0}$ *(scalar)* gives $(\lambda - 9)(\lambda + 5) - (-6)(8) = 0$

Solving for λ we get $\lambda_1 = 1$ $\lambda_2 = 3$. *– show factorization*

W next find the bases for the eigenspaces using $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$ which is

$\begin{bmatrix} \lambda - 9 & 8 \\ -6 & \lambda + 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0}$ and substituting in λ_1 and λ_2 we obtain the matrices

$\begin{bmatrix} -8 & 8 \\ -6 & 6 \end{bmatrix}$ and $\begin{bmatrix} -6 & 8 \\ -6 & 8 \end{bmatrix}$ respectively. It can easily be seen that the $\mathbf{v_1}$ and $\mathbf{v_2}$ are

$\mathbf{v_1} = (1,1)$ and $\mathbf{v_2} = (4,3)$ which produce the matrix

$\mathbf{P} = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$ who's inverse is $\mathbf{P}^{-1} = \begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix}$. The matrix **B** is also easily found to be

$$\mathbf{B} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Finally to calculate $\mathbf{A}^3$ we find $\mathbf{B}^3 = \begin{bmatrix} 1 & 0 \\ 0 & 27 \end{bmatrix}$ and then calculate the equation $\mathbf{PB}^3\mathbf{P}^{-1}$,

which is $\begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 27 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 108 \\ 1 & 81 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 105 & -104 \\ 78 & -77 \end{bmatrix} = \mathbf{A}^3$.

4

6.1 (15 Points)

1. List the Axioms of an inner product and use them to determine if: $<u,v>=u_1v_1 - u_2v_2 + u_3v_3$ is an inner product on R^3 (10 points)

Solution

1. Symmetry

   <u,v> = <v,u>

2. Additive

   <u + v, z> = <u,z> + <v,z>

3. Homogeneity

   <ku,v> = k<u,v>

4. Positivity

   $<u,u> \geq 0$

   And <u,u> = 0 iff and only if u =0

$u_1v_1 - u_2v_2 + u_3v_3$ passes all of the axioms except the last one.
Positivity - because there is no way to get a negative term in the expression.

What does this mean? Too flippant. Show something.

2. Find <A,B>, d(A,B) (5 points)

ambiguous. These must be defined.

$$A=\begin{bmatrix} 1 & 4 \\ -6 & 3 \end{bmatrix} \quad B=\begin{bmatrix} 5 & -3 \\ 0 & 8 \end{bmatrix}$$

Solution

<A,B> = 5 – 12 + 0 + 18 = 11 – depends on choice of inner product

$$(A\text{-}B) = \begin{bmatrix} 4 & -7 \\ 6 & 5 \end{bmatrix}$$

$$D(A,B) = \sqrt{16+49+36+25} = \sqrt{126}$$

Section 6.2 (15 pts)

1. Find the cosine of the angle between A & B when: (3 pts each)

   a) A=(4,1,8) B=(1,0,-3)

   b) A= -1 +5x+2x^2 B= 2 + 4x + 9x^2

   c) $A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

   for these answer depends on choice of inner product - ambiguous questions

   $\cos\theta = \langle u,v\rangle/(\|u\|\|v\|)$

Solutions

a)

$\langle A,B\rangle = 4(1) + 1(0) + 8(-3) = -20$

$\|A\|\|B\| = \sqrt{4^2+1^2+8^2} * \sqrt{1^2+0^2+8^2} = 9\sqrt{10}$

? $=-20/(9\sqrt{10})$

b)

$\langle A,B\rangle = -1(2) + 5(4) + 2(-9) = 0$

$\|A\|\|B\| = \sqrt{-1^2+5^2+2^2} * \sqrt{2^2+4^2+-9^2} = \sqrt{3030}$

? $=0$

c)

$\langle A,B\rangle = 2(3) + 6(2) + 1(1) + -3(0) = 19$

$\|A\|\|B\| = \sqrt{2^2+6^2+1^2+-3^2} * \sqrt{3^2+2^2+1^2+0^2} = 10\sqrt{7}$

? $=19/(10\sqrt{7})$

2. Determine if the following sets are orthogonal (2pts each)

a) u=(a,b,c) v=(-a,b,-c)

b) u=(a,b,c,d) v=(-b,a,-d,c)

c) u=(0,2,3) v=(9,1/2,-1/3)

Solutions

a) $a(-a) + b(b) + c(-c) = -a^2+b^2-c^2$ **NO** - depends on a, b & c

b) $a(-b) + b(a) + c(-d) + d(c) = 0$ **YES**

c) $0(9) + 2(1/2) + 3(-1/3) = 0$ **YES**

Math Exam Key III

6.3

1. (2 points each) Are the following sets of vectors orthogonal with respect to the Euclidean inner product on R^2?

a) (0,1),(2,0) b) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Solution

a) 0*2 + 1*0 = 0 inner product = 0 → Yes

b) $-\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}} = 1$ → inner product $\neq 0$ → No

2. (2 points each) Are the above sets of vectors orthonormal?

Solution

— what is this? what are you doing?

a) First we find the Euclidean Norm $\left(\frac{0}{1}, \frac{1}{1}\right)\left(\frac{2}{2}, \frac{0}{2}\right)$ 0*2 + 1*0 = 0

inner product = 0 → Yes — the set above is not orthonormal!

b) $\left(\sqrt{-\frac{1}{\sqrt{2}}^2 + \frac{1}{\sqrt{2}}^2}\right) = 1$

Nothing changes, therefore, $-\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}} = 1$

inner product $\neq 0$ → No

3. (3 points each) Find the coordinate vector of w with respect to the orthonormal basis that has been given.

a) $w = (3,7)$; $u_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ $u_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ ✱

b) $w = (-1,0,2)$; $u_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$ $u_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ $u_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

✱ If these are really orthonormal, use — no row reduction necessary!

$\underline{w} = \langle \underline{w}, \underline{u}_1 \rangle \underline{u}_1 + \langle \underline{w}, \underline{u}_2 \rangle \underline{u}_2 \Rightarrow (\underline{w})_{Basis} = (\langle \underline{w}, \underline{u}_1 \rangle, \langle \underline{w}, \underline{u}_2 \rangle)$

$= \langle \frac{3}{\sqrt{2}} - \frac{7}{\sqrt{2}}, \frac{3}{\sqrt{2}} + \frac{7}{\sqrt{2}} \rangle = \langle -\frac{4}{\sqrt{2}}, \frac{10}{\sqrt{2}} \rangle = \langle -2\sqrt{2}, 5\sqrt{2} \rangle$.

Solution

– backwards!

a) $\begin{bmatrix} 3 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 7 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2\sqrt{2} \\ 0 & 1 & 5\sqrt{2} \end{bmatrix}$ $w_S = (-2\sqrt{2}, 5\sqrt{2})$

– backwards!

b) $\begin{bmatrix} -1 & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 2 & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & \frac{11}{5} \\ 0 & 0 & 1 & \frac{23}{5} \end{bmatrix}$ $W_S = (-4, \frac{11}{5}, \frac{23}{5})$

– This is a statement that is either true or false.

4. (2 points) Choose the correct answer: Every nonzero finite-dimensional inner product space has an orthonormal basis.

Always Sometimes Never

Solution

Always

Theorem 6.3.6 on page 323 in book.

5. (4 points) The subspace of R^3 spanned by the vectors $u_1=(4/5, 0, -3/5)$ and $u_2=(0,1,0)$ is a plane passing through the origin. Express $w=(1,2,3)$ in the form $w=w_1+w_2$, where w_1 lies in the plane, and w_2 is perpendicular to the plane.

Solution

– why is this the solution? Compute ⊥ projection and its complement.

$W_1 = (-\frac{4}{5}, 2, \frac{3}{5})$

$W_2 = (\frac{9}{5}, 0, \frac{12}{5})$

Section 6.5

1. Give the Bases B={u_1,u_2,u_3) B'={v_1,v_2,v_3} for R^3 (10 pts), where

and $u_1=\begin{bmatrix}2\\1\\1\end{bmatrix}$ $u_2=\begin{bmatrix}2\\-1\\1\end{bmatrix}$ $u_3=\begin{bmatrix}1\\2\\1\end{bmatrix}$; $v_1=\begin{bmatrix}3\\1\\-5\end{bmatrix}$ $v_2=\begin{bmatrix}1\\1\\-3\end{bmatrix}$ $v_3=\begin{bmatrix}-1\\0\\2\end{bmatrix}$,

A) Find the transistion matriz from B to B'

B) Find the coordinate vector $[w]_{B'}$

Solution

Setting up the Matrix

$$\left[\begin{matrix}3 & 1 & -1\\1 & 1 & 0\\-5 & -3 & 2\end{matrix}\right|\left.\begin{matrix}2 & 2 & 1\\1 & -1 & 2\\1 & 1 & 1\end{matrix}\right]$$

And row reducing gives us

$$\left[\begin{matrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{matrix}\right|\left.\begin{matrix}3 & 2 & 5-2\\-2 & -3 & -1/2\\5 & 1 & 6\end{matrix}\right]$$

$$P_{B\text{-}B'} = \begin{bmatrix}3 & 2 & 5-2\\-2 & -3 & -1/2\\5 & 1 & 6\end{bmatrix}$$

Since part B asks us to find $[w]_{B'}$ we need the transistion matrix from B' to B.

$$\left[\begin{matrix}3 & 2 & 5-2\\-2 & -3 & -1/2\\5 & 1 & 6\end{matrix}\right|\left.\begin{matrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{matrix}\right] \sim \left[\begin{matrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{matrix}\right|\left.\begin{matrix}35/2 & 19/2 & -15/2\\-19/2 & -11/2 & 7/2\\-13 & -7 & 5\end{matrix}\right]$$

– what is w? Ambiguous question it was not specified.

where did this come from?

To Compute $[W]_{B'}$ from the equation
$P^{-1}[w] = [w]_{B'}$

$$\begin{bmatrix} 35/2 & 19/2 & -15/2 \\ -19/2 & -11/2 & 7/2 \\ -13 & -7 & 5 \end{bmatrix} \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -7/2 \\ 0 & 1 & 0 & 23/2 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

$$[w]_{B'} = \begin{bmatrix} -7/2 \\ 23/2 \\ 6 \end{bmatrix}$$

this question does not have a clear antecedent, or is patently ambiguous.

2. What happens to P, n transition matrix, if the vectors $v_1, v_2, \ldots v_n$ of the basis B are reversed, (ie: $v_n, \ldots v_2, v_1$) (5 points)

Solution
Reversing the columns on B will reverse the rows on P

6.6 (20 pts)

1. What is the determinant of an orthogonal Matrix? (5 pts)

   1 or -1

2. What is the definition of an orthogonal Matrix (6 pts)

   a matrix A with the property $A^{-1}=A^{T}$.

3. If the Matrix is orthogonal, find its inverse. (8 pts Each)

State what you want done less ambiguously.

a) $A=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A*A^{T}=I$ So, $A^{-1}=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

b) $A=\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -2 & 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}=\begin{bmatrix} 3 & & \\ & & \\ & & \end{bmatrix}$$

$A*A^{T}\neq I$, So A is not Orthogonal

c) $A=\begin{bmatrix} 0 & 1 & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A*A^{T}=I$ So, $A^{-1}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

7.1

1. (3 points each) Find the characteristic equations of the following matrices:

a) $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ b) $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$

Solutions

a) $\lambda^2 - 2\lambda - 3 = 0$

b) $\lambda^2 - 8\lambda + 16 = 0$

2. (3 points for each part) Find the bases for the eigenspaces of the matrices in problem 1.

Solution

Basis for eigenspace corresponding to $\lambda = 3$ $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Basis for eigenspace corresponding to $\lambda = -1$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

3. Choose the appropriate answer(3 points) A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Always Sometimes Never

Solution

Always
See theorem 7.1.4 on page 365

1

Compute < p, q > using inner products.

a) $p = -3x + 2y - z$
   $q = y - 5z$
b) $p = 16r - 12s + 3u$
   $q = 2r + 12t - 11u$

? I suppose r, s, u, x, y, z denote orthonormal basis elements. Ambiguous question.

Solution:

a) < p, q > = -3(0) +2(1) +1(5) = 7
b) < p, q > = 16(2) –12(0) +12(0) +3(-11) = -1

2

Determine if the matrix is orthogonal. If it is orthogonal, find the inverse.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1/\text{sq.root}(2) & 1/\text{sq.root}(2) \\ 0 & -1/\text{sq.root}(2) & 1/\text{sq.root}(2) \end{vmatrix}$$

Solution:

$$AA^T = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1/\text{sq.root}(2) & 1/\text{sq.root}(2) \\ 0 & -1/\text{sq.root}(2) & 1/\text{sq.root}(2) \end{vmatrix} * \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1/\text{sq.root}(2) & -1/\text{sq.root}(2) \\ 0 & 1/\text{sq.root}(2) & 1/\text{sq.root}(2) \end{vmatrix}$$

$$= \begin{vmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+1/2+1/2 & 0-1/2+1/2 \\ 0+0+0 & 0-1/2+1/2 & 0+1/2+1/2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

This is orthogonal, and the inverse is $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1/\text{sq.root}(2) & -1/\text{sq.root}(2) \\ 0 & 1/\text{sq.root}(2) & 1/\text{sq.root}(2) \end{vmatrix}$.

3

Find a matrix P that diagonalizes A.

$$A = \begin{vmatrix} 4 & 1 & 0 \\ 0 & 2 & 0 \\ 3 & 3 & 3 \end{vmatrix}$$

Solution:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & -1 & 0 \\ 0 & \lambda-2 & 0 \\ -3 & -3 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-4)(\lambda-2)(\lambda-3) = 0$$

$\lambda=4$:

$$\begin{vmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ -3 & -3 & 1 \end{vmatrix} \begin{array}{l} \\ \\ \text{Move to } 1^{st} \text{ row and divide by -3} \end{array}$$

$$\begin{vmatrix} 1 & 1 & -1/3 \\ 0 & -1 & 0 \\ 0 & 2 & 0 \end{vmatrix} \begin{array}{l} \text{Add } 2^{nd} \text{ row} \\ \text{Multiply by -1} \\ \text{Add 2 times } 2^{nd} \text{ row} \end{array}$$

$$\begin{vmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\mathbf{p}_1 = \begin{vmatrix} 1/3 \\ 0 \\ 1 \end{vmatrix} \propto \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$ – use this instead

$\lambda=2$:

$$\begin{vmatrix} -2 & -1 & 0 \\ 0 & 0 & 0 \\ -3 & -3 & -1 \end{vmatrix} \begin{array}{l} \text{Divide by -2} \\ \text{Switch with } 3^{rd} \text{ row} \\ \text{Add 3 times new } 1^{st} \text{ row} \end{array}$$

$$\begin{vmatrix} 1 & 1/2 & 0 \\ 0 & -3/2 & -1 \\ 0 & 0 & 0 \end{vmatrix} \begin{array}{l} \text{Add } -1/2 \text{ times new } 2^{nd} \text{ row} \\ \text{Multiply by } -2/3 \\ \end{array}$$

$$\begin{vmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\mathbf{p}_2 = \begin{vmatrix} 1/3 \\ -2/3 \\ 1 \end{vmatrix} \propto \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$ — use this instead — life much easier if you actually compute $P^{-1}AP$.

$\lambda=3$:

$$\begin{vmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ -3 & -3 & 0 \end{vmatrix}$$ Multiply by -1 (row 1); Add 3 times new 1st row

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$ Add –1 times 2nd row (row 1)

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\mathbf{p}_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$$

$$\mathbf{P} = \begin{vmatrix} 1/3 & 1/3 & 0 \\ 0 & -2/3 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

don't use bold face for components of vectors in $\mathbb{R}^2$.

4

Verify the Cauchy-Schwarz Inequality for the following inner product and vectors.

$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}_1\mathbf{v}_1 + 2\mathbf{u}_1\mathbf{v}_2 + 2\mathbf{u}_2\mathbf{v}_1 + \mathbf{u}_2\mathbf{v}_2$ — this is probably not an inner product!

$\langle \underline{u}, \underline{u} \rangle = u_1^2 + 4u_1u_2 + u_2^2$

$= (u_1+u_2)^2 + 2u_1u_2$

$\mathbf{u} = (-1,2), \mathbf{v} = (3,4)$.

Solution

So choose $\underline{u} = (1,-1)$ to get $\langle \underline{u}, \underline{u} \rangle = -2 < 0$ — not an inner product!

Cauchy-Schwarz is $\langle \mathbf{u}, \mathbf{v} \rangle \le \|\mathbf{u}\|\|\mathbf{v}\|$ first evaluating the inner product $\langle \mathbf{u}, \mathbf{v} \rangle$.

probably violated for some $\underline{u}$'s and $\underline{v}$'s since not an inner product.

$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}_1\mathbf{v}_1 + 2\mathbf{u}_1\mathbf{v}_2 + 2\mathbf{u}_2\mathbf{v}_1 + \mathbf{u}_2\mathbf{v}_2$

$= -1(3) + 2(-1)(4) + 2(2)(3) + (2)(4) = -3 + -8 + 12 + 8$

$\langle \mathbf{u}, \mathbf{v} \rangle = 9$

$\|\mathbf{u}\| = (u_1^2 + u_2^2)^{1/2}$

$= ((-1)^2 + 2^2)^{1/2} = (1+4)^{1/2}$

$\|\mathbf{u}\| = \sqrt{5}$

$\|\mathbf{v}\| = (v_1^2 + v_2^2)^{1/2}$

$= (3^2 + 4^2)^{1/2} = (9+16)^{1/2} = (25)^{1/2}$

$\|\mathbf{v}\| = 5$

$\|u\| = \langle u, u \rangle^{1/2} = (u_1^2 + 4u_1u_2 + u_2^2)^{1/2}$ $= ((-1)^2 + 4(-1)\cdot(2) + 2^2)^{1/2}$ $= (1 - 8 + 4)^{1/2} = (-3)^{1/2}$ ∄ … or real number! — used ‖ ‖ not associated w/ ⟨ , ⟩ — In C.S. ‖ ‖ and ⟨ , ⟩ must be related!

Now using these results we get the inequality to be:

$\langle \mathbf{u}, \mathbf{v} \rangle \le \|\mathbf{u}\|\|\mathbf{v}\| \Rightarrow 9 \le 5\sqrt{5}$, which if it is not obvious we can square both sides to get:
$81 \le 25(5) \Rightarrow 81 \le 125$, validating ~~the Cauchy-Schwarz~~ Inequality.

5

Consider the bases $B = \{u_1, u_2\}$ and $B' = \{v_1, v_2\}$ when
$u_1 = (1,-2)$, $u_2 = (2,1)$, $v_1 = (3,-1)$, $v_2 = (1,1)$.

a) Find the transition matrix from B to B'.

b) If $[\mathbf{w}]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ find $\mathbf{w}$ in respect to B'.

Solution

a) Finding the transition matrix. First start by writing v_1 and v_2 in terms of B.

$v_1 = a * u_1 + b * u_2$ and $v_2 = c * u_1 + d * u_2$ which are then re-written with the vectors in place:

$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = a\begin{bmatrix} 1 \\ -2 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c\begin{bmatrix} 1 \\ -2 \end{bmatrix} + d\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ which is code for the system of equations:

3= a+2b	and	1= c+2d
-1=-2a+b		1=-2c+d

Once again can be re-written as:

so here directly from *

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

if $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, then the det($A$)=5, and $A^{-1} = \begin{bmatrix} 1/5 & -2/5 \\ 2/5 & 1/5 \end{bmatrix}$.

Solving for a, b, c, d using the inverse of A.

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/5 & -2/5 \\ 2/5 & 1/5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1/5 & -2/5 \\ 2/5 & 1/5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 3/5 \end{bmatrix}.$$

Finally the transition matrix is $P_{BB'} = [(v_1)_B | (v_2)_B] = \begin{bmatrix} 1 & -1/5 \\ 1 & 3/5 \end{bmatrix}$.

b) $[\mathbf{w}]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ to find $[\mathbf{w}]_B$ in respect to B', simply multiply $[\mathbf{w}]_B$ by $P_{BB'}$.

$$\begin{bmatrix} 1 & -1/5 \\ 1 & 3/5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3+2/5 \\ 3-6/5 \end{bmatrix} = \begin{bmatrix} 17/5 \\ 9/5 \end{bmatrix}.$$

6

What are the eigenvalues of the following matrices

$$A = \begin{bmatrix} 3/2 & 0 & 0 & 0 \\ 4 & -1 & 0 & 0 \\ 2 & 3 & 1/5 & 0 \\ 3 & 8 & -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}$$

Solution

A) By inspection, because it is a lower triangular matrix, the eigenvalues are the entries of the main diagonal. So 3/2, -1, 1/5, and 3.

B) For B use that $\det(\lambda I - A) = 0$.

$\det(\lambda I - A) = \det \begin{bmatrix} \lambda+2 & -1 \\ 3 & \lambda-2 \end{bmatrix}$, the determinate of a 2x2 matrix is just ad-bc, so

$$\det \begin{bmatrix} \lambda+2 & -1 \\ 3 & \lambda-2 \end{bmatrix} = (\lambda+2)(\lambda-2) - (3(-1)) = \lambda^2 - 4 + 3 = \lambda^2 - 1.$$

$\lambda^2 - 1 = 0$. Solving for lamda, we get the eigenvalues to be ± 1.

7

Find the least squares solution of **Ax** = **b** given by

$$\begin{aligned} 3x_1 - x_2 &= 0 \\ x_1 + 3x_2 &= -1 \\ 6x_1 - 3x_2 &= -4 \end{aligned}$$

also find the orthogonal projection of b on the column space of A.

Solution:

To find a least square solution **Ax** = **b**, solve the normal solution $A^T A\mathbf{x} = A^T\mathbf{b}$

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 6 & -3 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} \quad A^T = \begin{bmatrix} 3 & 1 & 6 \\ -1 & 3 & -3 \end{bmatrix}$$

$$A^T A\mathbf{x} = \begin{bmatrix} 3 & 1 & 6 \\ -1 & 3 & -3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 46 & -18 \\ -18 & 19 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 3 & 1 & 6 \\ -1 & 3 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -23 \\ 15 \end{bmatrix}$$

Think about a way to engineer this better — it is possible.

$$\left[\begin{array}{cc|c} 46 & -18 & -23 \\ -18 & 19 & 15 \end{array}\right] \sim \left[\begin{array}{cc|c} 10 & 20 & 7 \\ -18 & 19 & 15 \end{array}\right] \sim \left[\begin{array}{cc|c} 10 & 20 & 7 \\ -18 & 19 & 15 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 7/10 \\ -18 & 19 & 15 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|c} 1 & 2 & 7/10 \\ -18 & 19 & 15 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 7/10 \\ 10 & 55 & 138/5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 7/10 \\ 10 & 1 & 138/275 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -167/550 \\ 0 & 1 & 138/275 \end{array}\right]$$

$x_1 = -167/550 \quad x_2 = 138/275$

To find the orthogonal projection of **b** on the column space of A use thm. 6.4.2 which states $\text{proj}_w\mathbf{b} = A\mathbf{x}$ if w is the column space of A and **x** is the least squares solution.

$$\text{proj}_{\mathbf{w}}\mathbf{b} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} -167/550 \\ 138/275 \end{bmatrix} = \begin{bmatrix} -777/550 \\ 661/550 \\ -183/55 \end{bmatrix}$$

8

Good type of problem, but could be engineered better – pick Q and R and, so, A indirectly.

Find the QR decomposition of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

Solution:

Column vectors of A:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

You must convert to an orthogonal basis, then normalize to make orthonormal, do this using Grahm-Schmidt.

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ orthogonal basis

$\mathbf{v}_1 = \mathbf{u}_1 = (1,2,1)$

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{w}}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2}\mathbf{v}_1$$

$\langle \mathbf{u}_2, \mathbf{v}_1 \rangle = 2 + 18 + 0 = 20$
$\|\mathbf{v}_1\| = \sqrt{1+4+1} = \sqrt{6}$

$v_2 = (2,9,0) - 20/6\,(1,2,1) = (-4/3, 7/3, -10/3)$ *– change to (-4,7,-10) – makes life easier.*

$$\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{\mathbf{w}2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2}\mathbf{v}_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2}v_2$$

$\langle \mathbf{u}_3, \mathbf{v}_1 \rangle = 3+6+4 = 13$
$\langle \mathbf{u}_3, \mathbf{v}_2 \rangle = -4+7-210/3 = -31/3$
$\|\mathbf{v}_2\| = \sqrt{16/9+45/9+100/9} = \sqrt{165}/3$

$\mathbf{v}_3 = (3,3,4) - 113/6(1,2,1) - \dfrac{-31/3}{165/9}(-4/3, 7/3, -10/3)$

change to $(9,-2,-5)$

$= (3,3,4) = (13/6, 26/6, 13/6) + 31/55(-4/3, 7/3, -10/3) = (9/110, -1/55, -1/22)$

$\mathbf{v}_1 = (1,2,1)$ $\mathbf{v}_2 = (-4/3, 7/3, -10,3)$ $\mathbf{v}_3 = (9/110, -1/55, -1/22)$

$\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ orthonormal basis

$\|\mathbf{v}_1\| = \sqrt{1+4+1} = \sqrt{6}$ $\|\mathbf{v}_2\| = \sqrt{(4/3)^2 + (7/3)^2 + (-10/3)^2} = \sqrt{165/3}$

$\|\mathbf{v}_3\| = \sqrt{(9/110)^2 + (-1/55)^2 + (-1/22)^2} = \sqrt{81/(110^2) + 4/(110^2) + 25/(110^2)}$

$\sqrt{110/(110^2)} = \sqrt{1/110}$

$\|\mathbf{v}_3\| = 1/\sqrt{110}$

$\mathbf{q}_1 = \mathbf{v}_1/\|\mathbf{v}_1\| = 1/\sqrt{6}(1,2,1) = (1/\sqrt{6}, 2/\sqrt{6}, 1/\sqrt{6})$

$\mathbf{q}_2 = \mathbf{v}_2/\|\mathbf{v}_2\| = 3/\sqrt{165}(-4/3, 7/3, -10/3) = (-12/(3\sqrt{165}), 21/(3\sqrt{165}), -30/(3\sqrt{165}))$

$\mathbf{q}_3 = \mathbf{v}_3/\|\mathbf{v}_3\| = \sqrt{110}(9/110, -2/110, -5/110) = (9/\sqrt{110}, -2/\sqrt{110}, -5/\sqrt{110})$

$Q = [\mathbf{q}_1|\mathbf{q}_2|\mathbf{q}_3]$ $R = \begin{bmatrix} <u_1,q_1> & <u_2,q_1> & <u_3,q_1> \\ 0 & <u_2,q_2> & <u_3,q_2> \\ 0 & 0 & <u_3,q_3> \end{bmatrix}$

$<\mathbf{u}_1,\mathbf{q}_1> = 6/\sqrt{6}$ $<\mathbf{u}_2,\mathbf{q}_1> = 20/\sqrt{6}$ $<\mathbf{u}_3,\mathbf{q}_1> = 13/\sqrt{6}$ $<\mathbf{u}_2,\mathbf{q}_2> = 55/\sqrt{165}$

$<\mathbf{u}_2,\mathbf{q}_2> = -31/\sqrt{165}$ $<\mathbf{u}_3,\mathbf{q}_3> = 1/\sqrt{110}$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$ $Q = \begin{bmatrix} 1/\sqrt{6} & -4/\sqrt{165} & 9/\sqrt{110} \\ 2/\sqrt{6} & 7/\sqrt{165} & -2/\sqrt{110} \\ 1/\sqrt{6} & -10/\sqrt{165} & -5/\sqrt{110} \end{bmatrix}$

$R = \begin{bmatrix} 6/\sqrt{6} & 20/\sqrt{6} & 13/\sqrt{6} \\ 0 & 55/\sqrt{165} & -31/\sqrt{165} \\ 0 & 0 & 1/\sqrt{110} \end{bmatrix}$

9

Given the matrices

$A = \begin{bmatrix} 1 & 3 \\ 5 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$

a. Determine the inner product of <A,B>, <B,C>, and <A,C>.

b. Determine the norm of A, B, and C.

Again, this only makes sense to ask if you indicate what <,> is.

Solution:

a.

<A,B> = 1(4)+3(3)+5(-1)+0(2) = 8
<B,C> = 4(1)+3(2)+(-1)0+2(-3) = 4
<A,C> = 1(1)+3(2)+5(0)+0(-3) = 7

b.

$$||A|| = \langle A,A\rangle^{1/2} = \sqrt{1^2+3^2+5^2+0^2} = \sqrt{35}$$
$$||B|| = \langle B,B\rangle^{1/2} = \sqrt{4^2+3^2+(-1)^2+2^2} = \sqrt{30}$$
$$||C|| = \langle C,C\rangle^{1/2} = \sqrt{1^2+2^2+0^2+(-3)^2} = \sqrt{14}$$

10

Find an orthogonal matrix P that diagonalizes $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Solution:

Find eigen values of A

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda-1 \end{bmatrix} = (\lambda-1)^3 - (\lambda-1) = 0$$

$(\lambda-1)(\lambda-1)(\lambda-1)-(\lambda-1) = 0$

$(\lambda^2-2\lambda+1)(\lambda-1)-(\lambda-1) = 0$

$\lambda^3-2\lambda^2+\lambda-\lambda+2\lambda-1-(\lambda-1) = 0$

$\lambda^3-3\lambda^2+3\lambda-1-\lambda+1 = 0$

why do this when [this] is really obviously factorable?

$(\lambda-1)^3-(\lambda-1) = (\lambda-1)[(\lambda-1)^2-1]$
$= (\lambda-1)(\lambda^2-2\lambda)$
$= \lambda(\lambda-1)(\lambda-2)$

$\lambda^3 - 3\lambda^2 + 2\lambda = 0$

$\lambda(\lambda^2 - 3\lambda + 2) = 0$

$\lambda(\lambda-2)(\lambda-1) = 0$

$\lambda = 0,2,1$

Find eigen vectors for all values of λ

Nul $\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}$

$\lambda = 0$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = s \;\; x_2 = t \;\; x_3 = -s$ — where does this come from?

$$\mathbf{x} = \begin{bmatrix} s \\ t \\ -s \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ -s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$$

$$\mathbf{x} = s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

— can't be 2 dimensional — this is wrong

$\lambda = 1$

?, There has to be one!

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ **Not a basis, not an eigen space**

see?!

Nul $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$.

$\lambda = 2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = s \;\; x_2 = t \;\; x_3 = s$ — where does this come from?

Nul $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$

$$\mathbf{x} = \begin{bmatrix} s \\ t \\ s \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Basis for eigen space corresponding to $\lambda = 0$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- no way!

basis for eigen space corresponding to $\lambda = 2$

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{u}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- no way!

Can't have 4 vectors for Eigenspaces of 3x3 matrix!!!

$\mathbf{u}_2$ and $\mathbf{u}_4$ overlap so we use Grahm-Schmidt to normalize $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

try again.

with $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as column vectors we obtain

$$\mathbf{P} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$ which orthagonally diagonalizes A.

1. Find the least squares solution to this system of equations (15 pts.)

$$\begin{aligned} 2x_1 - 4x_2 + 4x_3 &= 12 \\ 2x_1 - x_2 &= 0 \\ x_2 - x_3 &= 6 \\ 4x_1 - 2x_3 &= 0 \end{aligned}$$

Solution

This system can be represented in matrix form as

$$\begin{bmatrix} 2 & -4 & 4 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \\ 4 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 6 \\ 0 \end{bmatrix}$$

Where $A = \begin{bmatrix} 2 & -4 & 4 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \\ 4 & 0 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 12 \\ 0 \\ 6 \\ 0 \end{bmatrix}$.

From Theorem 6.4.2 we know that for any system, A**x** = **b**, the system $A^T A\mathbf{x} = A^T\mathbf{b}$ is consistent and is the least squares solution to A**x** = **b**.

So

$$\begin{bmatrix} 2 & 2 & 0 & 4 \\ -4 & -1 & 1 & 0 \\ 4 & 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & -4 & 4 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \\ 4 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 & 4 \\ -4 & -1 & 1 & 0 \\ 4 & 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} 12 \\ 0 \\ 6 \\ 0 \end{bmatrix}$$

simplifies to

$$\begin{bmatrix} 24 & -10 & 0 \\ -10 & 18 & -17 \\ 0 & -17 & 21 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ -42 \\ 42 \end{bmatrix}$$

which has the augmented matrix

$$\left[\begin{array}{ccc|c} 24 & -10 & 0 & 24 \\ -10 & 18 & -17 & -42 \\ 0 & -17 & 21 & 42 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -5/12 & 0 & 1 \\ -10 & 18 & -17 & -42 \\ 0 & 1 & -21/17 & -42/17 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 1 & -5/12 & 0 & 1 \\ 0 & 83/6 & -17 & -32 \\ 0 & 1 & -21/17 & -42/17 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -5/12 & 0 & 1 \\ 0 & 0 & 9/17 & 222/17 \\ 0 & 1 & -21/17 & -42/17 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 12 & -5 & 0 & 12 \\ 0 & 17 & -21 & -42 \\ 0 & 0 & 9 & 222 \end{array}\right] \quad \text{so} \quad \begin{array}{l} x_1 = 38/3 \\ x_2 = 28 \\ x_3 = 74/3 \end{array} \quad \text{is a least squares solution to } A\mathbf{x}=\mathbf{b}.$$

— try to engineer better — pick solutions and A, then find b.

2. Find the transition matrices $P_{B'B}$ and $P_{BB'}$ for $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$, which are bases for vector space V, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{u}'_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \text{ and } \mathbf{u}'_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}. \text{ (15 pts.)}$$

Solution

A transition matrix $P_{BB'}$ is a matrix such that $P_{BB'}[v]_B = [v]_{B'}$ where $[v]_B$, $[v]_{B'}$ are the coordinate vectors of $v \in V$ with respect to B and B' respectively. By theorem 5.4.1 any arbitrary vector in V can be expressed as a linear combination of any basis for V. That is $\mathbf{v} = [\mathbf{u}_1|\mathbf{u}_2|\ldots|\mathbf{u}_n]\,[v]_B = [\mathbf{u}'_1|\mathbf{u}'_2|\ldots|\mathbf{u}'_n]\,[v]_{B'}$. So

$$\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}[v]_B = \begin{bmatrix} 1 & 0 \\ 4 & -2 \end{bmatrix}[v]_{B'}.$$

The inverse of $\begin{bmatrix} 1 & 0 \\ 4 & -2 \end{bmatrix}$ can be found by the following process

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 4 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & -2 & -4 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1/2 \end{array}\right]$$

So

$$\begin{bmatrix} 1 & 0 \\ 2 & -1/2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}[v]_B = \begin{bmatrix} 1 & 0 \\ 2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & -2 \end{bmatrix}[v]_{B'}$$

$$\begin{bmatrix} 2 & -1 \\ 5/2 & -2 \end{bmatrix}[v]_B = [v]_{B'}$$

So $P_{BB'} = \begin{bmatrix} 2 & -1 \\ 5/2 & -2 \end{bmatrix}$.

By theorem 6.5.1 $P^{-1}{}_{BB'} = P_{B'B}$. $P^{-1}{}_{BB'}$ can be found by the following process.

$$\left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 5/2 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1/2 & 1/2 & 0 \\ 5 & -4 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 0 & -3 & -5 & 4 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 8/6 & -4/6 \\ 0 & 1 & 5/3 & -4/3 \end{array}\right]$$

$\therefore P_{B'B} = \begin{bmatrix} 8/6 & -4/6 \\ 5/3 & -4/3 \end{bmatrix}$.

3. If A is an orthogonal matrix, prove the following.
   a) A^{-1} is orthogonal (5 pts.)
   b) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in R^n$ (5 pts.)
   c) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in R^n$ (5 pts.)

Solution

a) A matrix is orthogonal if $A^{-1} = A^T$ (*) by definition. So it follows that $(A^{-1})^T = (A^T)^{-1} = (A^{-1})^{-1}$ *= A, which is (*) with A replaced by A^{-1}.*

b) Because $A^{-1} = A^T$ it follows that $AA^T = A^T A = I$.
So $\|A\mathbf{x}\| = (A\mathbf{x} \cdot A\mathbf{x})^{1/2} = (\mathbf{x} \cdot A^T A\mathbf{x})^{1/2} = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \|\mathbf{x}\|$. *assuming result?*

c) Since $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in R^n$ and from theorem 4.1.6 $A\mathbf{x} \cdot A\mathbf{y}$ ~~= x·y~~ can be rewritten as

$$1/4\|A\mathbf{x} + A\mathbf{y}\|^2 - 1/4\|A\mathbf{x} - A\mathbf{y}\|^2 = 1/4\|A(\mathbf{x}+\mathbf{y})\|^2 - 1/4\|A(\mathbf{x}-\mathbf{y})\|^2$$
$$= 1/4\|(\mathbf{x}+\mathbf{y})\|^2 - 1/4\|(\mathbf{x}-\mathbf{y})\|^2 = \mathbf{x} \cdot \mathbf{y}.$$
using b)

4. Find the eigenvectors of the following matrix:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \text{ (15 pts.)}$$

Solution

Solve the equation det(λI – A) = 0. Because A is an upper-triangular matrix, only the trace needs to be calculated for the matrix: *not the trace — the determinant, $Tr(\lambda I - A) = \lambda - 2 + \lambda + 1 + \lambda + 1 \neq (\lambda-2)(\lambda+1)^2$*

$$B = \begin{bmatrix} \lambda - 2 & -3 & -4 \\ 0 & \lambda + 1 & -3 \\ 0 & 0 & \lambda + 1 \end{bmatrix}$$

which is equals the equation (λ-2)(λ+1)(λ+1) = 0. Solving that equation shows that the determinant equals 0 when λ = -1 and 2 (there are two eigenvalues at λ = -1). Plugging those values back into the matrix B yields:

$$B = \begin{bmatrix} 0 & -3 & -4 \\ 0 & 3 & -3 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & -3 & -4 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Solving BX = 0 will give the eigenvectors for A.

$$\begin{bmatrix} 0 & -3 & -4 \\ 0 & 3 & -3 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -3 & -4 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving those systems of equations yields eigenvectors:

? this is not an eigenvector

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

5 Determine if there is a matrix P that diagonalizes the matrix A and if so, find P.

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \text{(15 pts.)}$$

Solution

In the preceding problem, we determined that the eigenvectors for the matrix A are:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

These vectors are linearly dependent therefore there isn't a matrix P that diagonalizes the matrix A. – irrelevant

don't be attached to notation – what is "D"?

6. Calculate a matrix P that orthogonally diagonalizes matrix A and then find "D" if possible. (15 pts.)

$$A = \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix}$$

Solution

Find the eigenvectors of A by solving the equation det(λI – A) = 0. The solutions of

$$\det\left(\begin{bmatrix} \lambda-9 & -1 \\ -1 & \lambda-9 \end{bmatrix}\right) = 0$$

are where λ = 10 and 8. Plugging those eigenvalues into the matrix above creates the two matrices:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

Solving BX = 0 will give the eigenvectors:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

The two eigenvectors of A are linearly independent and therefore it is possible to find a matrix P to orthogonally diagonalizes A and "it" forms a basis of A.

— absolutely & entirely irrelevant!

To continue the process of finding a matrix P, we must apply the Gram-Schmidt process on the basis formed by the eigenvectors.

— only relevant when you find 2 or more eigenvectors for a single (repeated) eigenvalue!

u_1 = <1/2^.5, 1/2^.5> ; u_2 = <-1/2^.5, 1/2^.5>

Step 1:

$v_1 = u_1$ = <1/2^.5, 1/2^.5>

Step 2:

$v_2 = u_2 - \langle u_2, v_1 \rangle v_1 / \|v_1\|^2$

v_2 = <-1/2^.5, 1/2^.5> - <1/2^.5, 1/2^.5> * 0 = <-1/2^.5, 1/2^.5>

of course!

Finally, we must form the matrix P with the columns v_1, v_2 giving us

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Now to perform the orthogonal diagonalization, we must perform the operation $P^{-1}AP = D$.

$$P^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 8 \end{bmatrix}$$

7. Define the 4 axioms that must be satisfied for an association between 2 vector spaces to be considered an inner product. (10 pts.)

Solution

A. <**u**,**v**> = <**v**,**u**>	Symmetry
B. <**u** + **v**,**z**> = <**u**,**z**> + <**v**,**z**>	Additivity
C. <k**u**,**v**> = k<**u**,**v**>	Homogeneity
D. <**v**,**v**>≥ 0 and <**v**,**v**> = 0 if and only if **v** = 0	Positivity

8. Let M_{22} have the inner product

$$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \text{ and } V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

$$\langle U, V \rangle = tr(U^T V) = tr(V^T U) = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

Find the cosine of the angle between A and B. (15 pts.)

$$A = \begin{bmatrix} 2 & 9 \\ 6 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 3 \\ 5 & -4 \end{bmatrix}$$

Solution

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\|\|v\|} \text{ and } 0 \le \theta \le \pi$$

$$= \frac{(2*1 + 9*3 + 6*5 + 1*(-4))}{\sqrt{2^2 + 9^2 + 6^2 + 1^2}\sqrt{1^2 + 3^2 + 5^2 + (-4)^2}}$$

$$= \frac{55}{\sqrt{122}\sqrt{51}}$$

$$\cos\theta = \frac{55}{\sqrt{6222}}$$

9. Let R^3 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into an orthonormal basis. (20 pts.)

$u_1 = (1, -1, 0)$, $u_2 = (0, -3, 1)$, $u_3 = (-1, 0, 2)$

Solution

$v_1 = u_1 = (1, -1, 0)$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = (0,-3,1) - \frac{(0+3+0)}{(1^2+(-1)^2+0^2)}(1,-1,0)$$

$$v_2 = (-\tfrac{3}{2}, -\tfrac{3}{2}, 1)$$ – change to $(-3,-3,2)$ or $(3,3,-2)$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2 \qquad v_3 = (\tfrac{16}{11}, \tfrac{5}{11}, \tfrac{15}{11})$$ change to $(16,5,15)$

$$\|v_1\| = \sqrt{2}, \ \|v_2\| = \frac{11}{\sqrt{22}}, \ \|v_3\| = \frac{46}{\sqrt{506}}$$

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{3\sqrt{22}}{22}, -\frac{3\sqrt{22}}{22}, \frac{\sqrt{22}}{11}\right)$$

$$= (-1,0,2) - \frac{(-1+0+0)}{(1^2+(-1)^2+0^2)}(1,-1,0) - \frac{(3/2+0+2)}{((-3/2)^2+(-3/2)^2+1^2)}(-\tfrac{3}{2}, -\tfrac{3}{2}, 1)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \left(\frac{8\sqrt{506}}{253}, \frac{5\sqrt{506}}{506}, \frac{15\sqrt{506}}{506}\right)$$

– inconclusive!!!! if $\det Q = \pm 1$

10. Determine if the following matrix is orthogonal by taking its determinant:

$$Q = \begin{bmatrix} 1 & 2 & 4 \\ 7 & 6 & 5 \\ 2 & 2 & 2 \end{bmatrix} \quad \text{(15 pts.)}$$

not so – converse true.

Solution

Q is orthogonal if the determinant of Q is 1 or -1. Since

for example, choose $Q = \begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}$.

$$|Q| = \begin{vmatrix} 1 & 2 & 4 \\ 7 & 6 & 5 \\ 2 & 2 & 2 \end{vmatrix} = 1\begin{vmatrix} 6 & 5 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 7 & 5 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 7 & 6 \\ 2 & 2 \end{vmatrix} = 1(2) - 2(4) + 4(2) = 2 \neq 1, -1$$

matrix Q is not orthogonal. good use of converse (finally).

1) What criteria does an orthonormal set of vectors have to meet (5 points)?

Solution:

1) Each vector in the set has to be orthogonal to one another (their ~~Euclidean~~ Inner Product must be zero).
2) Each vector in the set must have a norm of one.

2) Determine the eigenvalues of matrix A (10 points):

$$A = \begin{pmatrix} -1 & 7 \\ 3 & 3 \end{pmatrix}$$

Solution:

$$\det(\lambda I - A) = \det\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} -1 & 7 \\ 3 & 3 \end{pmatrix} \right)$$
$$= \det\begin{pmatrix} \lambda+1 & -7 \\ -3 & \lambda-3 \end{pmatrix}$$
$$= (\lambda+1)(\lambda-3) - (-3)(-7)$$
$$= (\lambda^2) - 2\lambda - 3 - 21$$
$$= (\lambda^2) - 2\lambda - 24$$
$$= (\lambda-6)(\lambda+4)$$

λ = -4, 6 (eigenvalues) — yes/no question

3) Show whether matrix B is diagonalizable or not (20 points).

$$B = \begin{pmatrix} 1 & -6 \\ -6 & 1 \end{pmatrix}$$ — symmetric matrices are always diagonalizable — done!

Solution:

$$\det(\lambda I - B) = \det\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & -6 \\ -6 & 1 \end{pmatrix} \right)$$
$$= \det\begin{pmatrix} \lambda-1 & 6 \\ 6 & \lambda-1 \end{pmatrix}$$
$$= (\lambda-1)(\lambda-1) - (6)(6)$$
$$= (\lambda^2) - 2\lambda + 1 - 36$$
$$= (\lambda^2) - 2\lambda - 35$$
$$= (\lambda-7)(\lambda+5)$$

λ = 7, -5 (eigenvalues) — no need!

Now substitute the eigenvalues into the equation ($\lambda I - B$) and row reduce to obtain the dimension (the number of basis vectors). Note that "R2" is referring to "row 2" during row reduction.

$$(7I - B) = \left(\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} - \begin{pmatrix} 1 & -6 \\ -6 & 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix} \quad (R2) - 6(R1),\ (R1) * (1/6)$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ Rank = 1, Nullity = 1, Dimension = 1 (1 basis vector)}$$

$$(-5I - B) = \left(\begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix} - \begin{pmatrix} 1 & -6 \\ -6 & 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} -6 & 6 \\ 6 & -6 \end{pmatrix} \quad (R2) + 6(R1),\ (R1) * (-1/6)$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \text{ Rank = 1, Nullity = 1, Dimension = 1(1 basis vector)}$$

Since each of the two eigenvalues have a basis vector, the matrix is diagonalizable. – guaranteed since A nxn had n distinct eigenvalues – you worked too hard.

4) Determine the norm (length) of each vector and the distances between them using the Euclidean inner product (15 points).

u = (4, 5, 6)
v = (7, 3, 9)
w = (8, 1, 0)

Solution:

norm u = ((4^2) + (5^2) + (6^2)) ^ .5
= (16 + 25 + 36) ^ .5
= **(77) ^ .5**
norm v = ((7^2) + (3^2) + (9^2)) ^ .5
= (49 + 9 + 81) ^ .5
= **(139) ^ .5**
norm w = ((8^2) + (1^2) + (0^2)) ^ .5
= (64 + 1 + 0) ^ .5
= **(65) ^ .5**

distance between u and v = ((7-4)^2 + (3-5)^2 + (9-6)^2) ^.5
= ((3)^2 + (-2)^2 + (3)^2) ^.5
= (9 + 4 + 9) ^.5
= **(22) ^.5**
distance between u and w = ((8-4)^2 + (1-5)^2 + (0-6)^2) ^.5
= ((4)^2 + (-4)^2 + (-6)^2) ^.5
= (16 + 16 + 36) ^.5
= **2 * ((17) ^.5)**

distance between v and w = ((8-7)^2 + (1-3)^2 + (0-9)^2) ^.5
= ((1)^2 + (-2)^2 + (-9)^2) ^.5
= (1 + 4 + 81) ^.5
= **(86) ^.5**

5) Calculate the Euclidean inner product <u,v>

a. u = [1 3] v = [3 2]
   [-6 2] [-1 2] *– this does not have a (standard) Euclidean inner product – not in $\mathbb{R}^n$.*

b. Calculate the inner product <p,q>
   p= 6 + x + 3x^2 *&* q= -1 + 4x – 3x^2 *(what is the inner product? Define it!)*

Solution:

a. <u,v> = (3 + 6 + 6 +8) = 23
b. <p,q> = (-6 + 4 -9) = -11

6) Which vectors and matrices are orthogonal with the Euclidean inner product?

a. u = (6, 3, 9) v = (-2, -2, 2)
b. A = [1 3 1]
   [0 1 0]
   [2 6 1]
c. u = (1, 2, -4) v = (3, 2, 2)

Solution:

a. <u,v> = -12 – 6 + 18 = 0
Because the inner product is 0, it is orthogonal
b. det (A) = (1 – 2) = -1
Because the determinate is 1 or -1, it is orthogonal *– not so, e.g. A = [1 5; 0 -1] is not orthogonal! (and neither is yours!)*
c. <u,v> = (3 + 4 – 8) = -1
Because the inner product is not 0, it is *(they are)* NOT orthogonal

7) Consider the vector space R3 with the Euclidean inner product. Use the Gram-Schmidt process to transform the basis vectors u1 = (2, 3, 1), u2 = (0, 1, 0), u3 = (-2, 1, 3) into an orthogonal basis {v1,v2,v3}; then normalize the orthogonal basis vectors to obtain an orthonormal basis {q1,q2,q3}.

Solution:

v1 = u1 = (2, 3, 1)

$$v2 = u2 - \frac{\langle u2,v1\rangle v1}{\|v1\|^2}$$

= (0, 1, 0) – 3/14(2, 3, 1) = (-3/7, 5/14, -3/14) *– use (-6,5,-3) or (6,-5,3)*

$$v3 = u3 - \frac{\langle u3,v1\rangle v1}{\|v1\|^2} - \frac{\langle u3,v2\rangle v2}{\|v2\|^2}$$

= (-2, 1, 3) – 1/7(2, 3, 1) – 8/5(-3/7, 5/14, -3/14)
= (-8/5, 0, 16/5) *use ~~(-56, 0, 16)~~ (-1, 0, 2) or (1, 0, -2).*

These vectors form an orthogonal basis for R3:
$v1 = (2, 3, 1)$ $v2 = (-3/7, 5/14, -3/14)$ $v3 = (-8/5, 0, 16/5)$

The norms of these vectors are:
$\|v1\| = \sqrt{14}$ $\|v2\| = \sqrt{5/14}$ $\|v3\| = 8/\sqrt{5}$

So an orthonormal basis for R3 is:

$$q1 = \frac{v1}{\|v1\|} = (2/\sqrt{14},\ 3/\sqrt{14},\ 1/\sqrt{14})$$

$$q2 = \frac{v2}{\|v2\|} = \left(\frac{-3\sqrt{14}}{7\sqrt{5}},\ \frac{5\sqrt{14}}{14\sqrt{5}},\ \frac{-3\sqrt{14}}{14\sqrt{5}} \right)$$

$$q3 = \frac{v3}{\|v3\|} = (1/\sqrt{5},\ 0,\ 2/\sqrt{5})$$

– messy because you didn't simplify before (but you could simplify now also)

8) Given the Matrix

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{bmatrix}$$

Find:

a) The characteristic Equation.
b) The eigenvalue(s).
c) Find the eigenvector(s).

Solution:
The characteristic equation can be found by the equation $\det[\lambda I - A] = 0$.

From this we have

$$\det\left[\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{bmatrix}\right] = 0$$

$$\Rightarrow \det\begin{bmatrix} \lambda-5 & 0 & -1 \\ -1 & \lambda-1 & 0 \\ 7 & -1 & \lambda \end{bmatrix} = 0$$

$$\Rightarrow (\lambda-5)(\lambda-1)(\lambda) - 1 - [-7(\lambda-1)] = 0$$

$$\Rightarrow (\lambda^2 - 5\lambda - 1\lambda + 5)(\lambda) + 7\lambda - 7 - 1 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

Which is the characteristic equation for the matrix A.

b) The characteristic equation can be factored as follows

$(\lambda - 2)^3 = 0$

$\Rightarrow \lambda = 2$

The eigenvalue for matrix A is thus.

c) The eigenvector is found by substituting λ=2 into the equation

$[\lambda I - A][x] = 0$

Which is done as follows

$$\begin{bmatrix} 2-5 & 0 & -1 \\ -1 & 2-1 & 0 \\ 7 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 0 & -1 \\ -1 & 1 & 0 \\ 7 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From which we form the augmented matrix to solve for x_1, x_2 and x_3 respectively:

$$\left[\begin{array}{ccc|c} -3 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 7 & -1 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} -3 & 0 & -1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & -3 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} -3 & 0 & -1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Which is code for

$$\begin{aligned} -3x_1 - x_3 &= 0 \\ -3x_2 - x_3 &= 0 \Rightarrow \\ x_3 &= t \end{aligned} \qquad \begin{aligned} x_1 &= \frac{-1}{3}x_3 = \frac{-1}{3}t \\ x_2 &= \frac{-1}{3}x_3 = \frac{-1}{3}t \\ x_3 &= t \end{aligned}$$

Which can be written as

$$t\begin{bmatrix} -1/3 \\ -1/3 \\ 1 \end{bmatrix}$$ thus the eigenvector for λ=2 is $$\begin{bmatrix} -1/3 \\ -1/3 \\ 1 \end{bmatrix}$$ ✓ use $\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$

9) Let

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$$

a) Find the eigenvalues of A.
b) Find the geometric and algebraic multiplicity of each eigenvalue.
c) Tell whether A is diagonalizable or not. Justify your answer.

Solution:

a) To find the eigenvalues of A we begin by finding the characteristic equation of A as such:

$$\det[\lambda I - A] = 0 \Rightarrow \det\left[\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}\right] = \det\begin{bmatrix} \lambda+1 & 0 & -1 \\ 1 & \lambda-3 & 0 \\ 4 & -13 & \lambda+1 \end{bmatrix} = 0$$

$$\Rightarrow (\lambda+1)(\lambda+1)(\lambda-3)+1-[-4(\lambda-3)]=0$$
$$\Rightarrow (\lambda^2-2\lambda+1)(\lambda-3)+4\lambda-12+1=0$$
$$\Rightarrow \lambda^3-3\lambda^2+2\lambda^2-6\lambda+\lambda+4\lambda-12+1-3=0$$
$$\Rightarrow \lambda^3-\lambda^2-\lambda-14=0$$

$(-3)^3 = -14?$ I don't think so.

Which factors as $(\lambda-3)^3=0 \Rightarrow \lambda=3$

b) Because there is only one eigenvalue from a 3x3 matrix its algebraic multiplicity is 3.

The geometric multiplicity can be found by substituting λ=3 into the matrix [λ I-A] and finding the nullspace.

Which is done by

$$\begin{bmatrix} 2+1 & 0 & -1 \\ 1 & 2-3 & 0 \\ 4 & -13 & 2+1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -1 \\ 1 & -1 & 0 \\ 4 & -13 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -1 \\ 0 & -3 & -1 \\ 0 & -39 & 13 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

The null~~space~~ity of the above matrix ~~is defined as~~ turns out to be the number of rows without a leading 1, which is the third column, so the null~~space~~ity is 1.

The geometric multiplicity is then also 1.

d) By theorem if the geometric multiplicity of A does not equal the algebraic multiplicity of A, then the matrix a is not diagonalizable.

10) Let $\mathbb{R}^3$ have the Euclidean inner product, and let *W* be the subspace spanned by the orthonormal vectors $v_1=(0,1,0)$ and $v_2=(-\frac{4}{5}, 0, \frac{3}{5})$. Find the orthogonal projection of $u=(1,1,1)$ onto *W*.

Solution:

$$\begin{aligned}
proj_W \boldsymbol{u} &= \langle \boldsymbol{u}, \boldsymbol{v}_1 \rangle \boldsymbol{v}_1 + \langle \boldsymbol{u}, \boldsymbol{v}_2 \rangle \boldsymbol{v}_2 \\
&= \langle (1,1,1), (0,1,0) \rangle (0,1,0) + \langle (1,1,1), (-\tfrac{4}{5}, 0, \tfrac{3}{5}) \rangle (-\tfrac{4}{5}, 0, \tfrac{3}{5}) \\
&= ((1)(0) + (1)(1) + (1)(0))(0,1,0) \\
&\quad + ((1)(-\tfrac{4}{5}) + (1)(0) + (1)(\tfrac{3}{5}))(\tfrac{4}{25,} 1, -\tfrac{3}{25}) \\
&= (1)(0,1,0) + (-\tfrac{1}{5})(-\tfrac{4}{5}, 0, \tfrac{3}{5}) \\
&= (\tfrac{4}{25,} 1, -\tfrac{3}{25})
\end{aligned}$$

- since vectors are orthonormal! Say this!

Why? Because this formula doesn't hold in just "any old case."

8

343 Exam Key   Midterm 3

1. If **u** and **v** are vectors in a real inner product space and k is any scalar, prove $<\mathbf{u}, k\mathbf{v}> = k<\mathbf{u},\mathbf{v}>$. (10 points)

Solution:

$$
\begin{aligned}
<\mathbf{u}, k\mathbf{v}> &= <k\mathbf{v},\mathbf{u}> && \text{[By symmetry]}\\
&= k<\mathbf{v},\mathbf{u}> && \text{[By homogenity]}\\
&= k<\mathbf{u},\mathbf{v}> && \text{[By symmetry]}\ .
\end{aligned}
$$

2. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Determine if $<\mathbf{u},\mathbf{v}> = 3u_1 v_1 + u_2 v_2 + 6u_3 v_3$ is an inner product ~~space~~ on R^3. If it is not, state the axioms that fail. (15 points)

Solution:
In order to be an inner product ~~space~~ the four inner product axioms must be satisfied.

Axiom 1

$<\mathbf{u},\mathbf{v}> = 3u_1 v_1 + u_2 v_2 + 6u_3 v_3 = 3v_1 u_1 + v_2 u_2 + 6v_3 u_3 = <\mathbf{v},\mathbf{u}>$

Axiom 2

Let $\mathbf{z} = (z_1, z_2, z_3)$.

$$
\begin{aligned}
<\mathbf{u}+\mathbf{v},\mathbf{z}> &= 3(u_1+v_1)z_1 + (u_2+v_2)z_2 + 6(u_3+v_3)z_3\\
&= 3u_1 z_1 + 3v_1 z_1 + u_2 z_2 + v_2 z_2 + 6u_3 z_3 + 6v_3 z_3\\
&= (3u_1 z_1 + u_2 z_2 + 6u_3 z_3) + (3v_1 z_1 + v_2 z_2 + 6v_3 z_3)\\
&= <\mathbf{u},\mathbf{z}> + <\mathbf{v},\mathbf{z}>
\end{aligned}
$$

Axiom 3

$$
\begin{aligned}
<k\mathbf{u},\mathbf{v}> &= 3(ku_1) v_1 + (ku_2) v_2 + 6(ku_3) v_3\\
&= k3u_1 v_1 + ku_2 v_2 + k6u_3 v_3\\
&= k(3u_1 v_1 + u_2 v_2 + 6u_3 v_3)\\
&= k<\mathbf{u},\mathbf{v}>
\end{aligned}
$$

Axiom 4

$<\mathbf{v},\mathbf{v}> = 3v_1^2 + v_2^2 + 6v_3^2 \geq 0$ for all values of v_1, v_2, and v_3.
$<\mathbf{v},\mathbf{v}> = 3v_1^2 + v_2^2 + 6v_3^2$ is only equal to 0 if each term is equal to zero thus $\mathbf{v} = \mathbf{0}$.

All four axioms are satisfied and thus it is an inner product ~~space.~~

3. Let W be the subspace of R^5 spanned by the vectors $\mathbf{w}_1 = (1, 0, 1, 0, 1)$, $\mathbf{w}_2 = (4, 3, 4, 0, 7)$, $\mathbf{w}_3 = (0, 0, 0, 2, -2)$, and $\mathbf{w}_4 = (8, 6, 8, 0, 14)$. Find a basis for the orthogonal complement of W. (15 points)

Solution:

The space W spanned by $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ is the same as the row space of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 4 & 3 & 4 & 0 & 7 \\ 0 & 0 & 0 & 2 & -2 \\ 8 & 6 & 8 & 0 & 14 \end{bmatrix}$$

The nullspace of A is the orthogonal complement of W. To find the nullspace we can create an augmented matrix for $A\mathbf{x} = \mathbf{0}$ and row reduce.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 3 & 4 & 0 & 7 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 \\ 8 & 6 & 8 & 0 & 14 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 \\ 8 & 6 & 8 & 0 & 14 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 6 & 0 & 0 & 6 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

these aren't ⊥ – something's wrong.

Which is code for $x_1 + x_3 + x_5 = 0$, $x_2 + x_5 = 0$, and $x_4 - x_5 = 0$. Setting $x_5 = t$ and $x_3 = s \Rightarrow$

$x_1 = -s - t$
$x_2 = -t$
$x_3 = s$
$x_4 = -t$
$x_5 = t$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s-t \\ -t \\ s \\ -t \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

$\Rightarrow$ The vectors v1 = (-1, 0, 1, 0, 0) and v2 = (-1, -1, 0, -1, -1) form a basis for the nullspace of A and thus are also a basis for the orthogonal complement of W.

Good Problem

4. Determine whether or not the following set of matrices is orthonormal with respect to the inner product on M_{22} where $\langle U,V\rangle = tr(U^TV) = tr(V^TU) = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$.
(15 points)

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -2/3 & 1/3 \\ 0 & 2/3 \end{bmatrix}, \begin{bmatrix} 1/3 & -2/3 \\ 0 & 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 & 2/3 \\ 0 & 1/3 \end{bmatrix}$$

Solution:
First, in order for this set to be orthonormal, it must be an orthogonal set, se we must show that the inner product for all combinations is zero.

For simplicity, ~~let~~ write A = $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, B = $\begin{bmatrix} -2/3 & 1/3 \\ 0 & 2/3 \end{bmatrix}$, C = $\begin{bmatrix} 1/3 & -2/3 \\ 0 & 2/3 \end{bmatrix}$ and D = $\begin{bmatrix} 2/3 & 2/3 \\ 0 & 1/3 \end{bmatrix}$.

$\langle A,B\rangle$ = 0*-2/3 + 0*1/3 + 1*0 + 0*2/3 = 0 + 0 + 0 + 0 = 0
$\langle A,C\rangle$ = 0*1/3 + 0*-2/3 + 1*0 + 0*2/3 = 0 + 0 + 0 + 0 = 0
$\langle A,D\rangle$ = 0*2/3 + 0*2/3 + 1*0 + 0*1/3 = 0 + 0 + 0 + 0 = 0
$\langle B,C\rangle$ = -2/3*1/3 + 1/3*-2/3 + 0*0 + 2/3*2/3 = -2/9 + -2/9 + 4/9 = 0
$\langle B,D\rangle$ = -2/3*2/3 + 1/3*2/3 + 0*0 + 2/3*1/3 = -4/9 + 2/9 + 2/9 = 0
$\langle C,D\rangle$ = 1/3*2/3 + -2/3*2/3 + 0*0 + 2/3*1/3 = 2/9 + -4/9 + 2/9 = 0

Because the inner product of each combination is 0, we know that the set is orthogonal. Now, we must show that the norm of each matrix is 1.

$\|A\| = \langle A,A\rangle^{1/2} = (0*0 + 0*0 + 1*1 + 0*0)^{1/2} = (1)^{1/2} = 1$
$\|B\| = \langle B,B\rangle^{1/2} = (-2/3*-2/3 + 1/3*1/3 + 0*0 + 2/3*2/3)^{1/2} = (4/9 + 1/9 + 4/9)^{1/2} = (1)^{1/2} = 1$
$\|C\| = \langle C,C\rangle^{1/2} = (1/3*1/3 + -2/3*-2/3 + 0*0 + 2/3*2/3)^{1/2} = (1/9 + 4/9 + 4/9)^{1/2} = (1)^{1/2} = 1$
$\|D\| = \langle D,D\rangle^{1/2} = (2/3*2/3 + 2/3*2/3 + 0*0 + 1/3*1/3)^{1/2} = (4/9 + 4/9 + 1/9)^{1/2} = (1)^{1/2} = 1$

Because the norm of every matrix is 1 and the set is orthogonal, then the set is orthonormal.

Good type - re-engineer to get integer solutions.

5. Find the orthogonal projection of **u** onto the subspace of R^3 spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
(10 points)

U = (1, 3, 2); $\mathbf{v}_1$ = (0, 1, 2), $\mathbf{v}_2$ = (1, 3, 1)

Solution:
The subspace W of R^3 spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is the column space of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 2 & 1 \end{bmatrix}$$

To find the orthogonal projection of **u** on W we can find a least squares solution **Ax** = **u** and then calculate $\text{proj}_w\, \mathbf{u} = \mathbf{Ax}$ from the least squares solution. The system **Ax** = **u** is

$$\begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

so

$$A^T A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 11 \end{bmatrix} \text{ and } A^T u = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

The normal system $A^T\mathbf{A}\mathbf{x} = A^T\mathbf{u}$ in this case is

$$\begin{bmatrix} 5 & 5 \\ 5 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

the solutions to this matrix are $x_1 = 17/30$ $x_2 = 5/6$. So

$$\text{proj}_w\mathbf{u} = \mathbf{Ax} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 17/30 \\ 5/6 \end{bmatrix} = \begin{bmatrix} 5/6 \\ 46/15 \\ 59/30 \end{bmatrix}$$

or, in horizontal notation (which is consistent with the original phrasing of the problem), $\text{proj}_w\, \mathbf{u} = (5/6, 46/15, 59/30)$.

6. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for R^2, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{u}'_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \text{and } \mathbf{u}'_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

(a) Find the transition matrix from B' to B. (15 points)

(b) Compute the coordinate vector $[\mathbf{w}]_B$, where

$$\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{(10 points)}$$

Solution:

(a) By inspection,

$$\mathbf{u}'_1 = 3\mathbf{u}_1 - \mathbf{u}_2$$

$$\mathbf{u}'_2 = 4\mathbf{u}_1 + 2\mathbf{u}_2$$

$$[\mathbf{u}'_1]_B = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{and} [\mathbf{u}'_2]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Therefore the transition matrix is

$$P = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

— the book does this alchemy, but I explain what is going on (i.e. my method is rather self explanatory and transparent / enlightening)

(b) Using the transition matrix found in part (a) yields:

$$[\mathbf{w}]_B = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

7. Is the following matrix orthogonal? If so, find the inverse. (15 points)

$$A = \begin{bmatrix} \frac{2}{3} & \frac{11}{15} & \frac{2}{15} \\ \frac{11}{15} & \frac{-46}{75} & \frac{-22}{75} \\ \frac{2}{15} & \frac{-22}{75} & \frac{71}{75} \end{bmatrix}$$

Solution:

?,

This matrix is orthogonal. A matrix is orthogonal if it is square and its inverse is equal to its transpose. Therefore any matrix can by multiplied by its transpose. If the identity matrix results, then the original matrix is deemed as orthogonal. For this problem, the following would result:

$$AA^T = \begin{bmatrix} \frac{2}{3} & \frac{11}{15} & \frac{2}{15} \\ \frac{11}{15} & \frac{-46}{75} & \frac{-22}{75} \\ \frac{2}{15} & \frac{-22}{75} & \frac{71}{75} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{11}{15} & \frac{2}{15} \\ \frac{11}{15} & \frac{-46}{75} & \frac{-22}{75} \\ \frac{2}{15} & \frac{-22}{75} & \frac{71}{75} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer?

don't use bold face for scalars.

8. Find the real numbers that **a** and **b** may equal for the following matrix to be orthogonal. (15 points)

$$A = \begin{bmatrix} \mathbf{a} & \frac{1}{\sqrt{2}} \\ -\mathbf{a} & \mathbf{a}+\mathbf{b} \end{bmatrix}$$

Solution:

By definition, a square matrix is orthogonal if $A^{-1} = A^T$. From this definition, it follows that A is orthogonal if and only if $AA^T = A^TA = I$. Therefore,

$$AA^T = \begin{bmatrix} \mathbf{a} & \frac{1}{\sqrt{2}} \\ -\mathbf{a} & \mathbf{a}+\mathbf{b} \end{bmatrix}\begin{bmatrix} \mathbf{a} & -\mathbf{a} \\ \frac{1}{\sqrt{2}} & \mathbf{a}+\mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In order to satisfy this equation and according to the definition of matrix multiplication, the following three equations, which make up a system of equations, must therefore be true:

1.
$$(\mathbf{a})^2 + (\tfrac{1}{\sqrt{2}})^2 = 1$$
$$\mathbf{a}^2 + \tfrac{1}{2} = 1$$
$$\mathbf{a}^2 = \tfrac{1}{2}$$
$$\mathbf{a} = \pm\tfrac{1}{\sqrt{2}}.$$

2.
$$(\mathbf{a})(-\mathbf{a}) + (\tfrac{1}{\sqrt{2}})(\mathbf{a}+\mathbf{b}) = 0$$
$$-\mathbf{a}^2 + \tfrac{1}{\sqrt{2}}\mathbf{a} + \tfrac{1}{\sqrt{2}}\mathbf{b} = 0.$$

3.
$$(-\mathbf{a})^2 + (\mathbf{a}+\mathbf{b})^2 = 1$$
$$\mathbf{a}^2 + \mathbf{a}^2 + 2\mathbf{ba} + \mathbf{b}^2 = 1$$
$$2\mathbf{a}^2 + 2\mathbf{ba} + \mathbf{b}^2 = 1.$$

Because **a** can equal $\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$ and still satisfy equation **1**, it may be substituted into equations **2** and **3** to obtain:

2.
$$-\tfrac{1}{2} + \tfrac{1}{\sqrt{2}}(\pm\tfrac{1}{\sqrt{2}}) + \tfrac{1}{\sqrt{2}}\mathbf{b} = 0$$
$$-\tfrac{1}{2} \pm \tfrac{1}{2} + \tfrac{1}{\sqrt{2}}\mathbf{b} = 0$$
$$\mathbf{b} = \sqrt{2} \text{ (When } \mathbf{a} = -\tfrac{1}{\sqrt{2}}),$$
$$\mathbf{b} = 0 \text{ (When } \mathbf{a} = \tfrac{1}{\sqrt{2}}).$$

3.
$$1 + 2\mathbf{b}(\pm\tfrac{1}{\sqrt{2}}) + \mathbf{b}^2 = 1$$
$$\pm\tfrac{2}{\sqrt{2}}\mathbf{b} + \mathbf{b}^2 = 0$$
$$\mathbf{b}(\pm\tfrac{2}{\sqrt{2}} + \mathbf{b}) = 0$$
$$\mathbf{b} = 0 \text{ (When a} = \pm\tfrac{1}{\sqrt{2}}),$$
$$\mathbf{b} = \tfrac{2}{\sqrt{2}} \text{ (When a} = -\tfrac{1}{\sqrt{2}}),$$
$$\mathbf{b} = \tfrac{2}{\sqrt{2}} \text{ (When a} = \tfrac{1}{\sqrt{2}}).$$

In order for this system of equations to be consistent, $a = \frac{1}{\sqrt{2}}$ and $b = 0$.

9. a) Find the eigenvalues and their corresponding bases for the eigenspaces for the following matrix. (10 points)

$$A = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

b) Find the matrix P that diagonalizes A – if possible. (5 points)

Solution:

a) From Theorem 7.1.1 we know that the eigenvalues of A are just the entries along the main diagonal of A because A is a lower triangular matrix. Therefore the eigenvalues are $\lambda = 10, \lambda = 2, \lambda = 1, \lambda = 5$.

By definition $\boldsymbol{x}$ is an eigenvector of A corresponding to λ if and only if $\boldsymbol{x}$ is a nontrivial solution of $(\lambda I - A)\,\boldsymbol{x} = \mathbf{0}$ or in matrix form:

$$\begin{bmatrix} \lambda-10 & 0 & 0 & 0 \\ -1 & \lambda-2 & 0 & 0 \\ -1 & 0 & \lambda-1 & 0 \\ -1 & 0 & 0 & \lambda-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If $\lambda = 10$ then this becomes

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 8 & 0 & 0 \\ -1 & 0 & 9 & 0 \\ -1 & 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Which by inspection yields

$$x_1 = t \quad x_2 = \frac{1}{8}t \quad x_3 = \frac{1}{9}t \quad x_4 = \frac{1}{9}t$$

Thus the eigenvectors of A corresponding to $\lambda = 10$ are the non zero vectors of the form

$$t\begin{bmatrix} 1 \\ \frac{1}{8} \\ \frac{1}{9} \\ \frac{1}{9} \end{bmatrix} = x \quad \text{and} \quad \begin{bmatrix} 1 \\ \frac{1}{8} \\ \frac{1}{9} \\ \frac{1}{9} \end{bmatrix}$$ is the basis for the eigenspace — choose $\begin{bmatrix} 72 \\ 9 \\ 8 \\ 8 \end{bmatrix}$

If $\lambda = 2$ then this becomes

$$\begin{bmatrix} -8 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Which by inspection yields

$$x_1 = 0 \quad x_2 = t \quad x_3 = 0 \quad x_4 = 0$$

Thus the eigenvectors of A corresponding to $\lambda = 2$ are the non zero vectors of the form

$t\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = x$ and $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ is the basis for the eigenspace

If $\lambda = 1$ then this becomes

$$\begin{bmatrix}-9 & 0 & 0 & 0\\-1 & -1 & 0 & 0\\-1 & 0 & 0 & 0\\-1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Which by inspection yields

$x_1 = 0 \quad x_2 = 0 \quad x_3 = s \quad x_4 = t$

Thus the eigenvectors of A corresponding to $\lambda = 1$ are the non zero vectors of the form

$s\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + t\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = x$ and $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ ~~are the~~ form a basis for the eigenspace

b) The matrix P is just a matrix with the columns as the eigenspace basis from part a.

$$p = \begin{bmatrix}1 & 0 & 0 & 0\\ \frac{1}{8} & 1 & 0 & 0\\ \frac{1}{9} & 0 & 1 & 0\\ \frac{1}{9} & 0 & 0 & 1\end{bmatrix}$$

10. For the following statements circle true or false and then support your choice with a logical argument, a theorem, or a counter example.

a) Matrices with repeated eigenvalues are always diagonalizable. (3 points)

b) The matrix $A = \begin{bmatrix}a & 0 & 0 & 0\\2 & b & 0 & 0\\3 & 1 & c & 0\\4 & 2 & 1 & d\end{bmatrix}$ where $a \neq b \neq c \neq d$ and a, b, c, d are real or complex numbers is diagonalizable. (4 points)

$1 \neq 2 \neq 1 \neq 2$ (!!)

c) For every eigenvalue of some n x n matrix A, the geometric multiplicity and the algebraic multiplicity are the same. (4 points)

d) All orthogonally diagonalizable are symmetric matrices. (4 points)

Solution:

a) False, $J_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has repeated eigenvalues of 1, but solving for its eigenvectors gives $(\lambda I - J_2)x = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} x = 0$, the solution of which is that $x_1 = t \quad x_2 = 0$. Thus every eigenvector of J_2 is a multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and since the eigenspace has 1 dimension and n =2, the matrix is not diagonalizable.

b) ~~True~~ False, theorem 7.2.3 states that if an n x n matrix has n distinct eigenvalues, then it is diagonalizable. It does not matter if the eigenvalues are real or complex. but this is not what you said! *

c) False, for an n x n matrix geometric multiplicity of an eigenvalue, λ_0, is the dimensions of the eigenspace corresponding to λ_0. Algebraic multiplicity for an eigenvalue, λ_0, is the number of times that the factor $\lambda - \lambda_0$ appears in the characteristic equation. The matrix $J_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has the eigenvalue of $\lambda = 1$ and the characteristic equation of $(\lambda - 1)(\lambda - 1) = 0$, so the algebraic multiplicity is 2, but $(\lambda I - I)x = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} x = 0$ yields the solution $x_1 = t \quad x_2 = 0$. Thus every eigenvector of J_2 is a multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and its geometric multiplicity does not equal its algebraic multiplicity

d) True, suppose that

$P^T A P = D$

where P is an orthogonal matix and D is a diagonal matrix.

Since P is orthogonal $PP^T = P^T P = I$,

so it follows that $A = P^T D P$.

We also know that $D = D^T$

so transposing both sides yields

$A^T = (P^T D P)^T = (P^T)^T D^T P^T = P D P^T = A$

* you said $a \neq b \neq c \neq d \not\Rightarrow a \neq d$, for example

9

Exam 2

1. (15 points) If $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ are two 2x2 matrices with an inner product defined as $\langle A,B \rangle := a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$, then compute the following when $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$.

This is the first well-written question of this type that I have had the pleasure to see.

(a) $\langle A,B \rangle$

(b) $\| A \|$

(c) $\| B \|^2$

(d) $\| A - B \|$

(e) $d(A,B)$

Solution:

(a) $\langle A,B \rangle$ = 1(0) +2(1)+ 3(-1)+2(3) = 0+2-3+6 = 5

(b) $\| A \| = \langle A, A \rangle^{1/2} = \sqrt{1(1)+2(2)+3(3)+2(2)} = \sqrt{1+4+9+4} = \sqrt{18}$

(c) $\| B \|^2 = \langle B, B \rangle$ = 0(0)+1(1)+(-1)(-1)+3(3) = 0 + 1 + 1 + 9 = 11

(d) $\| A - B \| = \left\| \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1-0 & 2-1 \\ 3+1 & 2-3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} \right\| =$

$\left\langle \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} \right\rangle^{1/2} = \sqrt{1(1)+1(1)+4(4)-1(-1)} = \sqrt{1+1+16+1} = \sqrt{19}$

(e) $d(A,B) = \| A - B \|$ = (solution to part (d)) = $\sqrt{19}$

2. (20 points) Consider the bases $S = \{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ and $S' = \{\underline{u}_1', \underline{u}_2', \underline{u}_3'\}$ in P^2, where $\underline{u}_1 = 1,\ \underline{u}_2 = x,\ \underline{u}_3 = x^2$, and $\underline{u}_1' = 2,\ \underline{u}_2' = 3x - 1,\ \underline{u}_3' = 2x^2 + x$.

(a) Find the transition matrix from S' to S.

(b) Find $\underline{p}_S$ for polynomial $\underline{p}_{S'} = \{2 - x\}$.

does this mean coordinate vector? If so, what does ✱ mean?

Solution:

(a) First we must find the coordinate vectors for the new basis vectors u_1', u_2', u_3' relative to the old basis S.

$$\begin{pmatrix} \underline{u}_1' \\ \underline{u}_2' \\ \underline{u}_3' \end{pmatrix} = \begin{pmatrix} a\underline{u}_1 + b\underline{u}_2 + c\underline{u}_3 \\ d\underline{u}_1 + e\underline{u}_2 + f\underline{u}_3 \\ g\underline{u}_1 + h\underline{u}_2 + i\underline{u}_3 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3x-1 \\ 2x^2+x \end{pmatrix} = \begin{pmatrix} a+bx+cx^2 \\ d+ex+fx^2 \\ g+hx+ix^2 \end{pmatrix}$$

By inspection we can see that $\begin{pmatrix} 2 \\ 3x-1 \\ 2x^2+x \end{pmatrix} = \begin{pmatrix} 2+(0)x+(0)x^2 \\ -1+3x+(0)x^2 \\ (0)+x+2x^2 \end{pmatrix}$. Since $[\underline{u}_1']_{S'} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$,

$[\underline{u}_2']_{S'} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$ and $[\underline{u}_3']_{S'} = \begin{bmatrix} g \\ h \\ i \end{bmatrix}$; $[\underline{u}_1]_{S'} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $[\underline{u}_2]_{S'} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$ and $[\underline{u}_3]_{S'} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

In addition the transition matrix from S' to S is denoted $\mathrm{P} = \left[[\underline{u}_1']_S \middle| [\underline{u}_2']_S \middle| [\underline{u}_3']_S \right]$.

Thus the transition matrix from S' to S is: $\mathrm{P} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

(b) We first express $\underline{p}_{S'}$ as the column matrix $[\mathbf{v}]_{S'} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. Using P from the

previous solution, $[\mathbf{v}]_S := P[\mathbf{v}]_{S'} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2(2)+(-1)(-1) \\ 3(-1) \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}$.

We now simply convert $[\mathbf{v}]_S$ to $\underline{p}_S$. $\begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} = 5 - 3x$.

3. (30 points) Find an orthogonal matrix P that diagonalizes $\mathrm{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

Solution:

We will first need to find a basis for all possible eigenspaces of A. The characteristic equation of A is det($\lambda I - A$) = 0.

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\right) = \begin{vmatrix} \lambda-2 & -1 & -1 \\ -1 & \lambda-2 & -1 \\ -1 & -1 & \lambda-2 \end{vmatrix}$$

$$= (\lambda-2)^3 - 1 - 1 - 3(\lambda-2) = \lambda^3 - 6\lambda^2 + 9\lambda - 4 = (\lambda-1)^2(\lambda-4)$$

From this we see the characteristic polynomial is $(\lambda-1)^2(\lambda-4) = 0$, and the eigenvalues of A are $\lambda=1$ and $\lambda=4$.

By definition $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A corresponding to λ if and only if $\mathbf{x}$ is a nontrivial solution of $(\lambda I - A)\mathbf{x} = \mathbf{0}$. In our situation that would be:

$$\begin{bmatrix} \lambda-2 & -1 & -1 \\ -1 & \lambda-2 & -1 \\ -1 & -1 & \lambda-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

If $\lambda=1$,

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim x_1 = -s-t,\ x_2 = s,\ x_3 = t.$$

From this we see that the eigenvectors of A corresponding to $\lambda = 1$ are the nonzero vectors of the form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s\underline{u}_1 + t\underline{u}_2 = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and

$\underline{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \underline{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. These vectors ($\underline{u}_1$ and $\underline{u}_2$) form a basis for the eigenspace corresponding to $\lambda = 1$.

✓ The first relevant use of this process in this context.

To convert these into orthonormal vectors we use the Gram-Schmidt process to first convert $\underline{u}_1$ and $\underline{u}_2$ into an orthogonal set of vectors, and then normalize the resulting vectors to obtain the orthonormal vectors.

Applying the Gram-Schmidt process:

$$\underline{v}_1 = \underline{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{v}_2 = \underline{u}_2 - \frac{\langle \underline{u}_2, \underline{v}_1 \rangle}{\|\underline{v}_1\|^2}\underline{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$ — change to $\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

To normalize we divide each vector by its magnitude resulting in the orthonormal vector set of $\underline{v}_1'$ and $\underline{v}_2'$.

$\|\underline{v}_1\| = \sqrt{1+1} = \sqrt{2}$

$\|\underline{v}_2\| = \sqrt{(-1/2)^2 + (-1/2)^2 + 1} = \sqrt{(6/4)} = \sqrt{6}/2$

$$\underline{v}_1' = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\underline{v}_2' = \frac{2}{\sqrt{6}}\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

If λ=4,

$$\begin{bmatrix} \lambda-2 & -1 & -1 \\ -1 & \lambda-2 & -1 \\ -1 & -1 & \lambda-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$x_1 = t,\ x_2 = t,\ x_3 = t.\quad \underline{x} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\underline{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$. Therefore $\underline{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$ is a basis vector for the eigenspace of A when λ = 4. To convert this into a third orthonormal vector ($\underline{v}_3'$), we use the Gram-Schmidt, which is simply $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and then normalize as before.

$\|\underline{v}_3\| = \sqrt{1+1+1} = \sqrt{3}$

$$\underline{v}_3' = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$$

Since P consists of $\underline{v}_1', \underline{v}_2'$ and $\underline{v}_3'$ as its column vectors, we conclude that

$$P = \left[\underline{v}_1' \middle| \underline{v}_2' \middle| \underline{v}_3'\right] = \begin{bmatrix}-1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 0 & 2/\sqrt{6} & 1/\sqrt{3}\end{bmatrix}$$

4. (15 points) Determine whether the following set of vectors is orthogonal with respect to the given inner product. If orthogonal, show that the Pythagorean Theorem holds for the vectors.

a) $\mathbf{p} = x, \quad \mathbf{q} = x^2, \quad \langle \mathbf{p}, \mathbf{q}\rangle := \int_{-1}^{1} p(x)q(x)dx$ — cool problem

b) $\mathbf{v} = (1, 2, 3), \quad \mathbf{u} = (-3, 3, -1), \quad \langle \mathbf{v}, \mathbf{u}\rangle := v_1u_1 + v_2u_2 + v_3u_3$

Solution:

If the vectors are orthogonal their inner product will be zero.

a) $\langle \mathbf{p}, \mathbf{q}\rangle = \int_{-1}^{1} x^2 x dx = \int_{-1}^{1} x^3 dx = \frac{1}{4}x^4\Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$

Because the inner product is 0, **p** and **q** are orthogonal.

To show the Pythagorean theorem holds:

$\|\mathbf{p}+\mathbf{q}\|^2 = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2 \sim$

$\langle \mathbf{p}+\mathbf{q}, \mathbf{p}+\mathbf{q}\rangle = \langle \mathbf{p}, \mathbf{p}\rangle + \langle \mathbf{q}, \mathbf{q}\rangle \sim$

$\langle x + x^2, x + x^2\rangle = \langle x, x\rangle + \langle x^2, x^2\rangle \sim$ cool.

$\int_{1}^{-1} (x+x^2)^2 dx = \int_{1}^{-1} (x)^2 dx + \int_{1}^{-1} (x^2)^2 dx \sim$

$\int_{1}^{-1} x^2 + 2x^3 + x^4 dx = \int_{1}^{-1} (x)^2 dx + \int_{1}^{-1} (x^4) dx \sim$

$\int_{1}^{-1} x^2 dx + \int_{-1}^{1} 2x^3 dx + \int_{-1}^{1} x^4 dx = \int_{1}^{-1} (x)^2 dx + \int_{1}^{-1} (x^4) dx \sim$

$\int_{-1}^{1} 2x^3 = 0 \sim$

$\int_{1}^{-1} x^2 + x^4 dx = \int_{1}^{-1} x^2 + x^4 dx \sim$

$\frac{16}{15} = \frac{16}{15}$

b) $\langle \mathbf{v}, \mathbf{u}\rangle = 1(-3) + 2(3) + 3(-1) = -3 + 6 - 3 = 0$

Because the inner product is 0, **v** and **u** are orthogonal.

To show the Pythagorean theorem holds:

$\|\mathbf{v}+\mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2,$

$\mathbf{v}+\mathbf{u} = (-2,5,2),$

$\|\mathbf{v}+\mathbf{u}\|^2 = 4+25+4 = 33,$

$\|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 = (1+4+9)+(9+9+1) = 33,$

$33 = 33$

5. (10 points) Determine if the following matrix is orthogonal by computing A^TA. Show that the $\det(A) = \pm 1$, and that $A^{-1} = A^T$.

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Solution:

The matrix is orthogonal if $A^TA=I$

$$A^TA = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2(\theta)+\sin^2(\theta) & \cos(\theta)\sin(\theta)-\cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta)-\cos(\theta)\sin(\theta & \cos^2(\theta)+\sin^2(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

— just showed that $A^T = A^{-1}$.

$$\det(A) = \cos^2(\theta)+\sin^2(\theta) = 1$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = A^T$$

— this is not a definition, but a formula/theorem — irrelevant to question asked

6. (15 points) Find the eigenvalue(s) and eigenvector(s) of the following matrix:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution:

We first determine the characteristic polynomial for A.

$$\det(\lambda I - A) \sim \det\begin{bmatrix} \lambda-2 & 0 & -1 \\ -2 & \lambda-2 & -1 \\ 0 & -1 & \lambda \end{bmatrix} \sim (\lambda-2)(\lambda-2)(\lambda)-2-(\lambda-2) \sim$$

$$\lambda^3 - 4\lambda^2 + 3\lambda \sim \lambda(\lambda-3)(\lambda-1)$$

From this we see that the characteristic polynomial is $\lambda(\lambda-3)(\lambda-1)$ =0, and that eigenvalues of A are $\lambda = 0,1,$ and 3.

To find the eigenvectors we compute $(\lambda I - A)\mathbf{x} = \mathbf{0}$ for all eigenvalues.

$(\lambda I - A)\mathbf{x} = \mathbf{0} \sim$

$$\begin{bmatrix} \lambda-2 & 0 & -1 \\ -2 & \lambda-2 & -1 \\ 0 & -1 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

when $\lambda = 0$

$$\begin{bmatrix} -2 & 0 & -1 \\ -2 & -2 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \sim (by\ row\ reduction) \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ *is an eigenvector*

when $\lambda = 3$

$$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \sim (by\ row\ reduction) \begin{bmatrix} 1 & -1/2 & \frac{1}{2} \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -\frac{3}{4} \\ 3 \\ 1 \end{bmatrix}$ *is an eigenvector* – change to $\begin{bmatrix} -3 \\ 12 \\ 4 \end{bmatrix}$

where does this come from?

when $\lambda = 1$

$$\begin{bmatrix} -1 & 0 & -1 \\ -2 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \sim (by\ row\ reduction) \begin{bmatrix} 1 & 1/2 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ *is an eigenvector*

7. (8 points) If A is an orthogonal matrix, prove that the inverse of A is orthogonal.

Solution:

If A is an orthogonal matrix, then $A^{-1} = A^T$.

$A^{-1} = A^T \sim (A^{-1})^{-1} = (A^T)^{-1} \sim (A^{-1})^{-1} = (A^{-1})^T$ therefore (A^{-1}) is also orthogonal.

8. (12 points) Answer the following questions using the following system of equations:

$$x = 1$$
$$y = 2$$
$$x + y = 3.001$$

nice problem, even with decimal.

a. Is the system of equations consistent?
b. Find the least squares solution of this system

Solution:

a. The system is not consistent because $1 + 2 \neq 3.001$
b. We begin by converting the system to the form A**x**=**b**, for our system that would be:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3.001 \end{bmatrix}$$

To find the least squares solution we solve the normal system $A^T A\underline{x} = \underline{A^T b}$.

$$A^TA = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}, \text{ and } A^Tb = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}\begin{vmatrix} 1 \\ 2 \\ 3.001 \end{vmatrix} = \begin{vmatrix} 4.001 \\ 5.001 \end{vmatrix}$$

$$A^T A\underline{x} = \underline{A^T b} \sim \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4.001 \\ 5.001 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 4.001 \\ 1 & 2 & 5.001 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.001/3 \\ 0 & 1 & 6.001/3 \end{bmatrix}$$

Therefore, x = 3.001/3 and y = 6.001/3.

9. (15 points) Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Solution:

First find the eigenvalues of A.

$$\det(\lambda I - A) = \det\begin{vmatrix} \lambda-2 & 0 & 0 \\ 1 & \lambda-2 & 1 \\ -1 & 0 & \lambda-1 \end{vmatrix} = (\lambda-2)^2(\lambda-1)$$

From this we see that the characteristic polynomial is $(\lambda - 2)^2(\lambda-1) = 0$.

The eigenvalues of A are $\lambda_{1,2} = 2,\ \lambda_3 = 1$.

Next we find the independent eigenvectors of A by solving $(\lambda I - A)\underline{x} = 0$, for each value of λ (assuming $\underline{x}$ is a nontrivial solution).

If $\lambda = 1$,

$$\begin{bmatrix} \lambda-2 & 0 & 0 \\ 1 & \lambda-2 & 1 \\ -1 & 0 & \lambda-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we obtain $x_1 = 0,\ x_2 = t,\ x_3 = t$. From this we see that,

$\underline{x} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Therefore an eigenvector if $\lambda = 1$ is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

If $\lambda = 2$,

$$\begin{bmatrix} \lambda-2 & 0 & 0 \\ 1 & \lambda-2 & 1 \\ -1 & 0 & \lambda-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we obtain $x_1 = 0,\ x_2 = 0,\ x_3 = t$. From this we see that,

$\underline{x} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Therefore an eigenvector if $\lambda = 2$ is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Since A is a 3x3 matrix and there are only two basis vectors, we conclude that A is <u>not</u> diagonalizable.

10. (10 points) If A is an nxn orthogonal matrix, prove that $A\underline{x} \bullet A\underline{y} = \underline{x} \bullet \underline{y}$.

Solution:

We prove this by making use of the fact that $A^T A = I$ for an orthogonal matrix, as well as applying basic transposition and associative properties of matrices.

$$A\underline{x} \bullet A\underline{y} = (A\underline{y})^T(A\underline{x}) = (\underline{y}^T A^T)(A\underline{x}) = \underline{y}^T(A^T A)\underline{x} = \underline{y}^T I \underline{x} = \underline{y}^T \underline{x} = \underline{x} \bullet \underline{y}$$

1) a. Find the least squares solution of the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned} x_1 - x_2 &= 4 \\ 3x_1 + 2x_2 &= 1 \\ -2x_1 + 4x_2 &= 3 \end{aligned}$$

b. Find the orthogonal projection of $\mathbf{b}$ on the column space of A.

Solution:

a) We have

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

Using the associated normal system from Theorem 6.4.2, we have

$$A^T A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

so the normal system $A^T A\mathbf{x} = A^T\mathbf{b}$ is

$$\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

Solving for x_1 and x_2 gives the least squares solution of engineer better

$$x_1 = \tfrac{17}{95} \text{ and } x_2 = \tfrac{143}{285}$$

b) The orthogonal projection of $\mathbf{b}$ on the column space of A is

$$Ax = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \frac{17}{95} \\ \frac{143}{285} \end{bmatrix} = \begin{bmatrix} \frac{-92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{bmatrix}$$

2) Find the orthogonal projection of the vector $\mathbf{u} = (-3,-3,8,9)$ on the subspace of R^4 spanned by the vectors

$\mathbf{u}_1 = (3,1,0,1)$ $\mathbf{u}_2 = (1, 2, 1, 1)$ $\mathbf{u}_3 = (-1, 0, 2, -1)$

The subspace W spanned by the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ is the column space of the matrix

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Solution:
With $\mathbf{u}$ expressed as a column vector, we can find the orthogonal projection of $\mathbf{u}$ on W by finding the least squares solution of the system $A\mathbf{x} = \mathbf{b}$ and then calculating $\text{proj}_W\mathbf{u} = A\mathbf{x}$. We have

$$A\mathbf{x} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 8 \\ 9 \end{bmatrix}$$

so since

$$A^T A = \begin{bmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{bmatrix}$$

and

$$A^T\mathbf{u} = \begin{bmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 10 \end{bmatrix})$$

The normal system $A^T A\mathbf{x} = A^T\mathbf{b}$ for this case is

$$\begin{bmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 10 \end{bmatrix}$$

so

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

and

$$\text{proj}_w\mathbf{u} = A\mathbf{x} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \\ 0 \end{bmatrix}$$

which can also be written as (-2, 3, 4, 0).

3) Find the eigen values of the following matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ -2 & \frac{2}{3} & 0 & 0 \\ -1 & 1 & \frac{-1}{4} & 0 \\ 4 & -8 & 1 & \frac{1}{3} \end{bmatrix}$$

Solution:

The eigen values of a lower triangular matrix are just the values of the coefficients along the diagonal. So by inspection

$$\lambda = \tfrac{1}{2},\ \lambda = \tfrac{2}{3},\ \lambda = \tfrac{-1}{4},\ \text{and}\ \lambda = \tfrac{1}{3}.$$

4) Given the transpose P^T of a matrix P that transforms upon matrix multiplication a coordinate vector relative to the basis B' = {$\mathbf{u}_1$', $\mathbf{u}_2$', ...$\mathbf{u}_r$'} to a coordinate vector relative to the basis B = {$\mathbf{u}_1$, $\mathbf{u}_2$... $\mathbf{u}_r$}, and given the linear combination of the vectors of B yielding $\mathbf{u}_j$',

$$\mathbf{u}_j' = k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \ldots + k_r\mathbf{u}_r,$$

which element of P^T is the k_m of the equation above?

Solution: The element in the jth row and mth column of P^T is k_m. This is because the columns of P and so the rows of P^T are the coordinate vectors of each vector of B' relative to the basis B, so the mth element of this row/column is the multiple of $\mathbf{u}_m$ in the linear combination above.

"I recognize the lion by his paw." – Jacob Bernoulli (Regarding Isaac Newton)

5) Given a finite-dimensional inner-product space V with basis U = {$\mathbf{u}_1$, $\mathbf{u}_2$... $\mathbf{u}_r$} with $W_{\{\mathbf{u}(n)\}}$ a subspace of V spanned by the vectors in {$\mathbf{u}_{(n)}$}, a collection of some possible orthogonal $\mathbf{u}_{(n)}$'s where $\mathbf{u}_{(n)}$ denotes the nth basis vector of V with n contained in {1, 2, ... dim(V)}, what is the set of vectors written in terms of the basis of V given above that describe an orthogonal basis for V?

Solution: We use Gram-Schmidt decomposition to solve. First we define the set

$Z = \{Z_1, Z_2 \dots Z_m \mid$ for all k Z_k is uniquely equal to some $W_{\{v(n)\}}\}$.

So the Gram-Schmidt process yields a set T of orthogonal vectors all of which are of the form

$\mathbf{t}_m = \mathbf{u}_m$ - proj(span{$\mathbf{u}_1$, $\mathbf{u}_2$... $\mathbf{u}_{m-1}$} , $\mathbf{u}_m$)

or equivalently

$\mathbf{t}_m = \mathbf{u}_m - \Sigma_j \text{proj}(\mathbf{u}_j, \mathbf{u}_m)$

where the sum is over all j's that are not elements of any Z_k's of which $\mathbf{u}_m$ is an element.

So the above equation for the Euclidean inner product * is

$\mathbf{t}_m = \mathbf{u}_m - \Sigma_j (\mathbf{u}_j * \mathbf{u}_m)\ \mathbf{u}_j / (\mathbf{u}_j * \mathbf{u}_j)^{1/2}$

6) What is the size of the largest possible (having the most elements) sets of orthogonal nonzero vectors in an n-dimensional inner product space S?

Solution: The largest such sets have n vectors. By Theorem 6.3.3 the elements in a set of orthogonal vectors in an inner product space are linearly independent (which is easily seen when considering a vector's inner product with linear combinations of vectors orthogonal to it), and by Theorem 5.3.3 a set of vectors in n-dimensional space is linearly dependent if it has more than n vectors.

7) Let R^2 and R^3 have the Euclidean inner product. Find the cosine of the angle between v and w for each part given:

a.) v = (-1, 2), w = (3, -4).

b.) v = (-1, 8, -4), w = (6, -2, 3).

Solution: Using the formula (8) given to us in section 6.2, we can see that $\cos\theta = \dfrac{\langle v, w\rangle}{\|v\|\|w\|}$.

a.) Solving for cosθ, using the formula for v and w, yields

$$\cos\theta = \frac{\langle v, w\rangle}{\|v\|\|w\|} = \frac{(-1*2)+(3*-4)}{\sqrt{(-1)^2+(2)^2}\ \ \sqrt{(3)^2+(-4)^2}} = \frac{-14}{5\sqrt{5}}$$

where $\dfrac{-14}{5\sqrt{5}}$ is the cosine of the angle between v and w.

b.) Using the same formula, we find that

$$\cos\theta = \frac{\langle v, w\rangle}{\|v\|\|w\|} = \frac{(-1*8*4)+(6*-2*3)}{\sqrt{(-1)^2+(8)^2+(4)^2}\ \sqrt{(6)^2+(-2)^2+(3)^2}} = \frac{-68}{63}$$

where $\frac{-68}{63}$ is the cosine of the angle between v and w.

8) Let U be a subspace of R^3 spanned by the vectors

$\mathbf{u}_1 = (6, 4, 10, 16, 28)$ $\mathbf{u}_2 = (1, 1, 4, 2, 6)$ $\mathbf{u}_3 = (3, 2, 5, 8, 14)$

Find a basis for the orthogonal complement of U.

Solution: The space that is spanned by the vectors, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$, can be expressed in the following matrix:

$$\begin{bmatrix} 1 & 1 & 4 & 2 & 6 \\ 3 & 2 & 5 & 8 & 14 \\ 6 & 4 & 10 & 16 & 28 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 4 & 2 & 6 \\ 3 & 2 & 5 & 8 & 14 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 4 & 2 & 6 \\ 0 & -1 & -7 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & -3 & 4 & 2 \\ 0 & 1 & 7 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving for the nullspace of the matrix is the orthogonal complement of U. We find the given values of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5$.

$\mathbf{x}_1 = 3s - 4t - 2j$, $\mathbf{x}_2 = -7s + 2t - 4j$, $\mathbf{x}_3 = s$, $\mathbf{x}_4 = t$, $\mathbf{x}_5 = j$

We shall express these vectors in the following notation as vectors for the basis for this nullspace which is the orthogonal complement:

$\mathbf{v}_1 = (3, -7, 1, 0, 0)$, $\mathbf{v}_2 = (-4, 2, 0, 1, 0)$ $\mathbf{v}_3 = (-2, -4, 0, 0, 1)$.

9) Show that the following matrix is orthogonal, and then give three properties of an orthogonal matrix,

$$A = \begin{bmatrix} 2/7 & 3/7 & 6/7 \\ 3/7 & -6/7 & 2/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix}.$$

Solution:

According to the definition of an orthogonal matrix, the inverse of matrix A, is equal to the transpose of the same matrix, $A^{-1} = A^T$. We can show that the matrix is the inverse by multiplying the original matrix by its transpose. Like so,

$$A^TA = \begin{bmatrix} 2/7 & 3/7 & 6/7 \\ 3/7 & -6/7 & 2/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix} \begin{bmatrix} 2/7 & 3/7 & 6/7 \\ 3/7 & -6/7 & 2/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore we can deduce that $A^T = A^{-1}$, and by definition state that matrix A is orthogonal. Three other properties that can make up an orthogonal matrix, can be included in the following: (b) The row vectors of A form an orthonormal set in R^n with Euclidean inner product, (c) The column vectors of A form an orthonormal set in R^n with Euclidean inner product, (d) The inverse of an orthogonal matrices is orthogonal (e) A product of orthogonal matrices in orthogonal, (f) If A is orthogonal, the det(A) = 1 or det(A) = -1.

10) Find a weighted Euclidean inner product in R^2 that satisfies all four axioms.

<u>Solution</u>: Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in R^2. A weighted Euclidean inner product could be:

$$\langle u, v\rangle = 5u_1v_1 + 3u_2v_2$$

Verify:

a) If **u** and **v** are interchanged, the right side remains the same, thus:

$$\langle u, v\rangle = \langle v, u\rangle$$

b) If $\mathbf{z} = (z_1, z_2)$, then

$$\langle u+v, z\rangle = 5(u_1+v_1)z_1 + 3(u_2+v_2)z_2 = (5u_1z_1 + 5v_1z_1) + (3u_2z_2 + 3v_2z_2)$$
$$= (5u_1z_1 + 3u_2z_2) + (5v_1z_1 + 3v_2z_2) = \langle u, z\rangle + \langle v, z\rangle$$

c) $\langle ku, v\rangle = 5(ku_1)v_1 + 3(ku_2)v_2 = k(5u_1v_1 + 3u_2v_2) = k\langle u, v\rangle$

d) $\langle v, v\rangle = 5v_1v_1 + 3v_2v_2 = 5v_1^2 + 3v_2^2$ By inspection: $\langle v, v\rangle = 5v_1^2 + 3v_2^2 \geq 0$ and $\langle v, v\rangle = 5v_1^2 + 3v_2^2 = 0$ iff $v_1 = v_2 = 0$ or iff $v = (v_1, v_2) = 0$

1) If **u** = (3,5,8) and **v** = (2,7,0)
a) what is the norm of **u**?
b) what is the norm of **v**?
c) What is d(**u**,**v**)?

Solution
a) $\|\mathbf{u}\| = <\mathbf{u},\mathbf{u}>^{1/2} = (\mathbf{u} \bullet \mathbf{u})^{1/2} = (3^2 + 5^2 + 8^2)^{1/2} = (98)^{1/2}$
b) $\|\mathbf{v}\| = <\mathbf{v},\mathbf{v}>^{1/2} = (\mathbf{v} \bullet \mathbf{v})^{1/2} = (2^2 + 7^2 + 0^2)^{1/2} = (53)^{1/2}$
c) $\|\mathbf{u} - \mathbf{v}\| = <\mathbf{u-v},\mathbf{u-v}> = [(\mathbf{u-v}) \bullet (\mathbf{u-v})] = [(3-2)^2 + (5-7)^2 + (8-0)^2] = (69)^{1/2}$

2) If **u** = (1,4,6) and **v** = (8,5,9), what is the angle between these vectors?

Solution
$\mathrm{Cos}\theta = \mathbf{u} \bullet \mathbf{v}/(\|\mathbf{u}\| \|\mathbf{v}\|) = [(1x8) + (4x5) + (6x9)]/[(1^2 + 4^2 + 6^2)^{1/2}(8^2 + 5^2 + 9^2)^{1/2}]$

$\mathrm{Cos}\theta = 82/[(53)^{1/2}(170)^{1/2}]$

$\theta = \mathrm{Cos}^{-1}(0.863876) = 30.25°$

3) $U = \begin{bmatrix} 9 & -2 \\ -4 & 3 \end{bmatrix}$ $V = \begin{bmatrix} 2 & 4 \\ 2 & 1 \end{bmatrix}$ $Q = \begin{bmatrix} 3 & 1 \\ -2 & -6 \end{bmatrix}$

Which of the following are orthogonal vectors?
a) <U,V>
b) <U,Q> - what is inner product?
c) <V,Q>
- answer depends on this!

Solution
a) <U,V> = 9(2) – 2(4) – 4(2) + 3(1) = 5, not orthoganol
b) <U,Q> = 9(3) – 2(1) – 4(-2) + 3(-6) = 15, not orthoganol
c) <V,Q> = 2(3) + 4(1) + 2(-2) + 1(-6) = 0, Orthoganol

4. (15 pts) Find the least square solution of the linear system $A\mathbf{x} = \mathbf{b}$ and find the orthogonal projection of b in to the column space A.

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -2 \\ 3 & 2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix}$$

, Good problem

Solution:

A^(T)A**x** = A^(T)**b**

$$A^{(T)}A\mathbf{x} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & -2 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 14 & 0 \\ 0 & 12 \end{pmatrix}$$

$$A^{(T)}\mathbf{b} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 0 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \end{pmatrix} \quad \left(\begin{array}{cc|c} 14 & 0 & 2 \\ 0 & 12 & 12 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1/7 \\ 0 & 1 & 1 \end{array}\right) \quad \mathbf{x} = \begin{pmatrix} 1/7 \\ 1 \end{pmatrix}$$

Orthogonal Projection: $\text{proj}_W = A\mathbf{x} = \begin{pmatrix} -1 & 2 \\ 2 & -2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1/7 \\ 1 \end{pmatrix} = \begin{pmatrix} 13/7 \\ -12/7 \\ 17/7 \end{pmatrix}$

, don't be attached to a notation, only (correct/precise) language.

5 (15 pts.) Find the transition matrix from B' to B and find $[v]_{B'}$, given $[v]_B = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$\mathbf{u}_1 = (2, 3)$ $\mathbf{u}_2 = (3, 4)$ $\mathbf{u}'_1 = (-1, 0)$ $\mathbf{u}'_2 = (-1, -2)$

$\mathbf{u}'_1 = k_1\mathbf{u}_1 + k_2\mathbf{u}_2$ $\mathbf{u}'_1 = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $\left(\begin{array}{cc|c} 2 & 3 & -1 \\ 3 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 3 & -1 \\ 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 3 & -1 \\ 0 & -1 & 3 \end{array}\right)$

$k_1 = 4$, $k_2 = -3$ $\mathbf{u}'_1 = 4\mathbf{u}_1 - 3\mathbf{u}_2$

$\mathbf{u}'_2 = k_1\mathbf{u}_1 + k_2\mathbf{u}_2$ $\mathbf{u}_2' = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ $\left(\begin{array}{cc|c} 2 & 3 & -1 \\ 3 & 4 & -2 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 3 & -1 \\ 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 3 & -1 \\ 0 & -1 & -1 \end{array}\right)$

$k_1 = -2$, $k_2 = 1$ $\mathbf{u}'_1 = -2\mathbf{u}_1 + 1\mathbf{u}_2$ $[\mathbf{u}'_1]_B = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ $[\mathbf{u}'_2]_B = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Transition matrix from B' to B is $P = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

To find $[v]_{B'}$, given $[v]_B$, we use the following formula: $[v]_{B'} = P^{-1}[v]_B$

$$P^{-1} = \frac{1}{(4)(1) - (-2)(-3)} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -1/2 & -1 \\ -3/2 & -2 \end{pmatrix}$$

$$[v]_{B'} = \begin{pmatrix} -1/2 & -1 \\ -3/2 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

6. What Conditions must a and b satisfy for the matrix

$$A = \begin{bmatrix} a-b & a+b \\ a+b & a-b \end{bmatrix}$$

to be Orthogonal?

Solution:

We know that $A(A^T) = I$ =>

1) $(a-b)^2 + (a+b)^2 = 1$
2) $(a+b)(a-b) + (a-b)(a+b) = 0$ => $(a+b) = 0$ or $(a-b) = 0$

If $(a+b) = 0$ => $(a-b)^2 = 1$
If $(a-b) = 0$ => $(a+b)^2 = 1$

7. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 4 & 3 & 4 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

Solution:

$$\lambda I - A = \begin{bmatrix} \lambda -1 & -2 & -1 & -2 \\ 0 & \lambda -4 & -3 & -4 \\ 0 & 0 & \lambda -6 & -5 \\ 0 & 0 & 0 & \lambda -7 \end{bmatrix}$$

=> $\det(\lambda I - A) = (\lambda -1)(\lambda -4)(\lambda -6)(\lambda -7) = 0$

=> $\lambda = 1$ $\lambda=4$ $\lambda=6$ $\lambda=7$

8. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an orthogonal matrix and $d = \sqrt{3}/2$

Find the values of a, b and c if det (A) = 1

Solution:

We know that if A is orthogonal ~~since it~~ and is a 2 x 2 matrix and has a determinant of 1, it must be of the form :

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

since we know $d = \sqrt{3}/2$ and $\cos^{-1}(\sqrt{3}/2) = \pi/6$

we can find the values easily:

$a = \cos(\pi/6) = \sqrt{3}/2$
$b = -\sin(\pi/6) = 1/2$ and
$c = \sin(\pi/6) = 1/2$

Problem #9

Let A be the matrix

$$\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Let A be diagonalized by the matrix P =

$$\begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Find A^6

Solution:

We know that $A^k = P*D^k*P^{-1}$

We also know that $D = P^{-1}*A*P$

Thus

$A^6 = PD^6P^{-1}$

D = ←? how?

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So A^6 =

$$\begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2\text{^}6 & 0 & 0 \\ 0 & 2\text{^}6 & 0 \\ 0 & 0 & 1\text{^}6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -62 & 0 & -126 \\ 63 & 64 & 63 \\ 63 & 0 & 127 \end{bmatrix}$$

Problem #10

If A =

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Find an orthogonal matrix P that will diagonalize matrix A.

Solution:
Ma
We can get the characteristic equasion of A from

$$\text{Det}\,(\lambda I - A) = (\lambda - 2)^2\,(\lambda - 8) = 0$$

When $\lambda = 2$ we get the two vectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

When $\lambda = 8$ we get the eigen vector of

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

If we apply the Gram Schmidt process to the previous vectors we get the following orthonormal eigenvectors

$$\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -1\sqrt{6} \\ -1\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \qquad \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Then if we put those vectors into a matrix P as the columns we get the orthogonal Matrix

$$P = \begin{bmatrix} -1/\sqrt{2} & -1\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

12

1) Consider the basis S = {u_1, u_2, u_3} of $\Re^3$, where

$$u_1 = \begin{bmatrix}1\\1\\0\end{bmatrix},\ u_2 = \begin{bmatrix}1\\0\\1\end{bmatrix},\ u_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

Use Gram-Schmidt process to transform S to an orthonormal basis of $\Re^3$.

Solution:

Step 1: $v_1 = u_1$
Step 2:

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_2} \cdot v_1$$

$$= \begin{bmatrix}1/2\\-1/2\\1\end{bmatrix} \rightarrow \text{Clearing denominators we get } v_2 = \begin{bmatrix}1\\-1\\2\end{bmatrix}$$

Good!

$$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_2} \cdot v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} \cdot v_2$$

$$= \begin{bmatrix}-2/3\\2/3\\2/3\end{bmatrix} \rightarrow \text{Clearing denominators we get } v_3 = \begin{bmatrix}-2\\2\\2\end{bmatrix} \propto \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

Yeah!... but ...

Step 3: Normalize the vectors and get:

$$w_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, w_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}, w_3 = \frac{1}{\sqrt{12}}\begin{bmatrix}-2\\2\\2\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

as expected.

The set {w_1, w_2, w_3} is the required orthonormal basis.

2) Let $A = \begin{bmatrix}2 & -2\\-2 & 5\end{bmatrix}$

a) Find all Eigenvalues and Eigenvectors of the matrix A.
b) Diagonalize matrix A orthogonally.

Solution:

a) The characteristic polynomial of A is

$$tI - A = \det\begin{bmatrix} t-2 & 2 \\ 2 & t-5 \end{bmatrix} = (t-2)(t-5) - 4 = t^2 - 7t + 6 = (t-6)(t-1)$$

The eigenvalues of A are 1 and 6.

wrong use of "implies"

If t = 1 we get the eigenvector:

$$\begin{bmatrix} 1-2 & 2 \\ 2 & 1-5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow y = t, x = 2t \Rightarrow \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

If t = 6 we get the eigenvector:

$$\begin{bmatrix} 6-2 & 2 \\ 2 & 6-5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1/2 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \Rightarrow y = t, x = -1/2\, t \Rightarrow \begin{bmatrix} -1/2\, t \\ t \end{bmatrix} = t\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

b) We form the orthogonal matrix P after normalizing the eigenvectors:

$$P = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \text{ and } P^{-1}AP = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}.$$

3)

a) Show that A and A^T have the same eigenvalues.
b) If A is nonsingular and diagonalizable, then A^{-1} is also diagonalizable.

show this? is that the question/task?

Solution:

a) since det A = det A^T, then $\det(\lambda I_n - A^T) = \det(\{\lambda I_n - A\}^T) = \det(\lambda I_n - A)$

b) we have $P^{-1}AP = D$, where P is a matrix of column eigenvectors,

$PAP^{-1} = D$

$(PAP^{-1})^{-1} = PA^{-1}P^{-1} = D^{-1}$ (Note: D^{-1} is a diagonal matrix because D is diagonal)

4) Find the least square solution of the linear system Ax=b and the orthogonal projection of b onto the column space of A where $A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

– don't use ambiguous language

Solution:

$$A^T = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & 1 \end{bmatrix}$$

Multiply A by its transpose

$$A^T A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix}$$

Multiply transpose A by the b value

$$\begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

$A^T A\underline{x} = A^T b$

$$\begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

We form an augmented matrix and solve by row reduction

$$\begin{bmatrix} 14 & 0 & 6 \\ 0 & 6 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3/7 \\ 0 & 1 & -2/3 \end{bmatrix} \Rightarrow x_1 = 3/7, x_2 = -2/3$$

Orthogonal projection Ax=b

$$\begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3/7 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 46/21 \\ -5/21 \\ 13/21 \end{bmatrix}$$

5) For two orthogonal matrices A and B, prove the following:

A. $\det(A) = \pm 1$

B. AB is orthogonal

C. $\|Ax\| = \|x\|$

Solution:

A. $1 = \det(I) = \det(A'A) = \det(A')\det(A) = (\det(A))^2 \neq \pm 1$

if $(\det A)^2 = \pm 1$, then, in one instance, $\det A = \pm i$. Be careful with your thinking.

B. $(AB)'(AB) = B'A'AB = B'IB = B'B = I$

C. $\|Ax\| = (Ax \cdot Ax)^{\frac{1}{2}} = (x \cdot A'Ax)^{\frac{1}{2}} = (x \cdot x)^{\frac{1}{2}} = \|x\|$

6) If $\mathbf{p} = a + bx + cx^2$,

$\mathbf{q} = d + ex + fx^2$,

$\langle p, q \rangle = ad + be + cf$ (the inner-product),

this is not a sentence.

and $\|p\| = \langle p, p \rangle^{\frac{1}{2}}$ (the norm)

Find the cosine of the angle between **p** and **q**:

$\mathbf{p} = 5 + 6x + 2x^2$ $\qquad\qquad$ $\mathbf{q} = 3 + 2x + 2x^2$

Solution:

$$\cos\theta = \frac{\langle p,q\rangle}{\|p\|\|q\|} = \frac{(5)(3)+(6)(2)+(2)(2)}{(25+36+4)^{\frac{1}{2}}(9+4+4)^{\frac{1}{2}}} = .9326$$

7) Find the least squares solution of the system of linear equations.

$$\begin{aligned} 2x - 2y &= 2 \\ x - y &= -1 \\ 3x + y &= 1 \end{aligned}$$

Solution:

$$\begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix} \Rightarrow \qquad x = -2/3,\ y = 3/7$$

8) Complete the following.

a). Write out the Cauchy-Schwarz inequality in terms of u & v.

b). Verify that the Cauchy-Schwarz inequality holds for the given vectors.

i). u = (2, 1, 3,), v = (-1, 0, -3).

ii). u = (3, -12, 9), v = (1, -4, 3).

iii). u = v = (1, 1, 1).

iv). u = (0, 0, 0), v = (3, 3, 3).

Solution:

a). $\langle u, v\rangle^2 \le \|u\|^2\,\|v\|^2$

b). i). $\langle u, v\rangle^2 = (\sqrt{4+9})^2 = (\sqrt{13})^2 = 13.$

$\|u\|^2 = (\sqrt{4+1+9})^2 = (\sqrt{14})^2 = 14.$

$\|v\|^2 = (\sqrt{1+9})^2 = (\sqrt{10})^2 = 10.$

< 140. Therefore, the inequality is verified.

ii). $\langle u, v\rangle^2 = (\sqrt{3+48+27})^2 = (\sqrt{78})^2 = 78.$

$= (-2-9)^2 = (-11)^2 = 121$

what's with the square root?

this is all wrong.

121 < 140

$\|\|u\|\|^2$	=	$(\sqrt{9+144+81})^2 = (\sqrt{134})^2 = 134.$
$\|\|v\|\|^2$	=	$(\sqrt{1+16+9})^2 = (\sqrt{26})^2 = 26.$

$78^2 \le 3484$. Therefore, the inequality is verified. what's 78·78? Is it less than 3484?

iii).

$<u, v>^2$	=	$(\sqrt{1+1+1})^2 = (\sqrt{3})^2 = 9.$
$\|\|u\|\|^2$	=	$(\sqrt{1+1+1})^2 = (\sqrt{3})^2 = 3.$
$\|\|v\|\|^2$	=	$(\sqrt{1+1+1})^2 = (\sqrt{3})^2 = 3.$

$9 \le 9$. Therefore, the inequality is verified.

iv).

$<u, v>^2$	=	$(\sqrt{o})^2 = 0.$
$\|\|u\|\|^2$	=	$(\sqrt{o})^2 = 0.$
$\|\|v\|\|^2$	=	$(\sqrt{9+9+9})^2 = (\sqrt{27})^2 = 27.$

$0 \le 0$. Therefore, the inequality is verified.

9) Let R^3 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis (u_1, u_2, u_3) into an orthonormal basis.

$$u_1 = (2, 2, 2) \quad u_2 = (2, 2, 0) \quad u_3 = (2, 0, 0)$$

Solution:

$v_1 = u_1 = (2, 2, 2)$. $\propto (1,1,1)$ – use this

$v_2 = u_2 - (<u_2, v_1> / \|\| v_1 \|\|^2)\, v_1$

$= (2, 2, 0) - \frac{8}{12}(2, 2, 2)$

$= (2, 2, 0) - (\frac{4}{3}, \frac{4}{3}, \frac{4}{3})$ – use this

$= (\frac{2}{3}, \frac{2}{3}, -\frac{4}{3})$. $\propto (1,1,-2)$

$v_3 = u_3 - (<u_3, v_1> / \|\| v_1 \|\|^2)\, v_1 - (<u_3, v_2> / \|\| v_2 \|\|^2)\, v_2$

$= (2, 0, 0) - \frac{4}{12}(2, 2, 2) - \frac{4/3}{24/9}(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3})$

$= (2, 0, 0) - (\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) - (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$

$= (1, -1, 0)$.

Thus, $v_1 = (2, 2, 2)$, $v_2 = (\frac{2}{3}, \frac{2}{3}, -\frac{4}{3})$, $v_3 = (1, -1, 0)$ form an orthogonal basis for R_3.

$\|\| v_1 \|\| = 2\sqrt{3}, \quad \|\| v_2 \|\| = \frac{2}{3}\sqrt{6}, \quad \|\| v_3 \|\| = \sqrt{2}.$

An orthonormal basis for R_3 is

$q_1 = v_1 / \|\| v_1 \|\|$

$$= \frac{(2,2,2)}{2\sqrt{3}}$$

$$= (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}).$$ – as expected from choosing the simpler (1,1,1) or (1,1,-2).

$$q_2 = v_2 / \| v_2 \|$$

$$= \frac{(2/3, 2/3, -4/3)}{2\sqrt{6}/3}$$

$$= (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}).$$

$$q_3 = v_3 / \| v_3 \|$$

$$= \frac{(1,-1,0)}{\sqrt{2}}$$

$$= (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0).$$

10) Find the least squares solution of the linear system **Ax** = **b** given by

x + 2y = 7
3x – 4y = 1
-x +3y = 3.

Solution:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ -1 & 3 \end{bmatrix}, b = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}.$$

$$A^T A = \begin{bmatrix} 11 & -13 \\ -13 & 29 \end{bmatrix}.$$

$$A^T b = \begin{bmatrix} 7 \\ 19 \end{bmatrix}.$$

Thus, the system $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ is

$$\begin{bmatrix} 11 & -13 \\ -13 & 29 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 19 \end{bmatrix}.$$

Solving the equation results in x = 3, y = 2.

13

1. Let u = F(x) = $3 + 2x + x^2$ and v = G(x) = $1 + x + 2x^2$
Find <u,v> where <u,v> is defined to be $\int_{-1} F(x)G(x)dx$
and find $\|u\|$. (15 pts.)

Solution

$\int_{-1} (3+2x+x^2)(1+x+2x^2)dx$

$\int_{-1} 3 + 2x + x^2 + 3x + 2x^2 + x^3 + 6x^2 + 4x^3 + 2x^4\, dx$

$\int_{-1} 3 + 5x + 9x^2 + 5x^3 + 2x^4\, dx$ – simplify, use symmetry $= 2\int_0^1 3+9x^2+2x^4\,dx$

$3x + 5x^2/2 + 9x^3/3 + 5x^4/4 + 2x^5/5 \mid^1_{-1}$

$= (3 + 5/2 + 9/3 + 5/4 + 2/5) - (-3 + 5/2 - 9/3 + 5/4 - 2/5)$

=12.8 or 64/5

$\|u\| = \langle u,u\rangle^{1/2} = \langle F(x),F(x)\rangle = \sqrt{\int_{-1} F(x)F(x)dx}$

$=\sqrt{\int_{-1} (3+2x+x^2)(3+2x+x^2)dx}$

$= \sqrt{\int_{-1} 9+6x+3x^2+6x+4x^2+2x^3+3x^2+2x^3+x^4dx}$

$= \sqrt{\int_{-1} 9 + 12x + 10x^2 + 4x^3 + x^4\, dx}$ ditto

$= \sqrt{9x + 12x^2/2 + 10x^3/3 + 4x^4/4 + x^5/5 \mid^1_{-1}}$

$= \sqrt{(9 + 6 + 10/3 + 1 + 1/5) - (-9 + 6 - 10/3 + 1 - 1/5)}$

$= \sqrt{376/15} = \|u\|$

2. V is a subspace of R^3, v_1, v_2, v_3 are vectors in V:

$V_1 = (1,1,-2)$
$V_2 = (3,6,3)$
$V_3 = (3,7,6)$.

Find a basis for the orthogonal compliment of the subspace spanned by the vectors and check your work by finding the inner products (15 pts.)

watch the ambiguous or inaccurate language.

$$\begin{bmatrix} 1 & 1 & -2 \\ 3 & 6 & 3 \\ 3 & 7 & 6 \end{bmatrix}$$

why are you doing this? This development is "information free."

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & 9 \\ 0 & 4 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -12 & -36 \\ 0 & 12 & 36 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$X_1 = 5$
$X_2 = -3$
$X_3 = 1$

(5,-3,1) is a basis for the orthogonal compliment of V

Checking:

Excellent checking,

$\langle v_1,b\rangle = \langle(1,1,-2),(5,-3,1)\rangle = 5-3-2 = 0$

$\langle v_2,b\rangle = \langle(3,6,3)(5,-3,1)\rangle = 15-18+3 = 0$

$\langle v_3,b\rangle = \langle(3,7,6)(5,-3,1)\rangle = 15-21+6 = 0$

3. Verify that the following sets of vectors are orthogonal with respect to the Euclidean inner product; then convert it to an orthonormal set by normalizing the vectors. (15 pts.)
(1,0,-1), (2,0,2), (0,5,0)

just immediately change these to (1,0,1) and (0,1,0).

First verify that the set is orthogonal
<(1,0,-1), (2, 0,2)> = (2x1)+(0x0)+(2x-1)=0
<(2,0,2), (0,5,0)>=(2x0)+(5x0)+(2x0)=0
<(1,0,-1), (0,5,0)>=(1x0)+(0x5)+(-1x0)=0

Then to normalize the vectors

$$\frac{1}{\|v\|}v = \frac{1}{\langle v,v\rangle}v = \frac{1}{\langle(1,0,-1),(1,0,-1)\rangle}(1,0,-1) = \frac{1}{\sqrt{2}}(1,0,-1) = (\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}})$$

$$\frac{1}{\langle(2,0,2),(2,0,2)\rangle}(2,0,2) = \frac{1}{2\sqrt{2}}(2,0,2) = (\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})$$

$$\frac{1}{\langle(0,5,0),(0,5,0)\rangle}(0,5,0) = \frac{1}{5}(0,5,0) = (0,1,0)$$

Cool! Non-standard Gram-Schmidt! I like it. *

4. Let R^3 have the inner product <**u**,**v**> = $u_1v_1 + 2u_2v_2 + 3u_3v_3$. Use the Gram-Schmidt process to transform $u_1 = (1,1,1)$, $u_2 = (1,1,0)$ $u_3 = (1,0,0)$ into an orthonormal set. (15 pts.)

<(1,1,0),(1,1,1)> = 1·1 + 2·1·1 + 3·0·1 = 3 ≠ 2

$$u_1 = v_1 = (1,1,1)$$

$$v_2 = u_2 - \frac{\langle u_2,v_1\rangle}{\|v_1\|}v_1 = (1,1,0) - \frac{\langle(1,1,0),(1,1,1)\rangle}{\|(1,1,1)\|^2}(1,1,1) = (1,1,0) - \frac{2}{3}(1,1,1) = (\frac{1}{3},\frac{1}{3},\frac{-1}{3})$$

— change to (1,1,-1) (1)
<(1,1,1),(1,1,1)> = 1+2+3 = 6 ≠ 3

$$v_3 = u_3 - \frac{\langle u_3,v_1\rangle}{\|v_1\|}v_1 - \frac{\langle u_3,v_2\rangle}{\|v_2\|}v_2 = (1,0,0) - \frac{\langle(1,0,0),(1,1,1)\rangle}{\|(1,1,1)\|^2}(1,1,1) - \frac{\left\langle(1,0,0),(\frac{1}{3},\frac{1}{3},\frac{-1}{3})\right\rangle}{\left\|(\frac{1}{3},\frac{1}{3},\frac{-1}{3})\right\|^2}(\frac{1}{3},\frac{1}{3},\frac{-1}{3}) =$$

$$= (1,0,0) - \frac{1}{3}(1,1,1) - \frac{1}{3}(\frac{1}{3},\frac{1}{3},\frac{-1}{3}) = (1,0,0) - (\frac{1}{3},\frac{1}{3},\frac{1}{3}) - (\frac{1}{9},\frac{1}{9},\frac{-1}{9}) = (\frac{5}{9},\frac{4}{9},\frac{2}{9})$$

Now to Normalize

$$\|v_1\| = \sqrt{3}, \|v_2\| = \frac{1}{\sqrt{3}}, \|v_3\| = \frac{\sqrt{5}}{3}$$

$$q_1 = \frac{v_1}{\|v_1\|} = (\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}) \qquad q_2 = \frac{v_2}{\|v_2\|} = (\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{-\sqrt{3}}{3})$$

$$q_3 = \frac{v_3}{\|v_3\|} = (\frac{5}{3\sqrt{5}},\frac{4}{3\sqrt{5}},\frac{2}{3\sqrt{5}})$$

* ultimately it looks like you may have used the standard inner product – I am very disappointed.

5. Find the orthogonal projection of **u** onto the subspace of R^3 spanned by the vectors v_1, v_2, and v_3
$\mathbf{u} = (5,4,1)$ $v_1 = (2,1,2)$ $v_2 = (1,2,3)$ (15 pts.)

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}$$

✓ This is like saying "Pencil."
– i.e. it is information free.
(Explain)

And A**x** = **u** can be show as

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$$

So we find $A^T A$ and $A^T \mathbf{u}$

$$A^T A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 10 \\ 10 & 14 \end{bmatrix}$$

$$A^T \mathbf{u} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 16 \end{bmatrix}$$

Now we set $A^T A \mathbf{x}$ and $A^T \mathbf{u}$ equal to each other

$$\begin{bmatrix} 9 & 10 \\ 10 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 16 \end{bmatrix}$$

So

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{7}{13} & \frac{-5}{13} \\ \frac{-5}{13} & \frac{9}{26} \end{bmatrix}\begin{bmatrix} 16 \\ 16 \end{bmatrix} = \begin{bmatrix} \frac{32}{13} \\ \frac{-8}{13} \end{bmatrix}$$

Don't just give a recipe. Rather, explain what you are doing and why.

6. Consider the bases $B = \{u_1, u_2\}$ and $B' = \{v_1, v_2\}$ for R^2, where (no antecedent)

$$u_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -4 \\ 0 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 8 \\ -8 \end{bmatrix}, \text{ and } w = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Find the following:

a) The transition matrix from B\` to B. (5 pts.)

We can find the transition matrix by solving the augmented matrix

$$\left[\begin{array}{cc|c|c} 2 & -4 & 6 & 8 \\ 2 & 0 & 2 & -8 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & -2 & 3 & 4 \\ 0 & 4 & -4 & -16 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & -2 & 3 & 4 \\ 0 & 1 & -1 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

$$\Rightarrow P = \begin{bmatrix} 1 & -4 \\ -1 & -4 \end{bmatrix}$$

b) The transition matrix from B to B\`. (5 pts.)

$Q = P^{-1}$ can solve by augmented matrix

$$\left[\begin{array}{cc|cc} 1 & -4 & 1 & 0 \\ -1 & -4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -4 & 1 & 0 \\ 0 & -8 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -4 & 1 & 0 \\ 0 & 1 & -1/8 & -1/8 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & -1/2 \\ 0 & 1 & -1/8 & -1/8 \end{array}\right]$$

$$\Rightarrow Q = \begin{bmatrix} 1/2 & -1/2 \\ -1/8 & -1/8 \end{bmatrix}$$

c) The coordinate vector $[w]_B$. (5 pts.)

$$\left[\begin{array}{cc|c} 2 & -4 & -2 \\ 2 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 4 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$$\Rightarrow [w]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

d) The coordinate vector $[w]_{B'}$. (Solve by the equation $[w]_{B'} = P^{-1}[w]_B$, and then check your answer by solving for $[w]_{B'}$ directly.) (5 pts.)

Using the equation given, we find:

$$[w]_{B'} = \begin{bmatrix} 1/2 & -1/2 \\ -1/8 & -1/8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/4 \end{bmatrix}$$

Solving directly through the augmented matrix, we get the same answer

$$\left[\begin{array}{cc|c} 6 & 8 & -2 \\ 2 & -8 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 4 & -1 \\ 1 & -4 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 16 & -4 \\ 1 & -4 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 1 & -1/4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -1/4 \end{array}\right]$$

7. Determine which of the following matrices are orthogonal (5 pts. each)

a) $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

b) $\begin{bmatrix} a & -a \\ a & a \end{bmatrix}$

c) $\begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -2/3 & 1/3 \\ 1/2 & 1/3 & 1/3 \end{bmatrix}$

To solve these, remember that for an orthogonal matrix, $A^{-1} = A^T$

a) First we find A^{-1} $\left[\begin{array}{cc|cc} 1/\sqrt{2} & 1/\sqrt{2} & 1 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & \sqrt{2} & 0 \\ -1 & 1 & 0 & \sqrt{2} \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & \sqrt{2} & 0 \\ 0 & 2 & \sqrt{2} & \sqrt{2} \end{array}\right] \sim$

$\left[\begin{array}{cc|cc} 1 & 1 & \sqrt{2} & 0 \\ 0 & 1 & \sqrt{2}/2 & \sqrt{2}/2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} & 1/\sqrt{2} \end{array}\right]$ $A^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \rightarrow A^{-1} = A^T$

→ This matrix is orthogonal

b) $\left[\begin{array}{cc|cc} a & -a & 1 & 0 \\ a & a & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} a & -a & 1 & 0 \\ 0 & 2a & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} a & 0 & 1/2 & 1/2 \\ 0 & a & -1/2 & 1/2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/2a & 1/2a \\ 0 & 1 & -1/2a & 1/2a \end{array}\right] \rightarrow$

$A^{-1} = \begin{bmatrix} 1/2a & 1/2a \\ -1/2a & 1/2a \end{bmatrix}$ $A^T = \begin{bmatrix} a & a \\ -a & a \end{bmatrix}$ $A^{-1} \neq A^T \rightarrow$ not orthogonal unless $a = \pm \frac{1}{\sqrt{2}}$'

c) $\left[\begin{array}{ccc|ccc} 1/2 & 1/2 & 1/2 & 1 & 0 & 0 \\ 1/2 & -2/3 & 1/3 & 0 & 1 & 0 \\ 1/2 & 1/3 & 1/3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 0 \\ 3 & -4 & 2 & 0 & 6 & 0 \\ 3 & 2 & 2 & 0 & 0 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & -7 & -1 & -6 & 6 & 0 \\ 0 & -1 & -1 & -6 & 0 & 6 \end{array}\right] \sim$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 6 & 0 & 6 \\ 0 & 0 & 6 & 36 & 6 & 42 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -4 & -1 & -7 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 6 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 0 & -6 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 6 & 1 & 7 \end{array}\right]$

$A^{-1} = \begin{bmatrix} -4 & 0 & -6 \\ 0 & -1 & -1 \\ 0 & 1 & 7 \end{bmatrix}$ $A^T = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -2/3 & 1/3 \\ 1/2 & 1/3 & 1/3 \end{bmatrix}$ $\rightarrow A^{-1} \neq A^T \rightarrow$ not orthogonal

8. Find the eigenvalues of the following matrices. (5 pts. each)

a) $\begin{bmatrix} 6 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

To find the eigenvalues, use the formula det(λI – A) = 0.

a) $\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 0 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \lambda-6 & 0 & 0 \\ 2 & \lambda+3 & 0 \\ 1 & 0 & \lambda-2 \end{bmatrix}\right) = (\lambda-6)(\lambda+3)(\lambda-2)$

→ λ = 6, -3, 2

b) $\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \lambda-1 & 2 \\ 3 & \lambda-4 \end{bmatrix}\right) = \lambda 2 - 5\lambda + 4 - 6 = 0 \rightarrow \lambda 2 - 5\lambda - 2 = 0$

$\lambda = (5 + \sqrt{33})/2, \quad (5 - \sqrt{33})/2$

c) $\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \lambda & 0 & 2 \\ 0 & \lambda & 0 \\ 2 & 0 & \lambda \end{bmatrix}\right) = \lambda 3 - 4\lambda = \lambda(\lambda 2 - 4) = \lambda(\lambda+2)(\lambda-2)$

→ λ = 0, -2, 2

9. Find the matrix that diagonalizes the matrix $\begin{bmatrix} -2 & -2 \\ -3 & 3 \end{bmatrix}$ and then compute $P^{-1}AP$ (15 pts.)

a) To find the eigenvectors remember that det(λI – A) = 0.

$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & -2 \\ -3 & 3 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \lambda+2 & 2 \\ 3 & \lambda-3 \end{bmatrix}\right) = \lambda 2 - \lambda - 6 - 6 = 0 = \lambda 2 - \lambda - 12 =$

$(\lambda - 4)(\lambda + 3) \rightarrow \lambda = 4, -3$

For λ = 4 $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x_1 = -t,\ x_2 = 3t \rightarrow P_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

For λ = -3 $\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x_1 = 2t,\ x_2 = t \rightarrow P_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$$

b) P^{-1} : $-1/7\begin{bmatrix} 1 & -2 \\ -3 & -1 \end{bmatrix}$

$$P^{-1}AP = -1/7\begin{bmatrix} 1 & -2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} -2 & -2 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = -1/7\begin{bmatrix} 4 & -8 \\ 9 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = -1/7\begin{bmatrix} -28 & 0 \\ 0 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$$

10. Using the matrix $\begin{bmatrix} -2 & -2 \\ -3 & 3 \end{bmatrix}$ compute A^6 (10 pts.)

Remember $A^k = PD^kP^{-1}$

$$A^6 = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 4^6 & 0 \\ 0 & -3^6 \end{bmatrix}(-1/7)\begin{bmatrix} 1 & -2 \\ -3 & -1 \end{bmatrix}$$, let $4^6 = x$, $(-3)^6 = y$

$$A^6 = \begin{bmatrix} -x & 2y \\ 3x & y \end{bmatrix}(-1/7)\begin{bmatrix} 1 & -2 \\ -3 & -1 \end{bmatrix} = (-1/7)\begin{bmatrix} x-6y & 2x-2y \\ -3x-3y & -6x-y \end{bmatrix} = 1/7\begin{bmatrix} 278 & -6734 \\ 14475 & 25305 \end{bmatrix}$$

↑ each of these number better be divisible by 7, as A^6 clearly has integer entries.

$$\begin{array}{r} 39 \\ 7\,\overline{)\,278} \\ \underline{21} \\ 68 \\ \underline{63} \\ 5 \end{array}$$

→ you messed up somewhere.

1. Let the space vector P_2 have the inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$.
   a) Find $\|\mathbf{p}\|$ for $\mathbf{p}_1=1$, $\mathbf{p}_2=x$, and $\mathbf{p}_3=x^2$.
   b) Find $d(\mathbf{p},\mathbf{q})$ if $\mathbf{p}=1$ and $\mathbf{q}=x$.

Solution (13 Points):

a) $\|\mathbf{p}_1\| = \sqrt{\int_{-1}^{1} 1 \times dx} = \sqrt{x\Big|_{-1}^{1}} = \sqrt{1-(-1)} = \sqrt{2}$

$$\|\mathbf{p}_2\| = \sqrt{\int_{-1}^{1} (x \times x)dx} = \sqrt{\int_{-1}^{1} x^2 dx} = \sqrt{\frac{x^3}{3}\Big|_{-1}^{1}} = \sqrt{\frac{1^3}{3} - \frac{-1^3}{3}} = \sqrt{\frac{2}{3}}$$

$$\|\mathbf{p}_3\| = \sqrt{\int_{-1}^{1} (x^2 \times x^2)dx} = \sqrt{\int_{-1}^{1} x^4 dx} = \sqrt{\frac{x^5}{5}\Big|_{-1}^{1}} = \sqrt{\frac{1^5}{5} - \frac{-1^5}{5}} = \sqrt{\frac{2}{5}}$$

b) $d(\mathbf{p},\mathbf{q}) = \|\mathbf{p}-\mathbf{q}\| = \sqrt{\int_{-1}^{1} (x-1)^2 dx} = \sqrt{\int_{-1}^{1} (x-1)(x-1)dx} = \sqrt{\int_{-1}^{1} (x^2 - 2x + 1)dx}$

$$= \sqrt{\left(\frac{x^3}{3} - x^2 + x\right)\Big|_{-1}^{1}} = \sqrt{\left(\left(\frac{1}{3} - \frac{-1}{3}\right) - (1-1) + (1-(-1))\right)} = \sqrt{\frac{2}{3} + 2} = \sqrt{\frac{8}{3}}$$

unnecessary

$\int_{-1}^{1} (x-1)^2 dx = \frac{(x-1)^3}{3}\Big|_{-1}^{1} = -\frac{(-2)^3}{3} = \frac{8}{3}$

---

2. Let R^2 and R^3 have the Euclidean inner product. In each part, find the cosine angle between **u** and **v**.
   a) $\mathbf{u} = (1,3)$, $\mathbf{v} = (2,-4)$
   b) $\mathbf{u} = (0,1)$, $\mathbf{v} = (8,-3)$
   c) $\mathbf{u} = (1,5,-2)$, $\mathbf{v} = (-2,4,9)$
   d) $\mathbf{u} = (1,-3,0)$, $\mathbf{v} = (4,8,1)$

Solution(12 Points): Note the formula: $\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$ *and* $0 \le \theta \le \pi$. Using this, find $\langle \mathbf{u}, \mathbf{v} \rangle$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$, then plug these into the formula above.

a) $\langle \mathbf{u}, \mathbf{v} \rangle = ((1\times 2) + (3 \times -4)) = (2 + -12) = -10, \|\mathbf{u}\| = \sqrt{1^2+3^2} = \sqrt{10}, \|\mathbf{v}\| = \sqrt{2^2 + (-4)^2} = \sqrt{20}$

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \frac{-10}{\sqrt{10}\sqrt{20}} = \frac{-10}{\sqrt{200}} = \frac{-10}{10\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

(so $\theta = 135°$?)

b)
$\langle \mathbf{u}, \mathbf{v} \rangle = ((0 \times 1) + (8 \times -3)) = (0 + -3) = -3, \|\mathbf{u}\| = \sqrt{0^2+1^2} = \sqrt{1} = 1, \|\mathbf{v}\| = \sqrt{8^2 + (-3^2} = \sqrt{73}$

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \ \|\mathbf{v}\|} = \frac{-3}{1\sqrt{73}} = \frac{-3}{\sqrt{73}}$$

c) $\langle \mathbf{u}, \mathbf{v} \rangle = ((1\times -2)+(5\times 4)+(-2\times 9)) = (-2+20-18) = 0$, because the inner product is 0, we get

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \ \|\mathbf{v}\|} = \frac{0}{\cdots} = 0.$$ (so $\theta = 90°$?)

d) $\langle \mathbf{u}, \mathbf{v} \rangle = ((1\times 4)+(-3\times 8)+(0\times 1)) = (4-24+0) = -20, \|\mathbf{u}\| = \sqrt{1^2+(-3)^2+0^2} = \sqrt{10},$

$\|\mathbf{v}\| = \sqrt{4^2+8^2+1^2} = \sqrt{81} = 9.$

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \ \|\mathbf{v}\|} = \frac{-20}{9\sqrt{10}}$$

(I suppose these factors mean something? Say it.)

3. Use $A^k = PD^kP^{-1}$ to find A^3 and A^{10}, where $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

Solution(20 Points): First find the characteristic equation, eigenspaces and bases for the matrix as follows.

$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ and det (λI-A)=0 $\Rightarrow$ det $\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix}$ $\Rightarrow$ (watch use of "$\Rightarrow$")

$\lambda(\lambda-2)(\lambda-3)-(-1)(\lambda-2)(2) \Rightarrow \lambda^3-5\lambda^2+8\lambda-4 \Rightarrow (\lambda-1)(\lambda-2)^2 \Rightarrow \lambda = 1, 2, 2.$

Thus the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 2, and \lambda_3 = 2$, or two eigenspaces for A. — watch poor English

~~The Basis~~ Bases are found by plugging the eigenvalues back into the matrix for λ.

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If λ = 1, then $\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and this gives us x_1 = -2s, x_2 = s, and x_3 = s. The eigenvectors corresponding to λ = 1 are of the form

$\begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ so that ~~the~~ a basis is $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

Repeating the process again for $\lambda = 2$ results in the basis $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Next, we need to find matrix P that diagonalizes A. To do this, we use the three basis vectors in total which we found above to get $P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. To make sure that P is diagonalizable, we must show that $P^{-1}AP$ is a diagonal matrix. Thus

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

make P a diagonalizable matrix.

where did this come from?

Using the formula $A^k = PD^kP^{-1}$ given above, and remembering that $D = P^{-1}AP$, we can plug into the formula

$$A^3 = PD^3P^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^3 & 0 & 0 \\ 0 & 2^3 & 0 \\ 0 & 0 & 1^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -6 & 0 & -14 \\ 7 & 8 & 7 \\ 7 & 0 & 15 \end{bmatrix}.$$

To find A^{10}, all we need to do is replace 10 with 3 and solve: (Remember $2^{10} = 1024$)

$$A^{10} = PD^{10}P^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 1^{10} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1022 & 0 & -2046 \\ 1023 & 1024 & 1023 \\ 1023 & 0 & 2047 \end{bmatrix}.$$

---

4. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$, and $\langle A,B \rangle := A_{11}B_{11} + A_{12}B_{12} + A_{13}B_{13} + A_{21}B_{21} + A_{22}B_{22} + A_{23}B_{23} + A_{31}B_{31} + A_{32}B_{32} + A_{33}B_{33}$, find:

= Tr(A^T B)?

a) $\langle A,B \rangle$
b) $||A||$
c) $||A-B||$

where did this come from?

Solution(10 Points): First find the characteristic equation, eigenspaces and bases for the matrix as follows.

a. $\langle A,B \rangle = 3 + 4 + 3 + 24 + 25 + 24 + 3 + 4 + 3 = 93.$

b. $||A|| = \langle A,A \rangle^{1/2} = \sqrt{1+4+9+16+25+36+1+4+9} = \sqrt{105}$

c. $||A-B|| = \langle A-B, A-B \rangle^{1/2} =$

$\sqrt{-2^2+0+2^2+-2^2+0+2^2+0+-2^2+0+2^2} = \sqrt{4+4+4+4+4+4} = 2\sqrt{6}$

---

5. Are the following matrices orthogonal? If so, find the inverse of the matrix.

a. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b. $B = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

c. $C = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 0 \\ 1/2 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1 \end{bmatrix}$

Solution(10 Points):

a) Let e_1, e_2, e_3 be the column vectors of A. Matrix A will be orthogonal if $\langle e_1, e_2\rangle = 0$, $\langle e_1,e_3\rangle = 0$, $\langle e_2,e_3\rangle = 0$, $||e_1|| = 1$, $||e_2|| = 1$, and $||e_3|| = 1$.

$\langle e_1, e_2\rangle = 0 + 0 + 0 = 0$,
$\langle e_1, e_3\rangle = 0 + 0 + 0 = 0$,
$\langle e_2, e_3\rangle = 0 + 0 + 0 = 0$,
$||e_1|| = \langle e_1,e_1\rangle^{1/2} = \sqrt{1+0+0} = 1$,
$||e_2|| = \langle e_2,e_2\rangle^{1/2} = \sqrt{0+1+0} = 1$,
$||e3|| = \langle e_3,e_3\rangle^{1/2} = \sqrt{0+0+1} = 1$,

So A is orthogonal and $A^{-1} = A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

b) Let e_1, e_2, e_3 be the column vectors of B. Matrix B will be orthogonal if $\langle e_1, e_2\rangle = 0$, $\langle e_1,e_3\rangle = 0$, $\langle e_2,e_3\rangle = 0$, $||e_1|| = 1$, $||e_2|| = 1$, and $||e_3|| = 1$.

$\langle e_1, e_2\rangle = 0 + 0 + 0 = 0$,
$\langle e_1, e_3\rangle = 0 - 1/2 + 1/2 = 0$,
$\langle e_2, e_3\rangle = 0 + 0 + 0 = 0$,
$||e_1|| = \langle e_1,e_1\rangle^{1/2} = \sqrt{0+1/2+1/2} = 1$,
$||e_2|| = \langle e_2,e_2\rangle^{1/2} = \sqrt{1+0+0} = 1$,
$||e3|| = \langle e_3,e_3\rangle^{1/2} = \sqrt{0+1/2+1/2} = 1$,

So B is orthogonal and $B^{-1} = B^T = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

c) Let e_1, e_2, e_3 be the column vectors of C. Matrix C will be orthogonal if $\langle e_1, e_2\rangle = 0$, $\langle e_1,e_3\rangle = 0$, $\langle e_2,e_3\rangle = 0$, $\|e_1\| = 1$, $\|e_2\| = 1$, and $\|e_3\| = 1$.

$\langle e_1, e_2\rangle = 1/2\sqrt{2} - 1/2\sqrt{2} + 0 = 0$,
$\langle e_1, e_3\rangle = 0 + 0 + 1/\sqrt{2} = 1/\sqrt{2} \neq 0$,
So C is not orthogonal.

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6. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix}$.

Solution(15 Points):

$\lambda I - A = \begin{bmatrix} \lambda & 0 & 1 \\ 1 & \lambda & 0 \\ 1 & 1 & \lambda-1 \end{bmatrix}$ det $(\lambda I - A) = 0 = \lambda^3 - \lambda^2 - \lambda + 1 = (\lambda-1)^2(\lambda+1)$. Thus $\lambda = \pm 1$.

X is an eigenvector with eigenvalues $\lambda = 1$ iff. $\mathbf{X} \neq \mathbf{0}$, and $\mathbf{x} \in$ Null $[\lambda I - A]_{\lambda=1}$

$$Null \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = Null \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \not\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Thus the eigenvectors with eigenvalues $\lambda = 1$ are Span$\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}$.

X is an eigenvector with eigenvalues $\lambda = -1$ iff. $\mathbf{X} \neq \mathbf{0}$, and $\mathbf{x} \in$ Null $[\lambda I - A]_{\lambda=-1}$

$$Null \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} = Nul \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = Nul \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \not\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus the eigenvectors with eigenvalues $\lambda = -1$ are Span $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$.

---

7. Find the normal system associated with the given linear systems, and then find the least squares solution to each normal system.

a. $A = \begin{bmatrix}1 & 2\\-1 & 1\\3 & 2\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}1\\1\\2\end{bmatrix}$.

b. $A = \begin{bmatrix}1 & 1\\0 & 1\\0 & 1\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}4\\1\\2\end{bmatrix}$.

Solution(15 Points):

a. $A^T A\mathbf{x} = A^T\mathbf{b}$ is the normal system.

$A^T = \begin{bmatrix}1 & -1 & 3\\2 & 1 & 2\end{bmatrix}$ thus $A^T A = \begin{bmatrix}11 & 7\\7 & 9\end{bmatrix}$ and $A^T\mathbf{b} = \begin{bmatrix}6\\7\end{bmatrix}$.

$\left[\begin{array}{cc|c}11 & 7 & 6\\7 & 9 & 7\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 7/11 & 6/11\\1 & 9/7 & 1\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 7/11 & 6/11\\0 & 50/77 & 5/11\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 7/11 & 6/11\\0 & 1 & 7/10\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 0 & 1/10\\0 & 1 & 7/10\end{array}\right]$ Thus

$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1/10\\7/10\end{bmatrix}$.

b. $A^T A\mathbf{x} = A^T\mathbf{b}$ is the normal system.

$A^T = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\end{bmatrix}$ thus $A^T A = \begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}$ and $A^T\mathbf{b} = \begin{bmatrix}4\\7\end{bmatrix}$

$\left[\begin{array}{cc|c}1 & 1 & 4\\1 & 3 & 7\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 1 & 4\\0 & 2 & 3\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 1 & 4\\0 & 1 & 3/2\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 0 & 5/2\\0 & 1 & 3/2\end{array}\right]$ Thus $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}5/2\\3/2\end{bmatrix}$.

---

8. Prove that if matrices A and B are orthogonal, then

a) A^{-1} is orthogonal

b) AB is orthogonal

c) det A = ± 1

Solution(15 Points):

a. [illegible] A is orthogonal $\Leftrightarrow A^T A = I$ * and A is square $\Leftrightarrow I = A^{-1}(A^T)^{-1} = A^{-1}(A^{-1})^T \Leftrightarrow A^{-1}$ and $(A^{-1})^T$ are inverses $\Leftrightarrow (A^{-1})^T A^{-1} = I$ which is * with A replaced by A^{-1}.

Thus A^{-1} is orthogonal.

b. If A and B are orthogonal, then $A^T A = I = B^T B$

$\Leftrightarrow (AB)^T(AB) = B^T A^T AB = B^T IB = B^T B = I$

Thus AB is orthogonal.

c. $1 = \det I = \det(A^T A) = \det A^T \det A = (\det A)^2 = 1 \Leftrightarrow \det A = \pm 1$.

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9. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where

$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

a) Find the transition matrix from B' to B.

b) Compute the coordinate vector $[\mathbf{w}]_B$ where $\mathbf{w} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ and use the following equation to compute $[\mathbf{w}]_{B'}$ (coordinate vector for $\mathbf{w}$ in B') : $[\mathbf{w}]_{B'} = P^{-1}[\mathbf{w}]_B$.

Solution(20 Points):

why? just solve. Easy if you use my (transparent) formalism.

a. $\mathbf{v}_1 = k_1\mathbf{u}_1 + k_2\mathbf{u}_2$ $\quad 2k_1 + 4k_2 = 1$, $\quad 2k_1 - k_2 = 3$ $\quad$ By inspection, $k_1 = 13/10$, $k_2 = -2/5$

$\mathbf{v}_1 = \frac{13}{10}\mathbf{u}_1 - \frac{2}{5}\mathbf{u}_2$ $\quad$ Thus $[\mathbf{v}_1]_B = \begin{bmatrix} 13/10 \\ -2/5 \end{bmatrix}$

$\mathbf{v}_2 = k_1\mathbf{u}_1 + k_2\mathbf{u}_2$ $\quad 2k_1 + 4k_2 = -1$, $\quad 2k_1 - k_2 = -1$ $\quad$ By inspection, $k_1 = -1/2$, $k_2 = 0$

$\mathbf{v}_2 = -\frac{1}{2}\mathbf{u}_1 + 0\mathbf{u}_2$ $\quad$ Thus $[\mathbf{v}_2]_B = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$

Thus the transition matrix from B' to B is $P = \begin{bmatrix} 13/10 & -1/2 \\ -2/5 & 0 \end{bmatrix}$

b. $\mathbf{w} = k_1\mathbf{u}_1 + k_2\mathbf{u}_2$ $\quad 2k_1 + 4k_2 = 3$, $\quad 2k_1 - k_2 = -5$ $\quad$ By inspection, $k_1 = -17/10$, $k_2 = 8/5$

$$\mathbf{w} = -\tfrac{17}{10}\,\mathbf{u}_1 + \tfrac{8}{5}\,\mathbf{u}_2 \qquad \text{Thus } [\mathbf{w}]_B = \begin{bmatrix} -\tfrac{17}{10} \\ \tfrac{8}{5} \end{bmatrix}$$

We must now find P^{-1}.

$$\left[\begin{array}{cc|cc} \tfrac{13}{10} & -\tfrac{1}{2} & 1 & 0 \\ -\tfrac{2}{5} & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & -\tfrac{5}{2} \\ 1 & -\tfrac{5}{13} & \tfrac{10}{13} & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & -\tfrac{5}{2} \\ 0 & -\tfrac{5}{13} & \tfrac{10}{13} & \tfrac{5}{2} \end{array}\right] \sim$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & -\tfrac{5}{2} \\ 0 & 1 & -2 & -\tfrac{13}{2} \end{array}\right] \qquad \text{Thus } P^{-1} = \begin{bmatrix} 0 & -\tfrac{5}{2} \\ -2 & -\tfrac{13}{2} \end{bmatrix}$$

Again, there's a much more transparent way to do this — less alchemy.

Thus the coordinate vector for **w** in B', $[\mathbf{w}]_{B'}$, is

$$P^{-1}[\mathbf{w}]_B = \begin{bmatrix} 0 & -\tfrac{5}{2} \\ -2 & -\tfrac{13}{2} \end{bmatrix} \begin{bmatrix} -\tfrac{17}{10} \\ \tfrac{8}{5} \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$$

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10. Let R^2 have the Euclidean inner product. Let $\mathbf{u}_1 = (1,-4)$ and $\mathbf{u}_2 = (2,3)$.
   a.) Use the Gram-Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.
   b.) Find the norms of $\mathbf{v}_1$ and $\mathbf{v}_2$.
   c.) Then normalize the orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2\}$.
   d.) Draw basis vectors $\{\mathbf{q}_1, \mathbf{q}_2\}$ in the *xy*-plane.

Solution (20 Points):

a) Let:

$\mathbf{v}_1 = \mathbf{u}_1 = (1, -4)$

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{w1}\,\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (2, 3) - \frac{(2,3)\bullet(1,-4)}{(1,-4)\bullet(1,-4)}(1, -4)$$

$$= (2, 3) - \frac{-10}{17}(1, -4) = \left(\frac{44}{17}, \frac{11}{17}\right)$$

Thus,

$$\mathbf{v}_1 = (1, -4) \quad \text{and} \quad \mathbf{v}_2 = \left(\frac{44}{17}, \frac{11}{17}\right)$$ ↗ change to (4,1) ✱

b) Let:

$$\|\mathbf{v}_1\| = \sqrt{\langle \mathbf{v}_1, \mathbf{v}_1\rangle} = \sqrt{1^2 + (-4)^2} = \sqrt{17}$$

$$\|\mathbf{v}_2\| = \sqrt{\langle \mathbf{v}_2, \mathbf{v}_2\rangle} = \sqrt{\left(\frac{44}{17}\right)^2 + \left(\frac{11}{17}\right)^2} = \sqrt{\frac{121}{17}} = \frac{11}{\sqrt{17}}$$

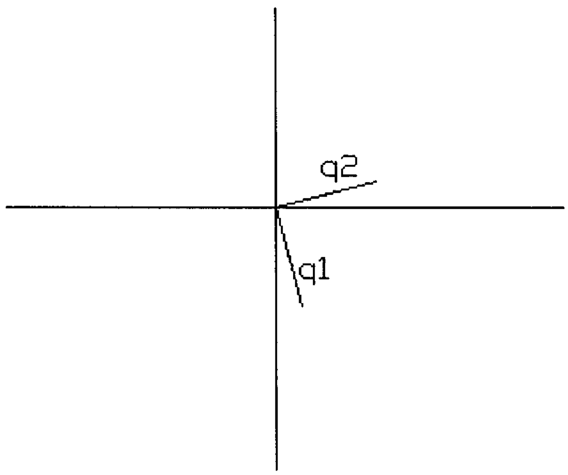
c) Let:

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, -4)}{\sqrt{17}} = \left(\frac{1}{\sqrt{17}}, \frac{-4}{\sqrt{17}}\right) \text{ or } (0.242535625, -0.9701425001)$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(\frac{44/17}{\sqrt{121/17}}, \frac{11/17}{\sqrt{121/17}}\right) \text{ or } (0.9701425001, 0.242535625)$$

$= \left(\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}}\right)$ (consistent with * — lots of good reasons to rescale early on.

d)



this is well-written. 15

Question 1:

For any linear system $A\underline{x} = \underline{b}$, the associated normal system $A^TA\underline{x} = A^T\underline{b}$ is consistent, and if W is the column space of A, and $\underline{x}$ is any least squares solution of $A\underline{x} = \underline{b}$, then the orthogonal projection of $\underline{b}$ on W is $\text{proj}_W\, \underline{b} = A\underline{x}$.

Find the orthogonal projection of $\underline{b}$ on ColA given by the following linear system $A\underline{x} = \underline{b}$.

$$\begin{aligned} 2x_1 + x_2 - 2x_3 &= 6 \\ x_1 \quad\quad - x_3 &= 3 \\ x_1 + x_2 \quad\quad &= 9 \\ x_1 + x_2 - x_3 &= 6 \end{aligned}$$

Solution 1:

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 6 \\ 3 \\ 9 \\ 6 \end{bmatrix},$$

$$A^TA = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}$$

$$A^T\underline{b} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 30 \\ 21 \\ -21 \end{bmatrix}$$

$$A^TA\underline{x} = A^T\underline{b} \Rightarrow \begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 21 \\ -21 \end{bmatrix} \not\Rightarrow \begin{bmatrix} 7 & 4 & -6 & 30 \\ 4 & 3 & -3 & 21 \\ -6 & -3 & 6 & -21 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}$$

$$A\underline{x} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 9 \\ 5 \end{bmatrix}.$$

Question 2:

Let **u**= (u_1, u_2, u_3) and **v**= (v_1, v_2, v_3). Determine if the inner product $\langle \mathbf{u}, \mathbf{v}\rangle = u_1 v_1 + u_3 v_3$ is an inner product on $\Re^3$ by verifying that the inner product axioms hold. If it does not hold, list the axiom(s) that do not hold.

Solution 2:

Axiom 1 states $\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle$ and for the given inner product $\langle \mathbf{u}, \mathbf{v}\rangle = u_1 v_1 + u_3 v_3$ and $\langle \mathbf{v}, \mathbf{u}\rangle = v_1 u_1 + v_3 u_3$ and $v_1 u_1 + v_3 u_3 = u_1 v_1 + u_3 v_3$ therefore $\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle$ and this demonstrates that axiom one holds for this inner product.

Axiom 2 states $\langle \mathbf{u}+\mathbf{v}, \mathbf{z}\rangle = \langle \mathbf{v}, \mathbf{z}\rangle + \langle \mathbf{u}, \mathbf{z}\rangle$. Let $\mathbf{z} = (z_1, z_2, z_3)$,

so $\langle \mathbf{u}+\mathbf{v}, \mathbf{z}\rangle =$

$\langle((u_1,u_2,u_3)+(v_1,v_2,v_3)),(z_1,z_2,z_3)\rangle = \langle((u_1+v_1)+(u_2+v_2)+(u_3+v_3)),(z_1,z_2,z_3)\rangle =$

$(u_1+v_1)z_1 + (u_3+v_3)z_3$ = $(u_1z_1 + u_3z_3) + (v_1z_1 + v_3z_3) = \langle u, z\rangle + \langle v, z\rangle$

$\neq u_1 + v_1 + z_1 + u_3 + v_3 + z_3$

$= (u_1, z_1 + u_3, z_3) + (v_1, v_3)$ — this one does hold.

$\neq \langle \mathbf{u}, \mathbf{z}\rangle + \langle \mathbf{v}, \mathbf{z}\rangle$ which demonstrates that axiom 2 does ~~not~~ hold for this inner product.

Axiom 3 states $\langle k\mathbf{u}, \mathbf{v}\rangle = k\langle \mathbf{u}, \mathbf{v}\rangle$. For the given inner product $k\langle \mathbf{u}, \mathbf{v}\rangle = ku_1 v_1 + ku_3 v_3$

? $\neq k(u_1, v_1 + u_3, v_3)$ — this one holds

$\neq k\langle \mathbf{u}, \mathbf{v}\rangle$, which demonstrates that axiom 3 does ~~not~~ hold for this inner product.

Axiom 4 states $\langle \mathbf{v}, \mathbf{v}\rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v}\rangle = 0$ if and only if $\mathbf{v} = 0$. For the given inner product $\langle \mathbf{v}, \mathbf{v}\rangle = v_1^2 + v_3^2$, because both terms are squared $\langle \mathbf{v}, \mathbf{v}\rangle \geq 0$ and $v_1^2 + v_3^2$ implies that both v_1 and v_3 must be zero, But v_2 does not have to equal 0 in order in order to make $\langle \mathbf{v}, \mathbf{v}\rangle = 0$ and therefore $\langle \mathbf{v}, \mathbf{v}\rangle \neq$ if and only if **v**=0 and thus this inner product is not an inner product on $\Re^2$. ?

Question 3:

If **u**= (1,4,7) and **v**= (0,1,1) and **u** and **v** are vectors in $\Re^3$, find the norm of **u** and the distance between vectors **u** and **v**.

Solution 3:

$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u}\rangle^{1/2} = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{1^2 + 4^2 + 7^2} = \sqrt{66}$

$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|\mathbf{w}\| = \langle \mathbf{w}, \mathbf{w}\rangle^{1/2} = \sqrt{1^2 + 3^2 + 6^2} = \sqrt{46}$

? what's the utility of introducing a new vector with no stated relationship to the old ones?

Question 4:
Find the cosine of the angle ☺ between the vectors u = (1,2,3,4) and v = (2,1,2,3)

Solution 4:

$\|u\| = \sqrt{1^2+2^2+3^2+4^2} = \sqrt{30}$ $\|v\| = \sqrt{2^2+1^2+2^2+3^2} = \sqrt{18}$

$\cos ☺ = \frac{\langle u,v\rangle}{\|u\|\|v\|} = \frac{22}{\sqrt{30}\sqrt{18}} = \frac{22}{6\sqrt{15}} = \frac{11}{3\sqrt{15}}$

why are you saying this?

Question 5:
For the set S = $\{v_1, v_2, v_3\}$ where S is an orthonormal basis for an inner product space V, and **v** and **u** are in V,

v_1= (1, 2, 1), v_2 = (3, -5, 2), v_3 = (-1, 1, 1) and u = (-1, 0, 2)

a) find the coordinate vector of u with respect to given basis. $\{v_1, v_3\}$?

b) Let S = $\{u_1, u_2\}$ where $u_1 = v_1$ and $u_2 = v_3$. -so S = Now consider the vector space $\Re^3$ with the Euclidean inner product. Apply the Gram-Schmidt process to transform basis vectors u_1 and u_2 in to an orthogonal basis $\{w_1, w_2\}$, then an orthonormal basis $\{q_1, q_2\}$.

basis for $\mathbb{R}^3$? Of course not.

this only holds for orthonormal basis

Solution 5: same as?

a) u = ⟨**u**, **v**$_1$⟩ v_1 + ⟨**u**, **v**$_2$⟩ v_2 + ⟨**u**, **v**$_3$⟩ v_3 therefore the coordinate vector $(u)_s$ = (⟨**u**, **v**$_1$⟩, ⟨**u**, **v**$_2$⟩, ⟨**u**, **v**$_3$⟩) for the given vectors ⟨**u**, **v**$_1$⟩ = -1 + 2 = 1
⟨**u**, **v**$_2$⟩ = -3 + 4= 1 and ⟨**u**, **v**$_3$⟩= 1+ 2 = 3 and $(u)_s$ = (1, 1, 3).

probably not!

b) $w_1 = u_1$ = (1, 2, 1)

$w_2 \propto u_2 - \text{Proj}_{span\{w_1\}} u_2 = u_2 - \frac{\langle u_2, w_1\rangle w_1}{\langle w_1, w_1\rangle}$

change to (-1,1,2)

= (-1, 1, 1) – (2/6)(1, 2, 1) ∝ (-1, 1, 1) -(2/6, 4/6, 2/6) = (-4/3, 1/3, 2/3) = w_2
Therefore the orthogonal basis = $\{(1,2,1), (-(4/3), (1/3), (2/3))\}$
And $q_1 = \frac{v_1}{\|v_1\|}$ = (1/ $\sqrt{6}$, 2/ $\sqrt{6}$, 1/ $\sqrt{6}$)

And $q_2 = \frac{v_2}{\|v_2\|} = (-(4/3)/\sqrt{(7/3)},\ (1/3)/\sqrt{(7/3)},\ (2/3)/\sqrt{(7/3)})$.

– simplify here, or, better, simplify earlier.

Question 6:

Find the Eigen values of A= $\begin{bmatrix} 2 & 2 & 1/4 & 27 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 5 & 3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution 6:

The characteristic polynomial of A is det(λ I-A)

$= \det \begin{bmatrix} \lambda-2 & -2 & -1/4 & -27 \\ 0 & \lambda-4 & 0 & -1 \\ 0 & 0 & \lambda-5 & -3/2 \\ 0 & 0 & 0 & \lambda-1 \end{bmatrix} = (\lambda-2)((\lambda-4)(\lambda-5)(\lambda-1)$ and the eigen values are the values of λ where the determinate is equal to zero;

so $\lambda = 2, 4, 5, 1$.

Question 7:

Show that if A and B are orthogonal then :

a) A^{-1} is orthogonal

b) AB is orthogonal

c) $\det(A) = \pm 1$

Solution 7:

a) ~~If~~ A is orthogonal $\Leftrightarrow A^T A = I \Leftrightarrow A^{-1}(A^T)^{-1} = I = A^{-1}(A^{-1})^T \Rightarrow$ that A^{-1} and $(A^{-1})^T$ are inverses $\Rightarrow (A^{-1})^T A^{-1} = I \Rightarrow A^{-1}$ is orthogonal.

b) ~~If~~ A is & B is orthogonal $\Leftrightarrow A^T A = I = B^T B$ so $(AB)^T(AB) = B^T A^T AB = B^T IB = B^T B = I \Rightarrow AB$ is orthogonal.

c) If $A^T A = I$, ~~and~~ then $1 = \det(I) = \det(A^T A) = \det(A^T)\det(A) = (\det(A))^2 \Leftrightarrow \det(A) = \pm 1$.

Question 8:

Find the orthogonal projection of $v = (2,1,3)$ onto the subspace w of $\mathbf{R}^3$ spanned by $u_1 = (1,1,0)$ and $u_2 = (1,2,1)$.

Solution 8:

The subspace of $\mathbf{R}^3$ spanned by u_1 and u_2 for the column space of matrix A is $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$. Using the least squares method $Ax = v$ is equal to $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$ and $A^T v = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$. The normal system $A^T Ax = A^T v$ is then $\begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$. v can then be found by augmenting $A^T A$ and $A^T v$ and performing Gaussian elimination.

$$\left[\begin{array}{cc|c} 2 & 3 & 3 \\ 3 & 6 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3/2 & 3/2 \\ 3 & 6 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3/2 & 3/2 \\ 0 & 3/2 & 5/2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3/2 & 3/2 \\ 0 & 1 & 5/3 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 5/3 \end{array}\right] \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 5/3 \end{bmatrix}$$ or $proj_w v = (-1, 5/3)$.

not correct if $\underline{x} = \begin{bmatrix} -1 \\ 5/3 \end{bmatrix}$ is the least squares solution. Rather $proj_{\underline{w}} \underline{v} = A\underline{x}$.

Question 9:

Let M_{22} have "the inner product." Find the cosine of the angle between A and B.

sounds like a dressee. Of course you want to indicate which particular inner product. What is it?

a) $A = \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$

b) $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 4 \\ -2 & -2 \end{bmatrix}$

Solution 9:

$$\cos\theta = \frac{\langle A, B\rangle}{\|A\|\|B\|}$$

a) $\cos\theta = \dfrac{-2+4+12}{\sqrt{25+1+4+16}\sqrt{4+4+9}} = \dfrac{14}{\sqrt{46}\sqrt{17}} = \dfrac{14}{\sqrt{782}}$.

b) $\cos\theta = \dfrac{5-4-6-2}{\sqrt{(25+1+9+1)}\sqrt{(1+16+4+4)}} = \dfrac{-7}{\sqrt{(36)}\sqrt{(25)}} = \dfrac{-7}{(6)(5)} = \dfrac{-7}{30}$.

Question 10:

Find the coordinate matrix for p relative to $S = \{p_1, p_2, p_3\}$.

a) $p = 4 + 5x + x^2$, $p_1 = 1 + 3x$, $p_2 = x - x^2$, $p_3 = x^2$

b) $p = x - x^2$, $p_1 = 1 + x$, $p_2 = 1 + x^2$, $p_3 = x + x^2$

fairly good explanation on indication why coefficients go into columns, but you can do better.

Solution10:

The polynomials can be written with their coefficients as $[p_1 \mid p_2 \mid p_3 \mid p]$, whose r.r.e.f yields the coordinate matrix for p relative to S.

a) $[p_1 \mid p_2 \mid p_3 \mid p] \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 \\ 3 & 1 & 0 & 5 \\ 0 & -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -6 \end{bmatrix} \Rightarrow$

$(p)_s = (4, -7, -6)$, $[p_s] = \begin{bmatrix} 4 \\ -7 \\ -6 \end{bmatrix}$

b) $[p_1 \mid p_2 \mid p_3 \mid p] \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow$

$(p)_s = (1, -1, 0)$, $[p_s] = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

1.)(a) Prove that if u and v are orthogonal vectors in an inner product space, then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Solution:

u and v are orthogonal vectors $\Rightarrow \langle u,v\rangle = 0$

$$\|u+v\|^2 = \langle u+v, u+v\rangle$$

$$= \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle, \quad \text{and } 2\langle u,v\rangle = 0$$

$$\|u+v\|^2 = \langle u,u\rangle + \langle v,v\rangle = \|u\|^2 + \|v\|^2$$

– irrelevant

(b) Also show that if u and v are not orthogonal vectors, then

$$\|u+v\| \le \|u\| + \|v\|$$

Solution:

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle$$

$$\|u+v\|^2 \le \langle u,u\rangle + |2\langle u,v\rangle| + \langle v,v\rangle = \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$\|u+v\|^2 \le (\|u\| + \|v\|)^2 \Leftrightarrow$$

$$\left(\|u+v\|^2\right)^{\frac{1}{2}} \le \left((\|u\| + \|v\|)^2\right)^{\frac{1}{2}}$$

$$\|u+v\| \le \|u\| + \|v\|$$

(c) Verify the theorem in part (a) with the following orthogonal vectors in P_2 with the inner product $\langle P_1, P_2\rangle = a_0b_0 + a_1b_2 + a_2b_2$ where $P_1 = a_0 + a_1x + a_2x^2$ and $P_2 = b_0 + b_1x + b_2x^2$

$P_1 = -1 + 3x + 2x^2$ and $P_2 = 4 + 2x - x^2$

Solution:

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

$$\|(4-1, 2+3, 2-1)\|^2 = \left((-1)^2 + 3^2 + 2^2\right) + \left(4^2 + 2^2 + (-1)^2\right)$$

$$\|(3,5,1)\|^2 = 1 + 9 + 4 + 16 + 4 + 1$$

$$\left(3^2 + 5^2 + 1^2\right) = 35$$

9+25+1=35

– irrelevant

(d) Verify the theorem in part (b) with the following non-orthogonal vectors in R^3 with the Euclidean inner product

$u = (-3,1,0)$ and $v = (2,-1,3)$

Solution: $\|(-3+2, 1+-1, 0+3)\| \le \sqrt{9+1} + \sqrt{4+1+9}$

$$\sqrt{1+9} \le \sqrt{10} + \sqrt{14}$$

2.) Let P_2 have the inner product, $\langle p,q\rangle = \int_0^1 p(x)q(x)dx$,

Find the angle between the following functions in P_2, $p(x)=2x$ and $q(x)=2$.

Solution:

$$\cos\theta = \frac{\langle u,v\rangle}{\|u\|\|v\|} \qquad \langle p,q\rangle = \int_0^1 4x\,dx = 2x^2\Big]_0^1 = 2$$

$$\|p\| = \langle p,p\rangle^{\frac{1}{2}} = \left(\int_0^1 4x^2dx\right)^{\frac{1}{2}} = \left(\frac{4}{3}x^3\Big]_0^1\right)^{\frac{1}{2}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

$$\|q\| = \langle q,q\rangle^{\frac{1}{2}} = \left(\int_0^1 4dx\right)^{\frac{1}{2}} = \left(4x\Big]_0^{2}\right)^{\frac{1}{2}} = \sqrt{4} = 2$$

$$\frac{\langle u,v\rangle}{\|u\|\|v\|} = \frac{2}{2\frac{2}{\sqrt{3}}} = \frac{\sqrt{3}}{2} \qquad \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} = 30^\circ$$

3.) Find an orthogonal matrix P that diagonalizes $A=\begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}$

Solution:

$$0 = \det(\lambda I - A) = \det\begin{bmatrix}\lambda-1 & -3\\ -3 & \lambda-1\end{bmatrix} = (\lambda-1)(\lambda-1)-9 = (\lambda^2-2\lambda+1)-9 = \lambda^2-2\lambda-8$$

$$=(\lambda-4)(\lambda+2) \Rightarrow \lambda=4, \lambda=-2$$

$$\begin{bmatrix}\lambda-1 & -3\\ -3 & \lambda-1\end{bmatrix}\begin{bmatrix}x_1\\ x_x\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \quad \text{when } \lambda=4, \begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$3x_1 - 3x_2 = 0, \text{ and } -3x_1 + 3x_2 = 0$$

$$x_1 = x_2 \Rightarrow \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}x_2\\ x_2\end{bmatrix} = x_2\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\text{When } \lambda=-2, \begin{bmatrix}-3 & -3\\ -3 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$-3x_1 - 3x_2 = 0, \text{ and } -3x_1 - 3x_2 = 0$$

$$x_1 = -x_2 \Rightarrow \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}-x_2\\ x_2\end{bmatrix} = x_2\begin{bmatrix}-1\\ 1\end{bmatrix}$$

$$u_1 = \begin{bmatrix}1\\ 1\end{bmatrix}, \text{ and } u_2 = \begin{bmatrix}-1\\ 1\end{bmatrix} \Rightarrow v_1 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}, \text{ and } v_2 = \begin{bmatrix}\frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \Rightarrow P = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

4.) Do vectors $\mathbf{v}_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $\mathbf{v}_2 = (0,1,0)$, $\mathbf{v}_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$ form an orthonormal basis and if so express the vector $\mathbf{u} = (3,-5,7)$ as a linear combination of the three.

for what?

Solution:

First we must verify that this is indeed an ortho~~normal~~ gonal basis.

$$\langle v_1, v_2 \rangle = \frac{1}{\sqrt{2}} \cdot 0 + 0 \cdot 1 + \frac{1}{\sqrt{2}} \cdot 0 = 0$$

$$\langle v_1, v_3 \rangle = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 0 \cdot 0 + \frac{1}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}} = 0$$

$$\langle v_2, v_3 \rangle = 0 \cdot \frac{1}{\sqrt{2}} + 1 \cdot 0 + 0 \cdot -\frac{1}{\sqrt{2}} = 0$$

— now show the "normalization"

Since all inner products ~~spaces~~ are equal to 0 we know that this is an orthonormal basis. — not yet

Now we have to discover the inner products ~~spaces~~ of u ~~to~~ with each v so that we get the coefficients for the linear combination.

$$\langle u, v_1 \rangle = 3 \cdot \frac{1}{\sqrt{2}} + (-5) \cdot 0 + 7 \cdot \frac{1}{\sqrt{2}} = \frac{10}{\sqrt{2}}$$

$$\langle u, v_2 \rangle = 3 \cdot 0 + (-5) \cdot 1 + 7 \cdot 0 = -5$$

$$\langle u, v_3 \rangle = 3 \cdot \frac{1}{\sqrt{2}} + (-5) \cdot 0 + 7 \cdot -\frac{1}{\sqrt{2}} = -\frac{4}{\sqrt{2}}$$

So the linear combination is $\frac{10}{\sqrt{2}} \cdot v_1 - 5v_2 - \frac{4}{\sqrt{2}} \cdot v_3$.

refer to a general/big result

5.) Find the least squares solution of the linear system Ax = b given by:

$$\begin{aligned} 3x_1 - 2x_2 + 4x_3 &= -1 \\ x_1 + 2x_2 - 6x_3 &= 8 \\ 2x_1 + 3x_2 + 4x_3 &= 4 \end{aligned}$$

Solution:

Here

$$A = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 2 & -6 \\ 2 & 3 & 4 \end{bmatrix} \text{ and } b = \begin{bmatrix} -1 \\ 8 \\ 4 \end{bmatrix}$$

Since A has linearly independent columns we know that in advance there is a unique least squares solution. So now we get

and since A is square, this is the "real" solution, i.e., $A\underline{x} = b$ is consistent and one does not need (no thinking) least squares.

$$A^T A = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 2 & -6 \\ 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 2 \\ -2 & 2 & 3 \\ 4 & -6 & 4 \end{bmatrix} = \begin{bmatrix} 29 & -25 & 16 \\ -25 & 41 & -16 \\ 16 & -16 & 29 \end{bmatrix}$$

$$A^T \boldsymbol{b} = \begin{bmatrix} 3 & 1 & 2 \\ -2 & 2 & 3 \\ 4 & -6 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 30 \\ -36 \end{bmatrix}$$

So the normal system $A^TA\boldsymbol{x}=A^T\boldsymbol{b}$ is

$$\begin{bmatrix} 29 & -25 & 16 \\ -25 & 41 & -16 \\ 16 & -16 & 29 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 30 \\ -36 \end{bmatrix}$$

By solving this system we get the least squares solution:

$$x_1 = \frac{35415}{11236} \qquad x_2 = \frac{21343}{11236} \qquad x_3 = \frac{-5428}{2809}$$

are you serious?

6.)

Consider the Bases B = {$\mathbf{u}_1$,$\mathbf{u}_2$} and B' = {$\mathbf{u}_1'$,$\mathbf{u}_2'$} where

$\mathbf{u}_1 = (3,1)$; $\mathbf{u}_2=(1,1)$; $\mathbf{u}_1'=(8,2)$; $\mathbf{u}_2'=(1,7)$;

(a) Find the transition matrix from B' to B

(b) Find $[\mathbf{v}]_B$ if $[\mathbf{v}]_{B'} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

Solution:

(a) $\mathbf{u}_1' = (3)\mathbf{u}_1 + (-1)\mathbf{u}_2$
$\mathbf{u}_2' = (-3)\mathbf{u}_1 + (10)\mathbf{u}_2$

– by inspection? Do something generally possible, e.g. my (transparent) approach.

$[\mathbf{u}_1']_B = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ $\quad [\mathbf{u}_2']_B = \begin{bmatrix} -3 \\ 10 \end{bmatrix}$

$$P = \begin{bmatrix} 3 & -3 \\ -1 & 10 \end{bmatrix}$$

(b) $[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$

$$[\mathbf{v}]_B = \begin{bmatrix} 3 & -3 \\ -1 & 10 \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} -24 \\ 62 \end{bmatrix}$$

7.) By definition we know:
If: $A^{-1} = A^T$
then A is said to be an orthogonal matrix.

Is the following statement always true or sometimes false?

If A is an orthogonal nxn matrix. Then for some constant k, kA is also orthogonal.

(handwritten: "scalar / namely ±1." above "constant"; "*" after the statement)

Solution:

Sometimes false

False case:

If:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and } k=5$$

$$kA = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/5 \end{bmatrix} \quad A^T = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Then:

$(kA)^{-1} \neq (kA)^T$

And therefore kA is not orthogonal

True case:

If:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and } k=1$$

$$kA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Always true. The "Always" does not refer to different choices of k in the way that you worded this. — but don't choose this — choose k = ±1.

* This becomes what you evidently wanted to say if you write "Then for all choices of a ~~constant~~ scalar k, kA is ...". But then the answer is "sometimes false", not "false", even though it is true for k = ±1. Think about it.

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^{T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then:

$(kA)^{-1} = (kA)^{T}$

And therefore kA is orthogonal

8.) Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors in R^3 with the inner product defined by $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 4u_2v_2 + 5u_3v_3$. Find the distance between the points $\mathbf{u} = (7,11,20)$ and $\mathbf{v} = (5,5,15)$.

Solution:

The distance between two points is defined by $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$. Since $\|\mathbf{u}\| = \langle u, u \rangle^{\frac{1}{2}}$, we can express the distance between the two points as follows:

$$d((7,11,20),(5,5,15)) = \|(7,11,20) - (5,7,15)\| = \langle (2,4,5),(2,4,5) \rangle^{\frac{1}{2}} = (2 \cdot 2 \cdot 2 + 4 \cdot 4 \cdot 4 + 5 \cdot 5 \cdot 5)^{\frac{1}{2}} = \sqrt{197}$$

9.) What conditions must x and y satisfy for the matrix

$$\begin{bmatrix} x & x-4 & y+5 & x^2 \\ 0 & x-y & y^2 & \sqrt{x} \\ 0 & 0 & x+y & y \\ 0 & 0 & 0 & y \end{bmatrix}$$

to be orthogonal?

Solution:

The matrix will be orthogonal if it's determinant is 1 or -1. Since the matrix is triangular, it's determinant can be computed by multiplied the entries in the main diagonal.

$x(x-y)(x+y)y = \pm 1 \Rightarrow xy(x^2 - y^2) = \pm 1 \Rightarrow x^3y - y^3x = \pm 1$

not so! e.g. $A = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$ is not orthogonal.

10.) Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3\}$ where

$\mathbf{u}_1 = (1,1,1), \mathbf{u}_2 = (4,5,4), \mathbf{u}_3 = (6,14,18); \mathbf{u}'_1 = (1,4,9), \mathbf{u}'_2 = (2,6,14), \mathbf{u}'_3 = (3,9,13)$

Find the transition matrix from B' to B.

Solution:

The transition matrix P from B' to B can be expressed as follows:

$P = [[\mathbf{u}'_1]_B \mid [\mathbf{u}'_2]_B \mid [\mathbf{u}'_3]_B]$

In other words, we must form a matrix whose columns consist of the coordinate vectors from B' relative to the basis B. We can do this by row reducing one big augmented matrix as follows: *

$$\left[\begin{array}{ccc|c|c|c} 1 & 4 & 6 & 1 & 2 & 3 \\ 1 & 5 & 14 & 4 & 6 & 9 \\ 1 & 4 & 18 & 9 & 14 & 13 \end{array}\right] \sim \left[\begin{array}{ccc|c|c|c} 1 & 4 & 6 & 1 & 2 & 3 \\ 0 & 1 & 8 & 3 & 4 & 6 \\ 0 & 0 & 1 & 4 & 6 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 93 & 142 & 109 \\ 0 & 1 & 0 & -29 & -44 & -34 \\ 0 & 0 & 1 & 4 & 6 & 5 \end{array}\right]$$

The transition matrix P from B' to B is now the right half of the above row reduced augmented matrix.

$$P = \begin{bmatrix} 93 & 142 & 109 \\ -29 & -44 & -34 \\ 4 & 6 & 5 \end{bmatrix}$$

* A lot more obvious that this holds in my development.

1. Use the four axioms which define an inner product space to determine whether the following is or is not an inner product space:

$$\mathbf{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \mathbf{v} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, \in M_{22}$$

$\langle \mathbf{u}, \mathbf{v} \rangle = ae - bf + cg - dh$, with usual addition and multipication

Solution

Axiom 1: $\langle \mathbf{u}, \mathbf{v} \rangle = ae - bf + cg - dh = ea - fb + gc - hd = \langle \mathbf{v}, \mathbf{u} \rangle$, so the axiom holds.

Axiom 2: Let $\mathbf{z} = \begin{bmatrix} l & m \\ n & p \end{bmatrix}$. Then

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{z} \rangle = \left\langle \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}, \begin{bmatrix} l & m \\ n & p \end{bmatrix} \right\rangle$$

$$\begin{aligned} &= al + el - bm - fm + cn + gn - dp - hp \\ &= (al - bm + cn - dp) + (el - fm + gn - hp), \\ &= \langle \mathbf{u}, \mathbf{z} \rangle + \langle \mathbf{v}, \mathbf{z} \rangle \end{aligned}$$

so the axiom holds.

Axiom 3: $\langle k\mathbf{u}, \mathbf{v} \rangle$

$$\begin{aligned} &= \left\langle \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}, \begin{bmatrix} l & m \\ n & p \end{bmatrix} \right\rangle \\ &= kae - kbf + kcg - kdh \\ &= k(ae - bf + cg - dh) \quad , \\ &= k\langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

so the axiom holds.

Axiom 4: $\langle \mathbf{v}, \mathbf{v} \rangle = e^2 - f^2 + g^2 - h^2$, which is smaller than zero whenever $f^2 + h^2 > e^2 + g^2$. Also, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ whenever $f^2 + h^2 = e^2 + g^2$, e.g. $e = 2, f = 3, g = -3, h = -2$. Since axiom four fails to hold, this is not an inner product space.

2. (a) For the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 5u_2v_2$, are the vectors $\mathbf{u} = (5, -1)$ and $\mathbf{v} = (2, 4)$ orthogonal?

   (b) What is the relationship between the space spanned by $\mathbf{u}$, and the space spanned by $\mathbf{v}$?

   (c) For the same inner product, what is the angle between $\mathbf{u}' = (4, 3)$ and $\mathbf{v}' = (1, 2)$?

Solution

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = 2 * 5 * 2 + 5 * (-1) * 4 = 20 - 20 = 0$, so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal with respect to this ~~eigenvector~~ inner product.

(b) The spaces spanned by $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to each other, i.e. Span$\{\mathbf{u}\}$ =(Span$\{\mathbf{v}\})^\perp$, Span$\{\mathbf{v}\}$ =(Span$\{\mathbf{u}\})^\perp$.

(c)

$$\begin{aligned}\cos\theta &= \frac{\langle \mathbf{u}'.\mathbf{v}'\rangle}{\|\mathbf{u}\|*\|\mathbf{v}\|}\\ &= \frac{2*2*1+5*2*2}{\sqrt{2*2^2+5*5^2}\sqrt{2*1^2+5*2^2}}\\ &= \frac{4+20}{\sqrt{8+20}\sqrt{2+20}}\\ &= \frac{24}{\sqrt{616}}\\ \theta &= \cos^{-1}\left(\frac{24}{\sqrt{616}}\right)\end{aligned}$$

3. If $\mathbf{v}_1 = (0,3,0), \mathbf{v}_2 = (-3,0,6), \mathbf{v}_3 = (6,0,3)$ is an orthogonal bass for R^3 with the Euclidean inner product,
   (a) Find $\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3$ such that they form an orthonormal basis S for R^3.
   (b) Express $\mathbf{u} = (3,0,1)$ as a linear combination of $\mathbf{v}'_1, \mathbf{v}'_2$, and $\mathbf{v}'_3$.
   (c) Find the coordinate vector of $\mathbf{u}$ with respect to S, i.e. $(\mathbf{u})_S$.

Solution

(a) Normalizing, we have

$$\begin{aligned}\mathbf{v}'_1 &= \tfrac{1}{\sqrt{3^2}}(0,3,0) = \tfrac{1}{3}(0,3,0) = (0,1,0)\\ \mathbf{v}'_2 &= \tfrac{1}{\sqrt{45}}(-3,0,6) = \tfrac{1}{3\sqrt{5}}(-3,0,6) = (-\tfrac{1}{\sqrt{5}},0,\tfrac{2}{\sqrt{5}})\\ \mathbf{v}'_3 &= \tfrac{1}{\sqrt{45}}(6,0,3) = \tfrac{1}{3\sqrt{5}}(6,0,3) = (\tfrac{2}{\sqrt{5}},0,\tfrac{1}{\sqrt{5}}).\end{aligned}$$

-of course - re-scale first

(b) As S is orthonormal,

$$\begin{aligned}(3,0,1) &= \langle \mathbf{u},\mathbf{v}'_1\rangle\mathbf{v}'_1 + \langle \mathbf{u},\mathbf{v}'_2\rangle\mathbf{v}'_2 + \langle \mathbf{u},\mathbf{v}'_3\rangle\mathbf{v}'_3\\ &= 0\mathbf{v}'_1 - \tfrac{1}{\sqrt{5}}\mathbf{v}'_2 + \tfrac{7}{\sqrt{5}}\mathbf{v}'_3\end{aligned}$$

computations?

(c) By the definition of coordinate vector, and (b) above, $(\mathbf{u})_S = (0, -\frac{1}{\sqrt{5}}, \frac{7}{\sqrt{5}})$.

4. You have collected data of force and acceleration of an object with unknown mass m. Here is the data collect:

$$\begin{aligned}F_1 &= 12.0N & a_1 &= 4.1ms^{-2}\\ F_2 &= 18.1N & a_2 &= 6.2ms^{-2}\\ F_3 &= 9.0N & a_3 &= 3.2ms^{-2}\\ F_4 &= 5.9N & a_3 &= 1.8ms^{-2}\end{aligned}$$

- Cool problem, although I could do w/o the decimals (i.e. you could on a test)

Use the data and Newton's Second Law $F = ma$ to find the best approximation for m to three decimal places.
Hint: m is scalar, but you can consider it as as a vector $\mathbf{m} = \mathbf{m}_{1\times1}$.

Solution

We have the system of equations

$$\begin{bmatrix}4.1\\6.2\\3.2\\1.8\end{bmatrix}\mathbf{m} = \begin{bmatrix}12.0\\18.1\\9.0\\5.9\end{bmatrix}.$$

We can obtain the least squares solution from the normal equation $A^T A\mathbf{m} = A^T\mathbf{b}$. In this case, we have

$$A = \begin{bmatrix} 4.1 \\ 6.2 \\ 3.2 \\ 1.8 \end{bmatrix}, A^T = \left[4.16.23.21.8\right], \mathbf{b} = \begin{bmatrix} 12.0 \\ 18.1 \\ 9.0 \\ 5.9 \end{bmatrix}.$$

Then

$$A^T A = \begin{bmatrix} 4.1 \\ 6.2 \\ 3.2 \\ 1.8 \end{bmatrix} \left[4.16.23.21.8\right] = \left[68.73\right],$$

and

$$A^T\mathbf{b} = \left[4.16.23.21.8\right] \begin{bmatrix} 12.0 \\ 18.1 \\ 9.0 \\ 5.9 \end{bmatrix} = \left[200.84\right].$$

Thus $\left[68.73\right]\mathbf{m} = \left[200.84\right] \Rightarrow \mathbf{m} = \left[2.922\right] \Rightarrow m = 2.922kg.$

5. Consider the bases $Q = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of R^3.

   (a) What would you multiply the column vector $\mathbf{u}_{\{e_1,e_2,e_3\}}$ by to find $\mathbf{u}_{\{S\}}$?

   (b) What is the transition matrix P_{SQ} to change $\mathbf{u}_{\{S\}}$ to $\mathbf{u}_{\{Q\}}$?

Solution

(a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{u}_{\{e_1,e_2,e_3\}} = \left[\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3\ \right] \mathbf{u}_{\{S\}}$$

$$\mathbf{u}_{\{S\}} = \left[\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3\ \right]^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{u}_{\{e_1,e_2,e_3\}} = \left[\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3\ \right]^{-1} \mathbf{u}_{\{e_1,e_2,e_3\}}$$

- cool. (and transparent).

So multiply $\mathbf{u}_{\{e_1,e_2,e_3\}}$ by $\left[\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3\ \right]^{-1}$.

(b)

$$\left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ \right] \mathbf{u}_{\{Q\}} = \left[\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3\ \right] \mathbf{u}_{\{S\}}$$

$$\mathbf{u}_{\{Q\}} = \left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ \right]^{-1} \left[\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3\ \right] \mathbf{u}_{\{S\}}$$

So $P_{SQ} = \left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ \right]^{-1} \left[\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3\ \right]$. - ditto

6. (a) What are the coordinates of $A = 1 + x + x^2$ with respect to the basis $S = \{1, x, x^2\}$?

   (b) What are the coordinates of A with respect to the basis $\{1 + x, x + x^2, 1 + x^2\}$?

Solution

(a) By inspection, $A_{\{1,x,x^2\}} = (1, 1, 1)$.

(b) $(1+x)\alpha + (x+x^2)\beta + (1+x^2)\gamma = 1 + x + x^2$? say it.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right].$$

Thus $A_{\{1+x,x+x^2,x^2+1\}} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Great problem,

7. All 2-by-2 orthogonal matrices are of one the two forms

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \text{ or } B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

(a) Of which form are matrices of rotation in R^2?

(b) The product of two orthogonal matrices is also an orthogonal matrix. Of which form is the product if you multiply...

i. two matrices of the first form?

ii. one of the first form by one of the second (i.e., AB)?

iii. one of the second by one of the first (i.e., BA)?

iv. two of the second form?

(c) Do orthogonal matrices always commute?

Solution

(a) Matrices of rotation in R^2 are of form A.

(b) i. As $\det(A) = 1$ and $\det(B) = -1$, $\det(A_1A_2) = \det(A_1)\det(A_2) = 1 * 1 = 1$, so the product is of form A.

ii. $\det(A_1B_1) = \det(A_1)\det(B_1) = 1 * (-1) = -1$, so the product is of form B.

iii. $\det(B_1A_1) = \det(B_1)\det(A_1) = (-1) * 1 = -1$, so the product is again of form B.

iv. $\det(B_1B_2) = \det(B_1)\det(B_2) = (-1) * (-1) = 1$, so the product is of form A.

(c) No—for example, although standard matrices of both rotations and reflections are orthogonal, rotation by $\pi/6$ radians followed by reflection over the x-axis is not equivalent to reflection over the x-axis followed by rotation by $\pi/6$ radians: the first ordering moves $(1,0)$ to $(\sqrt{2}/2, -1/2)$, while the second moves it to $(\sqrt{2}/2, 1/2)$. Thus, the operations—and their associated matrices do not generally commute.

8. (a) Find the eigenvalues of $A = \begin{bmatrix} 0 & -2 & 0 \\ -1 & -1 & 0 \\ -3 & -1 & -2 \end{bmatrix}$.

(b) How many eigenspaces are there? Find bases for the eigenspaces.

Solution

(a) The eigenvalues can be found from the characteristic equation $\det(\lambda I - A) = 0$

$$\Rightarrow \det \begin{bmatrix} \lambda & 2 & 0 \\ 1 & \lambda + 10 & 0 \\ 3 & 1 & \lambda - 2 \end{bmatrix} = 0$$

By cofactor expansion on the third column,

$$\begin{aligned} 0 &= (\lambda - 2)\left(\lambda(\lambda + 1) - 1 * 2\right) \\ &= (\lambda - 2)(\lambda^2 + \lambda - 2) \\ &= (\lambda - 2)(\lambda + 2)(\lambda - 1) \end{aligned}$$

$\Rightarrow \lambda = 2, -2, 1.$

(b) As there are three eignvalues, there are three eigenspaces. To find them, we find the nullspaces of the matrix $\lambda I - A$ for each λ.

i. $\lambda = 2$

$$\begin{bmatrix} 2 & 2 & 0 \\ 1 & 3 & 0 \\ 3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ is a basis.}$$

ii. $\lambda = -2$

$$\begin{bmatrix} -2 & 2 & 0 \\ 1 & -1 & 0 \\ 3 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ is a basis.}$$

iii. $\lambda = 1$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \text{ is a basis.}$$

9. Mind a matrix P that diagonalizes $A = \begin{bmatrix} 1 & 8 & -18 \\ 0 & 3 & -15 \\ 0 & 0 & -2 \end{bmatrix}$.

Solution

(a) Solve the characteristic equation, $\det(\lambda I - A) = 0$

$$\det \begin{bmatrix} \lambda - 1 & -8 & 18 \\ 0 & \lambda - 3 & 15 \\ 0 & 0 & \lambda + 2 \end{bmatrix} = 0 \Rightarrow (\lambda - 1)(\lambda - 3)(\lambda - 2) = 0$$

So $\lambda = 1, 3, -2.$

(b) Find bases for eigenspaces of A

$$\lambda = 1 \leadsto \text{Null}\begin{bmatrix} 0 & -8 & 18 \\ 0 & -2 & 15 \\ 0 & 0 & 1 \end{bmatrix} = \text{Null}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\lambda = 3 \leadsto \text{Null}\begin{bmatrix} 2 & -8 & 18 \\ 0 & 0 & 15 \\ 0 & 0 & 5 \end{bmatrix} = \text{Null}\begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \end{bmatrix} = \text{Span}\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}.$$

$$\lambda = -2 \leadsto \text{Null}\begin{bmatrix} -3 & -8 & 18 \\ 0 & -5 & 15 \\ 0 & 0 & 0 \end{bmatrix} = \text{Null}\begin{bmatrix} 1 & 8/3 & -6 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \text{Null}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span}\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}.$$

$$\Rightarrow P = \begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

– Good problem – if you know the theorems, you don't never hard, but if you don't know them, you do.

10. Determine whether the following matrices are diagonalizable:

(a)

$$\begin{bmatrix} -2 & 5 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ -1 & 4 & 2 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Solution

(a) Yes, as it is a triangular matrix with distinct entries on the main diagonal.

(b) Yes, as it is symmetrical

(c) Yes, as it is symmetrical

18

Question #1

Find the eigenvalues of

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ k^3 & -3k^2 & 3k\end{bmatrix}$$

Solution #1

15 points

The eigenvalues can be found by calculating Det(λI-A), which in this case is

$$\lambda I-A=\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}-\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ k^3 & -3k^2 & 3k\end{bmatrix}=\begin{bmatrix}\lambda & -1 & 0\\ 0 & \lambda & -1\\ -k^3 & 3k^2 & \lambda-3k\end{bmatrix}$$

$$Det(\tilde{A})=\begin{vmatrix}\lambda & -1 & 0\\ 0 & \lambda & -1\\ -k^3 & 3k^2 & \lambda-3k\end{vmatrix}$$

Det($\tilde{A}$) can be calculated by a number of methods, such as altering it to upper triangular form and multiplying along the diagonal, or perhaps the most simple method in this case is expand the first two columns out and multiply across the threee diagonals. Either method will ultimately yield the equation:

$$Det(\tilde{A})=\lambda^3-3\lambda^2k+3\lambda k^2-k^3$$

Which can be reduced to

$$\lambda^3-3\lambda^2k+3\lambda k^2-k^3=\lambda^3-2\lambda^2k+\lambda k^2-\lambda 2+2\lambda k^2-k^3=(\lambda-k)(\lambda^2-2\lambda k+k^2)$$
$$=(\lambda-k)(\lambda-k)(\lambda-k)=(\lambda-k)^3$$

Therefore, the eigenvalues for this ~~system of equations will equal zero when~~ matrix are all $\lambda=k$.

(there was no system of eq's, and such do not have eigenvalues)

Question #2

Diagonalize the matrix A

$$A=\begin{bmatrix}-1 & 7 & -1\\ 0 & 1 & 0\\ 0 & 15 & -2\end{bmatrix}$$

Solution #2

15 points

$D = P^{-1}AP$, where P is a matrix composed of the bases for the eigenvectors.
The first step is to find the bases for the eigenvectors of this matrix.

$$Det(\lambda I - A) = \begin{bmatrix} \lambda+1 & -7 & 1 \\ 0 & \lambda-1 & 0 \\ 0 & -15 & \lambda+2 \end{bmatrix} = (\lambda+1)(\lambda-1)(\lambda+2)$$

So the eigenvalues for this matrix are 1, -1, and -2. Returning and substiting these values in turn yields the following bases

$$\lambda = -1 \Rightarrow \begin{bmatrix} 0 & -7 & 1 \\ 0 & -2 & 0 \\ 0 & -15 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t * \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 1 \Rightarrow \begin{bmatrix} 2 & -7 & 1 \\ 0 & 0 & 0 \\ 0 & -15 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/5 \\ 0 & 1 & -1/5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} t/5 \\ t/5 \\ t \end{bmatrix} = t * \begin{bmatrix} 1/5 \\ 1/5 \\ 1 \end{bmatrix}$$ – change to $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

$$\lambda = -2 \Rightarrow \begin{bmatrix} -1 & -7 & 1 \\ 0 & -3 & 0 \\ 0 & -15 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t * \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Matrix P is composed of these values.

$$P = \begin{bmatrix} 1 & 1/5 & 1 \\ 0 & 1/5 & 0 \\ 0 & 1 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 5 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$

D can now be calculated

$$D = P^{-1}AP = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 5 & 0 \\ 0 & -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1/5 & 1 \\ 0 & 1/5 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The final answer is $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Question #3

3. Using matrix A, and the answer to the previous problem, calculate A^{11}

Solution #3

15 points

Remember that $A^k = PD^kP^{-1}$, $A^{11} = PD^{11}P^{-1}$

$$A^{11} = \begin{bmatrix} 0 & 1/5 & 1 \\ 0 & 1/5 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}^{11} \cdot \begin{bmatrix} 1 & 4 & -1 \\ 0 & 5 & 0 \\ 0 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/5 & 1 \\ 0 & 1/5 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1^{11} & 0 & 0 \\ 0 & 1^{11} & 0 \\ 0 & 0 & -2^{11} \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & -1 \\ 0 & 5 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/5 & 1 \\ 0 & 1/5 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2048 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & -1 \\ 0 & 5 & 0 \\ 0 & -5 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 10237 & -2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{bmatrix}$$

Therefore, $A^{11} = \begin{bmatrix} -1 & 10237 & -2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{bmatrix}$

think about how to say this accurately.

Question #4

"Minimize the following linear system" **Ax**=**b**,

$$x - y = -1$$

$$3x + 2 = 2$$

$$-2 + 4 = 0$$

, and find the orthogonal projection of **b** on the column space of A.

well-stated

Solution #4

15 points

The normal system ($A^TAx = A^Tb$) will equal:

$$A^TA = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} x = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

And solving this system gives the least squares solution of.

$$\begin{bmatrix} 14 & -3 & 5 \\ -3 & 21 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3/14 & 5/14 \\ 0 & 1 & 85/285 \end{bmatrix}$$ so the minimized solutions are x = 56/133 and y = 17/57

$$\begin{bmatrix} 1 & 0 & 56/133 \\ 0 & 1 & 17/57 \end{bmatrix}$$

so the projection onto **b** on the column space of A is

$$\begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 56/133 \\ 17/57 \end{bmatrix}=\begin{bmatrix} -74/399 \\ 373/399 \\ 386/399 \end{bmatrix}$$

— Think about how to engineer this to give integer solutions – its possible.

Question #5

Find the transition matrix from B to B′.

$\mathbf{u}_1$=(1, 2, 1) $\mathbf{u}_2$=(2, 1, 1) $\mathbf{u}_3$=(1, 2, 2) $\mathbf{u'}_1$=(2, 1, 1) $\mathbf{u'}_2$=(1, 1, 2) $\mathbf{u'}_3$=(1, 2, 1)

what is B and what is B′?

Solution #5

15 points

$\mathbf{u'}_1$=a $\mathbf{u}_1$ + b $\mathbf{u}_2$ + c $\mathbf{u}_3$

$\mathbf{u'}_1$=d $\mathbf{u}_1$ + e $\mathbf{u}_2$ + f $\mathbf{u}_3$

$\mathbf{u'}_1$=g $\mathbf{u}_1$ + h $\mathbf{u}_2$ + i $\mathbf{u}_3$

the transitional matrix is represented by a-i corresponding to the system above. We can solve for this matrix as follows:

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & -3 & 0 & -3 & -1 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \end{bmatrix}$$ so the transition matrix from B-B′ = $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 1/3 & 0 \\ 0 & 4/3 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 4/3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 4/3 & 0 \end{bmatrix}$$

I think this is the transition matrix from B′ to B, but not obvious since you don't have a demonstration of what you mean by this phrase.

Question #6

Find the transition matrix from B' to B.

$\mathbf{u}_1=(1, 2, 1)$ $\mathbf{u}_2=(2, 1, 1)$ $\mathbf{u}_3=(1, 2, 2)$ $\mathbf{u}'_1=(2, 1, 1)$ $\mathbf{u}'_2=(1, 1, 2)$ $\mathbf{u}'_3=(1, 2, 1)$

Solution #6

15 points

$\mathbf{u}_1=a\,\mathbf{u}'_1 + b\,\mathbf{u}'_2 + c\,\mathbf{u}'_3$

$\mathbf{u}_1=d\,\mathbf{u}'_1 + e\,\mathbf{u}'_2 + f\,\mathbf{u}'_3$

$\mathbf{u}_1=g\,\mathbf{u}'_1 + h\,\mathbf{u}'_2 + i\,\mathbf{u}'_3$

the transitional matrix is represented by a-i corresponding to the system above. We can solve for this matrix as follows:

$$\begin{vmatrix} 2 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 & 2 \end{vmatrix}$$

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 1 & 1/2 \\ 0 & 1/2 & 3/2 & 3/2 & 0 & 3/2 \\ 0 & 3/2 & 1/2 & 1/2 & 0 & 3/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 1 & 1/2 \\ 0 & 1 & 3 & 3 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 3/4 \end{bmatrix}$$ so the transition matrix from B'-B = $\begin{vmatrix} 0 & 1 & -1/4 \\ 0 & 0 & 3/4 \\ 1 & 0 & 3/4 \end{vmatrix}$?

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1/4 \\ 0 & 1 & 0 & 0 & 0 & 3/4 \\ 0 & 0 & 1 & 1 & 0 & 3/4 \end{bmatrix}$$

Question #7

Use the transition matrix from question 5 and 6 to find: $[\mathbf{w}]_B$ and $[\mathbf{w}]_{B'}$ where $\mathbf{w}=(1, 2, -2)$

I don't think so since $w = w_{\{e_1,e_2,e_3\}} \neq w_{B'}$ presumably.

Solution #7

10 points

$$[\mathbf{w}]_B = \begin{vmatrix} 0 & -1 & 1 \\ 1 & 1/3 & 0 \\ 0 & 4/3 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 2 \\ -2 \end{vmatrix} = \begin{vmatrix} -4 \\ 5/3 \\ 8/3 \end{vmatrix}$$

you aren't even self-consistent with what you claimed on previous page – think about it. You will need to re-think/study coordinate vectors – ask me.

$$[\mathbf{w}]_{B'} = \begin{bmatrix} 0 & 1 & -1/4 \\ 0 & 0 & 3/4 \\ 1 & 0 & 3/4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -3/2 \\ -1/2 \end{bmatrix}$$

Question #8

Prove the following:

If a matrix is orthogonal, then its inverse is orthogonal

Solution #8

10 points

Good

Suppose we have an orthogonal matrix A. By theorem (or definition, depending on if you are referring to class or the book) $A^{-1} = A^T$. Since by theorem (or again by definition), an orthogonal matrix' columns and rows form an orthonormal set in R^n with the Euclidean inner product, and by the definition of a matrix' transpose, A^T's columns and rows also form orthonormal sets. By that same theorem, this implies that A^T is also orthogonal which implies that A^{-1} is orthogonal.

Question #9

Prove the following:

a- A product of orthogonal matrices is orthogonal

b- If a matrix A is orthogonal then ||A**x**||=||**x**||

Solution #9

15 points

a) Suppose we have two orthogonal matrices A and B. By theorem (or definition), if we multiply a matrix by it's transpose and end up with the identity matrix I, then that matrix is orthogonal. So say we have the matrix product AB. We can multiply by the transpose and get $AB(AB)^T = AB(B^T A^T) = AB(B^{-1}A^{-1}) = AB\,B^{-1}A^{-1} = AI\,A^{-1} = I$ which completes the proof.

b) Suppose we have an orthogonal matrix A so that $A^T A = I$. From the definition of the norm, and by using the Euclidean product for matrices we get

$\|A\mathbf{x}\| = (A\mathbf{x} \bullet A\mathbf{x})^{1/2} = (\mathbf{x} \bullet A^T A\mathbf{x})^{1/2} = (\mathbf{x} \bullet \mathbf{x})^{1/2} = \|\mathbf{x}\|$

Question #10

For the following matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 13 & -4 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

Find a) the characteristic equation; b) the eigenvalues; and c) a basis for the eigenspaces

Solution #10

25 points

a) We can find the characteristic equation of A by det($\lambda I - A$), which gives us

$$\det\begin{bmatrix} \lambda & 0 & 2 \\ -13 & \lambda+4 & -1 \\ 2 & -1 & \lambda-2 \end{bmatrix} = \lambda(\lambda+4)(\lambda-2) + 2(-13)(-1) - 2(2)(\lambda+4) - (-1)(-1)(\lambda)$$

$$= \lambda(\lambda^2+2\lambda-8) + 26 - (4\lambda+16) - \lambda = \lambda^3+2\lambda^2 - 8\lambda - 5\lambda+10 = \lambda^3+2\lambda^2-13\lambda+10$$

b) We can find the eigenvalues by solving the characteristic equation. We'll first try to find the integer solutions which can only be multiples of ten, namely ± 10, ± 5, ± 2, or ± 1. If we try 1 we will get 1+2-13+10 = 0 which shows that 1 is a factor. We then divide λ-1 into the equation.

$$\begin{array}{r} \lambda^2+3\lambda-10 \\ \lambda-1 \,\big)\, \lambda^3+2\lambda^2-13\lambda+10 \\ -\ \lambda^3-\lambda^2 \quad = \qquad\qquad \\ 0+3\lambda^2-13\lambda+10 \\ -\ 3\lambda^2-3\lambda = \qquad \\ 0\ -10\lambda+10 \\ -10\lambda+10 = \\ 0 \end{array}$$

So the equation factors like so: $(\lambda-1)(\lambda^2+3\lambda-10)$

Additionally, we can factor $\lambda^2+3\lambda-10$ into $(\lambda+5)(\lambda-2)$, giving us $(\lambda+5)(\lambda-2)(\lambda-1)$. If we set this equal to zero, we will get the eigenvalues 1, 2, -5

c) By plugging in the eigenvalues and solving the resulting homogenous system, we can get a basis for the eigenspaces.

For λ=1

$$\begin{bmatrix} 1 & 0 & 2 \\ -13 & 5 & -1 \\ 2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 25 \\ 0 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

which implies that x3=t, x2= -5t, and x1=-2t. The corresponding basis is then $\begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix}$

For λ=2

$$\begin{bmatrix} 2 & 0 & 2 \\ -13 & 6 & -1 \\ 2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 2 \\ 0 & 12 & 24 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Which implies that x3=t, x2=-2t, and x1=-t. The corresponding basis is then $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

For λ=-5

$$\begin{bmatrix} -5 & 0 & 2 \\ -13 & -1 & -1 \\ 2 & -1 & -7 \end{bmatrix} \sim \begin{bmatrix} -5 & 0 & 2 \\ 0 & -5 & -31 \\ 0 & -5 & -31 \end{bmatrix} \sim \begin{bmatrix} -5 & 0 & 2 \\ 0 & -5 & -31 \\ 0 & 0 & 0 \end{bmatrix}$$

Which implies that x3=t, x2= (-31/5)t, and x1= (2/5)t. The corresponding basis is then $\begin{bmatrix} 2/5 \\ -31/5 \\ 1 \end{bmatrix}$

- change to $\begin{bmatrix} 2 \\ -31 \\ 5 \end{bmatrix}$

Problem #1 (6.1) 10 points

Let $\langle u, v\rangle$ be the Euclidean inner product on R2, reduce $\langle 2\mathbf{u} + \frac{1}{2}\mathbf{v}, \mathbf{u} - 5\mathbf{v}\rangle$ to the simplest terms possible, and then calculate it using $\mathbf{u} = (2, 4)$ and $\mathbf{v} = (-1, 2)$.

Solution

$\langle 2\mathbf{u} + \frac{1}{2}\mathbf{v}, \mathbf{u} - 5\mathbf{v}\rangle \Rightarrow$
$\langle 2\mathbf{u}, \mathbf{u} - 5\mathbf{v}\rangle + \langle \frac{1}{2}\mathbf{v}, \mathbf{u} - 5\mathbf{v}\rangle$ by part (b) of theorem 6.1.1 $\Rightarrow$
$\langle 2\mathbf{u}, \mathbf{u}\rangle - \langle 2\mathbf{u}, 5\mathbf{v}\rangle + \langle \frac{1}{2}\mathbf{v}, \mathbf{u}\rangle - \langle \frac{1}{2}\mathbf{v}, 5\mathbf{v}\rangle$ by part (c) of theorem 6.1.1 $\Rightarrow$
$2\|\mathbf{u}\|^2 - 10\langle \mathbf{u}, \mathbf{v}\rangle + \frac{1}{2}\langle \mathbf{u}, \mathbf{v}\rangle - 5/2\|\mathbf{v}\|^2$ by part (c) of theorem 6.1.1 and by the definition of a norm $\Rightarrow$
$2\|\mathbf{u}\|^2 - 19/2\langle \mathbf{u}, \mathbf{v}\rangle - 5/2\|\mathbf{v}\|^2$ by axiom 1 of the definition of an inner product.

Plugging in the values for **u** and **v** we get: $2\|(2,4)\|^2 - 19/2\langle (2, 4), (-1, 2)\rangle - 5/2\|(-1, 3)\|^2$
$= 2(20) - 19/2(6) - 5/2(10) = 40 - 57 - 25 = -42$

Problem #2 (6.2) 10 points

Let R^3 have the Euclidean Inner Product. For which values of k and j are **u** and **v** orthogonal?
$\mathbf{u} = (k, -2, 0)$ $\qquad$ $\mathbf{v} = (6, j, 2)$

Solution

The inner product of **u** and **v** needs to equal zero. Therefore:
$0 = 6k - 2j + 0(2) \Rightarrow$
$0 = 6k - 2j \Rightarrow$
$2j = 6k \Rightarrow$
$j = 3k$
Every j that is three times k will make **u** and **v** orthogonal.

Problem #3 (6.3) 20 points

a) Verify that the following vectors form an orthogonal basis in R^3 with Euclidean Inner Product. b) create an Orthonormal basis from these vectors. c) express the vector T = (1, 2, 3) as a linear combination of this Orthonormal basis

$V_1 = (1, 0, -1)$ $\qquad$ $V_2 = (2, 0, 2)$ $\qquad$ $V_3 = (0, 5, 0)$

Solution:
a) These vectors are Linearly Independent ~~and~~ because F.Y.I.: none of them is 0 and...

$\langle V_1, V_2\rangle = 2 + 0 + -2 = 0$

$< V_1, V_3> = 0 + 0 + 0 = 0$
$< V_2, V_3> = 0 + 0 + 0 = 0$

Therefore $V_1 = (1, 0, -1)$, $V_2 = (2, 0 ,2)$, $V_3 = (0, 5, 0)$ form an Orthogonal Basis in R^3.

b) $U_1 = \dfrac{V_1}{\|V_1\|}$ $\quad U_2 = \dfrac{V_2}{\|V_2\|}$ $\quad U_3 = \dfrac{V_3}{\|V_3\|}$

$\|V_1\| = <V_1, V_1>^{1/2} = (1 + 0 + 1) = 2^{1/2}$
$\|V_2\| = <V_2, V_2>^{1/2} = (4 + 0 + 4) = 8^{1/2} = 2(3^{1/2})$
$\|V_3\| = <V_3, V_3>^{1/2} = (0 + 25 + 0) = 25^{1/2} = 5$

$U_1 = (1/(2)^{1/2}, 0, -1/(2)^{1/2})$ $\quad U_2 = (1/(2)^{1/2}, 0, 1/(2)^{1/2})$ $\quad U_3 = (0, 1, 0)$

c) $T = <T, U_1>U_1 + <T, U_2>U_2 + <T, U_3>U_3$

$<T, U_1> = 1/(2)^{1/2}, + 0 + -3/(2)^{1/2} = -2/(2)^{1/2}$
$<T, U_2> = 1/(2)^{1/2} + 0 + 3/(2)^{1/2} = 4/(2)^{1/2}$
$<T, U_3> = 0 + 2 + 0 = 2$

$T = -2/(2)^{1/2}\ U_1 + 4/(2)^{1/2}\ U_2 + 2\ U_3$

Problem #4 (6.3) 20 points

Using the Gram-Schmidt process transform the vectors:
$u_1 = (1, 2, 1)$ $u_2 = (2, 8, 6)$ $u_3 = (0, 1, 4)$ into an orthogonal basis $\{v_1, v_2, v_3\}$, and then into an orthonormal basis $\{v_1, v_2, v_3\}$.

Solution:
$v_1 = u_1 = (1, 2, 1)$

$$v_2 = u_2 - \frac{<u_2, v_1>}{\| v_1 \|^2} = (2, 8, 6) - \frac{(2*1 + 8*2 + 6*1)}{1^2 + 2^2 + 1^2} * (1, 2, 1) = (2, 8, 6) - \frac{24}{6} * (1, 2, 1) =>$$

$(2, 8, 6) – 4(1, 2, 1) = (2, 8, 6) – (4, 8, 4) = (-2, 0, 2) = v_2$ – change to (-1,0,1)

$$v_3 = u_3 - \frac{<u_3, v_1>*v_1}{\| v_1 \|^2} - \frac{<u_3, v_2>*v_2}{\| v_2 \|^2} =>$$

$$(0, 1, 4) - \frac{(0*1 + 1*2 + 4*1)}{1^2 + 2^2 + 1^2} * (1, 2, 1) - \frac{(0*-2 + 1*0 + 4*2)}{(-2)^2 + 0^2 + 2^2} * (-2, 0, 2) =>$$

$$(0, 1, 4) - \frac{6}{6} * (1, 2, 1) - \frac{8}{8} * (-2, 0, 2) = (0, 1, 4) - (1, 2, 1) - (-2, 0, 2) = (1, -1, 1) = v_3$$

The orthogonal basis for R^3 is: $v_1 = (1, 2, 1)$ $v_2 = (-2, 0, 2)$ $v_3 = (1, -1, 1)$

Now, find the orthonormal basis.

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{(1, 2, 1)}{\sqrt{1^2 + 2^2 + 1^2}} = \frac{(1, 2, 1)}{\sqrt{6}} = (1/\sqrt{6}, 2/\sqrt{6}, 1/\sqrt{6})$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{(-2, 0, 2)}{\sqrt{(-2)^2 + 0^2 + 2^2}} = \frac{(-2, 0, 2)}{\sqrt{8}} = (-1/\sqrt{2}, 0, 1/\sqrt{2})$$ - unsurprising - use $v_2 = (-1, 0, 1)$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{(1, -1, 1)}{\sqrt{1^2 + (-1)^2 + 1^2}} = \frac{(1, -1, 1)}{\sqrt{3}} = (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$$

~~The~~ An orthonormal basis for R^3 is:
$q_1 = (1/\sqrt{6}, 2/\sqrt{6}, 1/\sqrt{6})$ $q_2 = (-1/\sqrt{2}, 0, 1/\sqrt{2})$ $q_3 = (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$

Problem #5 (6.4) 15 points well-engineered

Find the least squares solution of the following linear system $A\mathbf{x} = \mathbf{b}$ and find the orthogonal projection of $\mathbf{b}$ onto the column space of A. well-said.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 2 \\ -4 \\ 4 \end{bmatrix}$$

Solution

From theorem 6.4.2/6.4.4, the system $A^TA\mathbf{x} = A^T\mathbf{b}$ is consistent and it has a unique solution which is the least squares solution. The orthogonal projection of $\mathbf{b}$ onto the column space of A is $A\mathbf{x}$.

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & -2 & -1 \\ 2 & 1 & 0 & 0 \end{bmatrix} \text{ and } A^TA = \begin{bmatrix} 3 & -3 & 3 \\ -3 & 6 & -2 \\ 3 & -2 & 5 \end{bmatrix} \text{ and } A^T\mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 12 \end{bmatrix}$$

Thus, the solution $\mathbf{x}$ of the equation $\begin{bmatrix} 3 & -3 & 3 \\ -3 & 6 & -2 \\ 3 & -2 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 12 \end{bmatrix}$ is the least squares solution.

Augment it and row-reduce.

$$\begin{bmatrix} 3 & -3 & 3 & 3 \\ -3 & 6 & -2 & -1 \\ 3 & -2 & 5 & 12 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -3 & 3 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 5 & 25 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

The least squares solution is $\mathbf{x} = \begin{bmatrix} -5 \\ -1 \\ 5 \end{bmatrix}$

The orthogonal projection is A**x** or

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -5 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

$\begin{bmatrix} 6 \\ 0 \\ -3 \\ 1 \end{bmatrix}$ is the orthogonal projection of **b** onto the column space of A.

Problem #6 (6.5) 15 points

Find the transition matrix for A to B if A = $\{u_1, u_2\}$ and B = $\{v_1, v_2\}$ in R^2 where $v_1 = u_2$ and $v_2 = u_1$.

Solution

If we let $u_1 = (a, b)$ and $u_2 = (c, d)$ then we can write A and B.

$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B=\begin{bmatrix} b & a \\ d & c \end{bmatrix}$ so, the transition matrix can be found with $A^{-1}B$ so we solve it:

$$\left[\begin{array}{cc|cc} a & b & b & a \\ c & d & d & c \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{b}{a} & 1 \\ c & d & d & c \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{b}{a} & 1 \\ 0 & \frac{ad-bc}{a} & \frac{ad-bc}{a} & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{b}{a} & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$ so the transition matrix from A to B is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. - which makes sense!

Problem #7 (6.6) 10 points

Determine whether the following matrix is Orthogonal.

[1 3 -2]
[3 1 -2]
[-2 -2 1]

Solution:

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 14 & 10 & -10 \\ 10 & 14 & -10 \\ -10 & -10 & 9 \end{bmatrix}$$

$AA^T \neq I$ Therefore, this is not an Orthogonal matrix.

Problem #8 (6.6) 20 points

For an orthogonal matrix A, A^TA and AA^T are both equal to the identity matrix. Use this fact to show that both the row vectors **r1**, **r2**, **r3** and column vectors **c1, c2**, **c3** form orthonormal sets, respectively, for an A_{3x3} matrix.

Solution

If row vectors are **r1**, **r2**, **r3** and the column vectors are **c1**, **c2**, **c3** then A is $\begin{bmatrix} \mathbf{r1} \\ \mathbf{r2} \\ \mathbf{r3} \end{bmatrix}$ or

$[\mathbf{c1} \quad \mathbf{c2} \quad \mathbf{c3}]$, and A^T is $\begin{bmatrix} \mathbf{c1} \\ \mathbf{c2} \\ \mathbf{c3} \end{bmatrix}$ or $[\mathbf{r1} \quad \mathbf{r2} \quad \mathbf{r3}]$

A^T A is then $\begin{bmatrix} \mathbf{r1 \bullet r1} & \mathbf{r1 \bullet r2} & \mathbf{r1 \bullet r3} \\ \mathbf{r2 \bullet r1} & \mathbf{r2 \bullet r2} & \mathbf{r2 \bullet r3} \\ \mathbf{r3 \bullet r1} & \mathbf{r3 \bullet r2} & \mathbf{r3 \bullet r3} \end{bmatrix}$

and likewise A A^T is $\begin{bmatrix} \mathbf{c1 \bullet c1} & \mathbf{c1 \bullet c2} & \mathbf{c1 \bullet c3} \\ \mathbf{c2 \bullet c1} & \mathbf{c2 \bullet c2} & \mathbf{c2 \bullet c3} \\ \mathbf{c3 \bullet c1} & \mathbf{c3 \bullet c2} & \mathbf{c3 \bullet c3} \end{bmatrix}$. These must equal the identity matrix I

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. So the corresponding entries of these matrices must be equal if the matrices are to be equal so $\mathbf{r1 \bullet r1} = \mathbf{r2 \bullet r2} = \mathbf{r3 \bullet r3} = 1$ and $\mathbf{c1 \bullet c1} = \mathbf{c2 \bullet c2} = \mathbf{c3 \bullet c3} =$ 1, since these dot products lie on the diagonal These shows that the both the row vectors and column vector of A are normalized.

To show that they are orthonormal, we must show that all the vectors are perpendicular. This is done by showing that their inner products are zero with respect to each other. All entries in the identity matrix that are not on the diagonal are zero. These means that all the row vectors

r1 • r2 = **r1 • r3** = **r2 • r3** = 0 and since the Euclidean inner product is commutative this shows that the row vectors form and orthonormal set.

c1 • c2 = **c1 • c3** = **c2 • c3** = 0, by the same token, the column vectors from an orthonormal set.

Since an orthogonal matrix multiplied by its transpose is the identity matrix, the row vectors form an orthonormal set as do the column vectors.

<u>Problem #9 (7.1) 15 points</u>

Find bases for the Eigenspace A. Show λ with each of it's corresponding bases.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & -2 & 0 \\ -1 & 0 & 4 \end{bmatrix}$$

<u>Solution:</u>

Take the determinate of the function "λI – A" to find the correct value for λ

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda - 2 & 0 & 1 \\ -1 & \lambda + 2 & 0 \\ -1 & 0 & \lambda - 4 \end{bmatrix}\right) = (\lambda + 2)(\lambda^2 - 6\lambda + 9)$$

$(\lambda + 2)(\lambda^2 - 6\lambda + 9) = 0$ then $\lambda = 3, -2$

$(\lambda - 3)^2$

now plug the values for λ back into the equation $\begin{bmatrix} \lambda - 2 & 0 & 1 \\ -1 & \lambda + 2 & 0 \\ -1 & 0 & \lambda - 4 \end{bmatrix}$

$$\lambda = 3 \Rightarrow \begin{bmatrix} 3-2 & 0 & 1 \\ -1 & 3+2 & 0 \\ -1 & 0 & 3-4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 5 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$$\lambda = -2 \Rightarrow \begin{bmatrix} -2-2 & 0 & 1 \\ -1 & -2+2 & 0 \\ -1 & 0 & -2-4 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & -6 \end{bmatrix}$$

using reduced row-echelon for we find that the bases are:

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 5 & 0 \\ -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow t\begin{bmatrix} -1 \\ -1/5 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ -1/5 \\ 1 \end{bmatrix} \text{ for } \lambda = 3$$

$$\begin{bmatrix} -4 & 0 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ for } \lambda = -2$$

for the corresponding values of λ.

Problem #10 (7.2) 15 points

Find if matrix A is diagonalizable:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

Solution

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-2 & 0 & 0 \\ -1 & \lambda-2 & 0 \\ -3 & 0 & \lambda-2 \end{bmatrix} = (\lambda - 2)^3$$

The characteristic polynomial of A is then : $(\lambda - 2)^3 = 0 \Rightarrow \lambda = 2$
Therefore the eigenvalue of A is 2. Now to find the eigenvector associated with $\lambda = 2$.

$$\begin{bmatrix} \lambda-2 & 0 & 0 \\ -1 & \lambda-2 & 0 \\ -3 & 0 & \lambda-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} 1x_1 + 0x_2 + 0x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 = 0s + 0t \\ x_2 = s \\ x_3 = t \end{matrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t$$

The eigenvectors for A with respect to $\lambda=2$ are: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \& \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Since A is a 3x3 Matrix and there is only two vectors A is not diagonalizable

5pts

1. Compute <**u**,**v**> where

$$u = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix} \text{ and } v = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

– need to specify inner product – such is "up to you"

Solution:

$$\langle u,v \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

<**u**,**v**>=3(-1)+(-2)3+4(1)+8(1)=3

15pts

2. Let R^2 and R^3 have the Euclidean inner product. Find the cosine of the angle between **u** and **v**.

a)
u = (1,-3), **v** = (2,4)

b)
u = (-1,5,2), **v** = (2,4,-9)

Solution:

a)

$$\cos\theta = \frac{\langle u,v \rangle}{\|u\|\|v\|}$$

<**u**,**v**>=1(2)+(-3)(4)=-10

$$\|u\| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$$
$$\|v\| = \sqrt{2^2 + 4^2} = \sqrt{20}$$

$$\cos\theta = \frac{-10}{\sqrt{10} * \sqrt{20}} = \frac{-10}{10\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

b)
<**u**,**v**> = -1(2) + 5(4) + 2(-9)=0

$$\|u\| = \sqrt{(-1)^2 + 5^2 + 2^2} = \sqrt{30}$$
$$\|v\| = \sqrt{2^2 + 4^2 + 9^2} = \sqrt{101}$$

$$\cos\theta = \frac{0}{\sqrt{30} * \sqrt{101}} = \frac{0}{\sqrt{3030}} = 0$$

– so θ = 90° ?

15pts

3. Find the least squares solution of the linear system **Ax**=**b**, and find the orthogonal projection of **b** onto the column space of A.

$$A=\begin{bmatrix}1&0&-1\\2&1&-2\\1&1&0\\1&1&-1\end{bmatrix}, b=\begin{bmatrix}6\\0\\9\\3\end{bmatrix}$$

Solution:

$$A^TA=\begin{bmatrix}1&2&1&1\\0&1&1&1\\-1&-2&0&-1\end{bmatrix}\begin{bmatrix}1&0&-1\\2&1&-2\\1&1&0\\1&1&-1\end{bmatrix}=\begin{bmatrix}7&4&-6\\4&3&-3\\-6&-3&6\end{bmatrix}$$

$$A^Tb=\begin{bmatrix}1&2&1&1\\0&1&1&1\\-1&-2&0&-1\end{bmatrix}\begin{bmatrix}6\\0\\9\\3\end{bmatrix}=\begin{bmatrix}18\\12\\-9\end{bmatrix}$$

The normal system $A^TA\mathbf{x}=A^T\mathbf{b}$ is:

$$\begin{bmatrix}7&4&-6\\4&3&-3\\-6&-3&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}18\\12\\-9\end{bmatrix}$$

Solving this system yields

$$\begin{bmatrix}7&4&-6&18\\4&3&-3&12\\-6&-3&6&-9\end{bmatrix}\sim\begin{bmatrix}28&16&-24&72\\-28&-21&21&-84\\-6&-3&6&-9\end{bmatrix}\sim\begin{bmatrix}42&24&-36&108\\0&-5&-3&-12\\-42&-21&42&-63\end{bmatrix}$$

$$\sim\begin{bmatrix}7&4&-6&18\\0&-5&-3&-12\\0&3&6&45\end{bmatrix}\sim\begin{bmatrix}7&4&-6&18\\0&1&2&15\\0&-5&-3&-12\end{bmatrix}\sim\begin{bmatrix}7&4&-6&18\\0&1&2&15\\0&0&7&63\end{bmatrix}\sim\begin{bmatrix}7&4&-6&18\\0&1&2&15\\0&0&1&9\end{bmatrix}$$

$$\sim \begin{bmatrix} 7 & 4 & 0 & 72 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 7 & 0 & 0 & 84 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{bmatrix}$$

therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -3 \\ 9 \end{bmatrix}$$

$x_1 = 12$, $x_2 = -3$, $x_3 = 9$

$$Ax = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 12 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 9 \\ 0 \end{bmatrix}$$

– anything to say about this?, Speak to your eager audience.

25pts

4. For the following matrix, find: characteristic equation, eigenvalues, bases for the eigenspaces.

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Solution

The characteristic equation for this matrix is the equation found by "using the fact that det [λI – A] = 0." First, we find the matrix λI – A: $\begin{bmatrix} \lambda-4 & 0 & 1 \\ -2 & \lambda-1 & 0 \\ -2 & 0 & \lambda-1 \end{bmatrix}$

(what does that mean?)

Next we take the determinant of this matrix:
$\text{Det}(\lambda I - A) = (\lambda-4)(\lambda^2 - 2\lambda + 1) + 2\lambda - 2$.

this makes sense— good.

This simplifies to: Det (λI – A) = $\lambda^3 - 6\lambda^2 + 11\lambda - 6$. This is the characteristic polynomial of the matrix A.

To find the eigenvalues, we solve the characteristic equation for λ:

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$.

Factoring yields $(\lambda - 3)(\lambda^2 - 3\lambda + 2) = 0$, or $(\lambda - 3)(\lambda - 2)(\lambda - 1)$. Solving this equation yields our **eigenvalues, λ = 3, λ = 2, and λ = 1.**

, indeed the null space of such.

To find the bases for the eigenspaces, we return to the matrix λI – A. For each eigenvalue we will replace λ with the eigenvalue, then solve the equation presented as follows:

If λ = 1: $\begin{bmatrix} \lambda-4 & 0 & 1 \\ -2 & \lambda-1 & 0 \\ -2 & 0 & \lambda-1 \end{bmatrix}$ becomes $\begin{bmatrix} -3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$. The equation $\begin{bmatrix} -3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(1) \\ x(2) \\ x(3) \end{bmatrix} = 0$ is

solved as follows: x(1) = 0 from the 2nd and 3rd equations. x(3) = 0 because of the 1st equation. No parameters are set on x(2) so x(2) = s. The basis then becomes

$\begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix}$ or $s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ So one basis is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

If λ = 2: $\begin{bmatrix} \lambda-4 & 0 & 1 \\ -2 & \lambda-1 & 0 \\ -2 & 0 & \lambda-1 \end{bmatrix}$ becomes $\begin{bmatrix} -2 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. The equation $\begin{bmatrix} -2 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(1) \\ x(2) \\ x(3) \end{bmatrix} = 0$ is solved as follows. Let x(1) = s. Then by the first row, x(3) = 2s. Also, by the 2^{nd} row, x(2) = 2s. Thus the basis becomes

$$\begin{bmatrix} s \\ 2s \\ 2s \end{bmatrix} \text{ or } s\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \text{ So another basis is } \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

If $\lambda = 3$ then $\lambda I - A$ becomes

$$\begin{bmatrix} -1 & 0 & 1 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix} \text{ The equation } \begin{bmatrix} -1 & 0 & 1 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x(1) \\ x(2) \\ x(3) \end{bmatrix} = 0 \text{ is}$$

solved as follows: Let x(1) = s. Then by the first and third rows, x(3) = x(1) = s. Then by the second row, x(2) = x(1) = s. Thus the basis becomes

$$\begin{bmatrix} s \\ s \\ s \end{bmatrix} \text{ or } s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus three bases of the eigenspaces are $\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{2} \end{bmatrix}$**, and** $\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}$**.**

Come by and we'll talk about using your equation editor more effectively.

30pts

5. For $A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$ find A^11.

Solution

To begin, we find the bases for the eigenspaces of A. This will help us to find P, the vector which diagonalizes A.

$$\lambda I - A = \begin{bmatrix} \lambda+1 & 7 & -1 \\ 0 & \lambda-1 & 0 \\ 0 & 15 & \lambda+2 \end{bmatrix} \text{ Det } \lambda I - A = (\lambda-1)(\lambda+1)(\lambda+2) = 0. \text{ By}$$

inspection we see that the eigenvalues for this matrix are -2, -1, and 1. Next we find the bases for the eigenspaces.

For $\lambda = -2$: $\begin{bmatrix} -1 & 7 & -1 \\ 0 & -3 & 0 \\ 0 & 15 & 0 \end{bmatrix}\begin{bmatrix} x(1) \\ x(2) \\ x(3) \end{bmatrix} = 0.$ By rows 2 and 3, we know that x(2) = 0.

From the first row, we see that x(1) = -x(3). So we set x(1) = s and x(3) = -s. This gives us our first basis:

$$\begin{bmatrix} s \\ 0 \\ -s \end{bmatrix} \text{ or } s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ So the first basis is } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For $\lambda = -1$: $\begin{bmatrix} 0 & 7 & -1 \\ 0 & -2 & 0 \\ 0 & 15 & 1 \end{bmatrix}\begin{bmatrix} x(1) \\ x(2) \\ x(3) \end{bmatrix} = 0.$ By row 2 we see that x(2) = 0. From that and row 3 we see that x(3) = 0. x(1) is never initialized, so it can be set to s. Thus the basis is as follows:

$$\begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} \text{ or } s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ So the second basis is } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 1$:

$$\begin{bmatrix} 2 & 7 & -1 \\ 0 & 0 & 0 \\ 0 & 15 & 3 \end{bmatrix}\begin{bmatrix} x(1) \\ x(2) \\ x(3) \end{bmatrix} = 0.$$

To solve, set x(2) = s. Then by row 3, x(3) = -5s.

Replacing x(2) = s and x(3) = -5s in row 1, we get x(1) = -6s. Thus the basis for this eigenspace is as follows:

$$\begin{bmatrix} -6s \\ s \\ -5s \end{bmatrix} \text{ or } s\begin{bmatrix} -6 \\ 1 \\ -5 \end{bmatrix} \text{ Thus the third basis is } \begin{bmatrix} -6 \\ 1 \\ -5 \end{bmatrix}$$

Putting these three bases together into one matrix gives us P, our matrix that diagonalizes A:

$$P = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 0 & 1 \\ -1 & 0 & -5 \end{bmatrix}.$$

Now we also will need the inverse of P, or P^-1. Computing this yields P^-1 to be equal to

$$\begin{bmatrix} 0 & -5 & -1 \\ 1 & 11 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We also will need D, where D is the diagonal matrix with $\lambda(1)$, $\lambda(2)$, and $\lambda(3)$ as its diagonal entries:

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now from formula 10 of section 7.2, we know that A^11 = P(D^11)P^-1. We will now proceed to calculate this.

$$P(D^{11}) = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 0 & 1 \\ -1 & 0 & -5 \end{bmatrix}\begin{bmatrix} -2048 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2048 & -1 & -6 \\ 0 & 0 & 1 \\ 2048 & 0 & -5 \end{bmatrix}$$

This matrix times P^-1 is as follows:

$$\begin{bmatrix} -2048 & -1 & -6 \\ 0 & 0 & 1 \\ 2048 & 0 & -5 \end{bmatrix}\begin{bmatrix} 0 & -5 & -1 \\ 1 & 11 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \mathbf{\begin{bmatrix} -1 & 10223 & 2047 \\ 0 & 1 & 0 \\ 0 & -10245 & -2048 \end{bmatrix} = A^{11}}$$

5 pts

6. What is the procedure for orthogonally diagonalizing a symmetric matrix? What is the theorem that this procedure is a consequence of?

Solution

Step 1. Find a basis for each eigenspace of A.
Step 2. Apply the Gram-Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.
Step 3. Form the matrix P whose columns are the basis vectors constructed in Step 2; this matrix orthogonally diagonalizes A.

This procedure is a result of the theorem that states that if A is a symmetric matrix, then

(a) The eigenvalues of A are all real numbers.
(b) Eigenvectors from different eigenspaces are orthogonal.
(c) there's "enough" of these, i.e. geometric multiplicity = algebraic mult.

10pts

7. Let $\mathbf{u} = (u_1,u_2)$ and $\mathbf{v} = (v_1,v_2)$ be vectors in R^2. Verify that the weighted Euclidean inner product

$$\langle \mathbf{u},\mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$$

satisfies the four inner product axioms.

Solution

If u and v are interchanged in this equation, the right side remains the same. Thus,

$$\langle \mathbf{u},\mathbf{v} \rangle = \langle \mathbf{v},\mathbf{u} \rangle$$

If $\mathbf{z} = (z_1,z_2)$, then

$$\begin{aligned} \langle \mathbf{u} + \mathbf{v},\mathbf{z} \rangle &= 3(u_1 + v_1)z_1 + 2(u_2 + v_2)z_2 \\ &= (3u_1z_1 + 2u_2z_2) + (3v_1z_1 + 2v_2z_2) \\ &= \langle \mathbf{u},\mathbf{z} \rangle + \langle \mathbf{v},\mathbf{z} \rangle \end{aligned}$$

which proves the second axiom.
Next,

$$\langle k\mathbf{u},\mathbf{v} \rangle = 3(ku_1)v_1 + 2(ku_2)v_2 = k(3u_1v_1 + 2u_2v_2) = k\langle \mathbf{u},\mathbf{v} \rangle$$

which proves the third axiom.
Last,

$$\langle \mathbf{v},\mathbf{v} \rangle = 3v_1v_1 + 2v_2v_2 = 3v_1^2 + 2v_2^2$$

Clearly, $\langle \mathbf{v},\mathbf{v} \rangle = 3v_1^2 + 2v_2^2 \geq 0$. Moreover, $\langle \mathbf{v},\mathbf{v} \rangle = 3v_1^2 + 2v_2^2 = 0$ iff $v_1 = v_2 = 0$, or, iff $\mathbf{v} = (v_1,v_2) = \mathbf{0}$. Thus the fourth axiom is satisfied.

25pts

8. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}_1', \mathbf{u}_2'\}$ for R^2, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \text{and } \mathbf{u}_2' = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

a) Find the transition matrix from B' to B. ($P_{BB'}$)

b) Find the transition matrix from B to B'. ($P_{B'B}$)

c) Compute the coordinate vector $[\mathbf{w}]_B$, where

$$\mathbf{w} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

and use the formula $[\mathbf{v}]_{B'} = P_{B'B}[\mathbf{v}]_B$ to calculate $[\mathbf{w}]_{B'}$.

d) Check your work by computing $[\mathbf{w}]_{B'}$ "directly". (By setting up an augmented matrix)

Solution:

There are several ways to do this, but the simplest way is to start with the equation:

$$\underline{\mathbf{v}} = \left[\mathbf{u}_1 \mid \dots \mid \mathbf{u}_n\right][\mathbf{v}]_B = \left[\mathbf{u}_1' \mid \dots \mid \mathbf{u}_n'\right][\mathbf{v}]_{B'} \quad \Leftrightarrow$$

$$[\mathbf{v}]_B = \left[\mathbf{u}_1 \mid \dots \mid \mathbf{u}_n\right]^{-1}\left[\mathbf{u}_1' \mid \dots \mid \mathbf{u}_n'\right][\mathbf{v}]_{B'} =: P_{BB'}[\mathbf{v}]_{B'}$$

$$\Rightarrow P_{BB'} = \left[\mathbf{u}_1 \mid \dots \mid \mathbf{u}_n\right]^{-1}\left[\mathbf{u}_1' \mid \dots \mid \mathbf{u}_n'\right]$$

a)

$$P_{BB'} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}.$$

b)

$$P_{B'B} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4/11 & 3/11 \\ -1/11 & 2/11 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4/11 & 3/11 \\ -1/11 & 2/11 \end{bmatrix}.$$

c)

$$[\mathbf{w}]_B = \left[\mathbf{u}_1 \mid \dots \mid \mathbf{u}_n\right]\mathbf{w} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

$$[\mathbf{w}]_{B'} = P_{B'B}[\mathbf{w}]_B = \begin{bmatrix} 4/11 & 3/11 \\ -1/11 & 2/11 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -3/11 \\ -13/11 \end{bmatrix}.$$

d) The corresponding augmented matrix to solve for $[\mathbf{w}]_{B'}$ is:

$$\left[\begin{array}{cc|c} 2 & -3 & 3 \\ 1 & 4 & -5 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -3 & 3 \\ 0 & 11/2 & -13/2 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -3 & 3 \\ 0 & 1 & -13/11 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -3/11 \\ 0 & 1 & -13/11 \end{array}\right]$$

$$\Rightarrow [\mathbf{w}]_{B'} = \begin{bmatrix} -3/11 \\ -13/11 \end{bmatrix}.$$

10pts

9. a) Explain why that if A is orthogonal, then A^T is orthogonal.
b) What is the normal system for $A\mathbf{x} = \mathbf{b}$ when A is orthogonal?

Solution:

a) Because A is orthogonal, we know that both the column vectors of A and the row vectors of A form an orthonormal set (Theorem 6.6.1). A^T is just A with its row and column vectors swapped. So, because both the column vectors and row vectors of A form an orthonormal set, the column vectors of A^T (which were the row vectors of A) and the row vectors of A^T (which were the column vectors of A) form orthonormal sets, and therefore A^T is orthogonal.

b) The normal system is given by the equation :

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

Because A is orthogonal, we know that $A^T = A^{-1}$ (definition of an orthogonal matrix). Therefore,

$$A^T A\mathbf{x} = A^T\mathbf{b} \rightarrow A^{-1}A\mathbf{x} = A^T\mathbf{b} \rightarrow I\mathbf{x} = A^T\mathbf{b} \rightarrow \mathbf{x} = A^T\mathbf{b}$$

10pts

10. Is the matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$ orthogonal? If so, find the inverse.

Solution:

The matrix A is orthogonal if $A^T = A^{-1}$. This implies that if $A^T A = I$, then A is orthogonal. So,

$$A^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}. \quad A^T A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow A \text{ is an orthogonal matrix. } A^{-1} = A^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}.$$

2)

Question 1 (10 pts)

Find the inner product of the following — ambiguous — you must define inner product in a) and b).

a)$\mathbf{u}=\begin{bmatrix} 3 & 7 \\ 2 & 9 \end{bmatrix}$ $\mathbf{v}=\begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$

b)$\mathbf{p}=-2+3x-4x^2$ $q=9x-5x^2$

c)$\mathbf{u}=[1,5,6,3]$ $\mathbf{v}=[2,2,5,1]$

a) 3*2+7*2+2*5+9*3
   6+14+10+27
   57
b) 3*9+4*5
   27+20
   57
c) 1*2+5*2+6*5+3*1
   2+10+30+3
   45

Question 2 (10 pts)

Let R^3 have the Euclidean inner product. Let (a, b) be any two real numbers. Find the cosine of the angle Θ between the vectors $\mathbf{u} = (1, 0, a)$ and $\mathbf{v} = (2, b, 1)$. Find "a" if Θ = 90?

<u>**Solution:**</u>

The cosine of the angle Θ between the vectors **u** and **v** is

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}.$$

We can calculate each part of the right hand side of the equation.

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \left(1 + a^2\right)^{1/2}$$

$$\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = \left(4 + b^2 + 1\right)^{1/2} = \left(b^2 + 5\right)^{1/2}$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = (1 \cdot 2 + 0 \cdot b + a \cdot 1) = (a + 2)$$

Knowing these values we can calculate that

$$\cos\theta = \frac{(a+2)}{\left(a^2+1\right)^{1/2}\left(b^2+5\right)^{1/2}}$$

If Θ = 90, then cos90 = 0. By inspection of the previous equation, we see that a = -2.

Question 3 (6 pts)

Good

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthogonal basis for an inner product space V. Show that if $\mathbf{w} \in$ V, then $\|\mathbf{w}\|^2 = \left(\frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|}\right)^2 + \left(\frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|}\right)^2 + \ldots + \left(\frac{\langle \mathbf{w}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|}\right)^2$.

Solution:

$$\|\mathbf{w}\|^2 = \langle \mathbf{w}, \mathbf{w} \rangle = \langle \mathbf{w}, c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n \rangle =$$

theorem

$$\left\langle \mathbf{w}, \frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{w}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle}\mathbf{v}_n \right\rangle =$$

$$\left\langle \mathbf{w}, \frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 \right\rangle + \left\langle \mathbf{w}, \frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 \right\rangle + \cdots + \left\langle \mathbf{w}, \frac{\langle \mathbf{w}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle}\mathbf{v}_n \right\rangle =$$

$$\frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\langle \mathbf{w}, \mathbf{v}_1 \rangle + \frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\langle \mathbf{w}, \mathbf{v}_2 \rangle + \cdots + \frac{\langle \mathbf{w}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle}\langle \mathbf{w}, \mathbf{v}_n \rangle =$$

$$\frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle^2}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} + \frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle^2}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} + \cdots + \frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle^2}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} = \frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle^2}{\|\mathbf{v}_1\|^2} + \frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle^2}{\|\mathbf{v}_1\|^2} + \cdots + \frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle^2}{\|\mathbf{v}_1\|^2} =$$

$$\left(\frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|}\right)^2 + \left(\frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|}\right)^2 + \ldots + \left(\frac{\langle \mathbf{w}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|}\right)^2$$

Question 4 (10 pts)

You are an analyst in the country Applemania, in their National Apple Department. Your boss believes that the quantity Q of apples demanded (in pounds) is proportional to the price P of the apples (per pound). ✱ She wants you to approximate the solution to the equation

$$Q = Px.$$

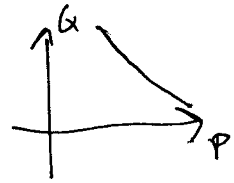
Where "Q" is the quantity of pounds of apples demanded per person and "P" is the price per pound. "x" is some proportional scalar. They have given you the following data from last years sales to help you estimate this equation. The table lists how many pounds of apples were demanded at various store prices.

What is the approximate value of x?

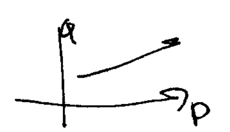
Price per Pound ($)	Quantity of Apples Demanded(lbs.)
2.01	.3
1.51	.54
1.22	.67
1.07	1.2
.94	1.73
.86	2.1

✱ That's a strange relationship – ask your economist colleagues.

→ your data looks like



, but your solution looks like →



→ so think about a more interesting/complicated model.

Solution:

We are trying to solve the inconsistent system Q=Px

.3	=	2.01x
.54	=	1.51x
.67	=	1.22x
1.2	=	1.07x
1.73	=	.94x
2.1	=	.86x

In other words, we must find the least squares approximation of the solution x in the inconsistent matrix equation Ax = B, where A=P and B=Q in our example.

$$A=\begin{bmatrix}2.01\\1.51\\1.22\\1.07\\.94\\.86\end{bmatrix},\ B=\begin{bmatrix}.3\\.54\\.67\\1.2\\1.73\\2.1\end{bmatrix}$$, and x is a scalar.

The normal equation is

$$A^TAx=A^TB\Rightarrow\begin{bmatrix}2.01 & 1.51 & 1.22 & 1.07 & .94 & .86\end{bmatrix}\begin{bmatrix}2.01\\1.51\\1.22\\1.07\\.94\\.86\end{bmatrix}x=\begin{bmatrix}2.01 & 1.51 & 1.22 & 1.07 & .94 & .86\end{bmatrix}\begin{bmatrix}.3\\.54\\.67\\1.2\\1.73\\2.1\end{bmatrix}$$

$$\Rightarrow[10.5767]x=[6.952]\Rightarrow x=6.952\cdot\frac{1}{10.5767}=.65729$$

By inspection, the least squares approximation is x = .65729. Therefore, the approximate proportion is Q=.657P – large error because model is bad.

Question 5 (10 points, 5 points each)

a. Find the coordinate vector for **w** relative to the basis S = {$\mathbf{u}_1$,$\mathbf{u}_2$} for R^2.

$\mathbf{u}_1$= (2, 5), $\mathbf{u}_2$= (1, 3); **w**= (5, 13)

b. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where

$$u_1 := \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad u_2 := \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$v_1 := \begin{pmatrix} 4 \\ 2 \end{pmatrix} \qquad v_2 := \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

Find the transition matrix from B to B'.

Solution:

Part A

$$u_1 := \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$u_2 := \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$w := \begin{pmatrix} 5 \\ 13 \end{pmatrix}$$

$$w = c_1 \cdot u_1 + c_2 \cdot u_2$$

$$5 = 2 \cdot c_1 + 1 \cdot c_2$$

$$13 = 5 \cdot c_1 + 3 \cdot c_2$$

Solve the corresponding augmented matrix

$$\begin{pmatrix} 2 & 1 & 5 \\ 5 & 3 & 13 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 5 \\ 1 & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

Therefore the coordinate vector is (c1,c2) = (2,1).

Part B

$u_1 := \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $u_2 := \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$v_1 := \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $v_2 := \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$v_1 = c_1 \cdot u_1 + c_2 \cdot u_2$

$v_2 = c_3 \cdot u_1 + c_4 \cdot u_2$

Which can be represented by the following matrix equation

$$\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 5 & 2 \end{pmatrix}$$

– I can't "see this", i.e., I don't see that it is relevant – but it may be – not obvious

This equation can be represented and solved using the following augmented matrix.

$$\begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 1 & 5 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 3 & 4 & 2 \\ 1 & -2 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 & 0 \\ 2 & 3 & 4 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 7 & 2 & 2 \end{pmatrix}$$

So c2=c4=2/7 and c1=1+2*(2/7)=11/7 and c3=2*(2/7)=4/7, therefore the transition matrix for B to B' is:

$$\begin{pmatrix} \frac{11}{7} & \frac{2}{7} \\ \frac{4}{7} & \frac{2}{7} \end{pmatrix}$$

Try $P_{B'B} = \begin{bmatrix} 4 & 5 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 2 & -5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -11 & 1 \\ 8 & -2 \end{bmatrix}$.

– probably not correct, but I can't follow your development. Hmm....

Question 6 (10 pts)

Consider the bases B={$\mathbf{u_1}$,$\mathbf{u_2}$,$\mathbf{u_3}$} and B'={$\mathbf{v_1}$,$\mathbf{v_2}$,$\mathbf{v_3}$} for R^3 – same type of problem as question 5.
Where

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \text{ and } \mathbf{v_1} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$$

Find the transition matrix from B to B'

Solution:
$v_1=3u_1+u_2+0u_3$
$v_2=1\ u_1+1/2\ u_2+0\ u_3$
$v_3=0+\ u_2+\ u_3$

– by inspection? Try something generally possible.

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1/2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 7 (10 points, 5 points each)
Are the following matrices orthogonal? Show your answer by showing the product of A^TA and by using the determinant.

a.

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$$

b.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Solution:

a. $A^TA = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}$. This is not equal to the identity matrix, therefore the transpose of A is not equal to the inverse of A and so the definition of orthogonality is not met.

   Det(A) = 2*(1)-2*(1) = 0. The determinant of A is not equal to 1 or -1, so the matrix is not orthogonal.

b. $A^TA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This is the identity matrix, so the transpose of A is equal to the inverse of A and the definition of orthogonality is satisfied. The matrix is orthogonal.

   Det(A) = (1)*(1) – 0= 1, therefore the matrix is orthogonal.

   necessary but not sufficient – e.g. $A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$.

Question 8 (10 points, 5 points)

Find the characteristic polynomials of the following matrices.

a.

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$$

b.

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 1 \\ -3 & 2 & 1 \end{pmatrix}$$

Solution:

Part A

$$\det(\lambda \cdot I - A) = 0$$

$$\det\begin{pmatrix} \lambda - 2 & 1 \\ 2 & \lambda - 1 \end{pmatrix} = 0$$

$$(\lambda - 2)\cdot(\lambda - 1) - 2 = 0$$

$$\lambda^2 - 3\cdot\lambda + 2 - 2 = 0$$

$$\lambda^2 - 3\cdot\lambda = 0$$

This is the characteristic ~~eigenvalue.~~ equation

Part B

$$\det(\lambda \cdot I - A) = 0$$

$$\det\begin{pmatrix} \lambda - 1 & -1 & 2 \\ 2 & \lambda + 2 & 1 \\ -3 & 2 & \lambda - 1 \end{pmatrix} = 0$$

$$(\lambda - 1)\cdot(\lambda + 2)(\lambda - 1) + 3 + 8 + 6\cdot(\lambda + 2) - 2\cdot(\lambda - 1) + 2\cdot(\lambda - 1) = 0$$

$$\left(\lambda^2 - 2\cdot\lambda + 1\right)\cdot(\lambda + 2) + 6\cdot(\lambda + 2) + 11 = 0$$

$$\left(\lambda^2 - 2\cdot\lambda + 7\right)\cdot(\lambda + 2) + 11 = 0$$

$$\left(\lambda^3 + 3\cdot\lambda + 14\right) + 11 = 0$$

$$\lambda^3 + 3\cdot\lambda + 25 = 0$$

This is the characteristic ~~polynomial~~ equation and can be solved for the eigenvalue.

Question 9 (10 pts)

Find a matrix P that Diagonalizes matrix A

$$A=\begin{bmatrix}2 & -2 & 14 & -37\\ 0 & 1 & 10 & -30\\ 0 & 0 & 3 & -6\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Solution:

First find the characteristic equation

det[λI-A]=0

$$\det\left(\begin{bmatrix}\lambda & 0 & 0 & 0\\ 0 & \lambda & 0 & 0\\ 0 & 0 & \lambda & 0\\ 0 & 0 & 0 & \lambda\end{bmatrix}-\begin{bmatrix}2 & -2 & 14 & -37\\ 0 & 1 & 10 & -30\\ 0 & 0 & 3 & -6\\ 0 & 0 & 0 & 1\end{bmatrix}\right)=0$$

$$\det\begin{bmatrix}\lambda-2 & 2 & -14 & 37\\ 0 & \lambda-1 & -10 & 30\\ 0 & 0 & \lambda-3 & 6\\ 0 & 0 & 0 & \lambda-1\end{bmatrix}=(\lambda-2)(\lambda-1)(\lambda-3)(\lambda-1)=0$$

λ=2,1,and 3

For λ=2

$$\text{null}\begin{bmatrix}0 & 2 & -14 & 37\\ 0 & 1 & -10 & 30\\ 0 & 0 & 3 & 6\\ 0 & 0 & 0 & 1\end{bmatrix}=\text{null}\begin{bmatrix}0 & 12 & -14 & 0\\ 0 & 1 & -10 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=$$

$$\text{null}\begin{bmatrix}0 & 12 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=\text{null}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$x_1=t$; $x_2=0$; $x_3=0$; $x_4=0$

$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda=1$

$$\text{null}\begin{bmatrix} -1 & 2 & -14 & 37 \\ 0 & 0 & -10 & 30 \\ 0 & 0 & -2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{null}\begin{bmatrix} -1 & 2 & -14 & 37 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{null}\begin{bmatrix} -1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} =$$

$$\text{null}\begin{bmatrix} 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1=2t-5s$; $x_2=t$; $x_3=-6s$; $x_4=s$

$\mathbf{p}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$; $\mathbf{p}_3 = \begin{bmatrix} -5 \\ 0 \\ 3 \\ 1 \end{bmatrix}$

For $\lambda=3$

$$\text{null}\begin{bmatrix} 1 & 2 & -14 & 37 \\ 0 & 2 & -10 & 30 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 2 & -14 & 0 \\ 0 & 2 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_1=4t$; $x_2=5t$; $x_3=t$; $x_4=0$

$\mathbf{p}_4 = \begin{bmatrix} 4 \\ 5 \\ 1 \\ 0 \end{bmatrix}$ $\qquad \mathbf{P} = \begin{bmatrix} 1 & 2 & 4 & -5 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Question 10 (14 pts)

Find an orthogonal matrix P that diagonalizes A and determine $P^{-1}AP$, given:

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Solution:

Step 1: Find the eigenvectors of A.

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\right) = \det\left[\begin{pmatrix} \lambda-3 & 1 \\ 1 & \lambda-3 \end{pmatrix}\right] = (\lambda-3)^2 - 1^2 = (\lambda-3-1)(\lambda-3+1)$$

$$(\lambda^2 - 6\lambda + 9) - 1 = \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2), \therefore \lambda = 4, 2$$

Step 2: Find the corresponding eigenvectors to these eigenvalues:

By definition,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is an eigenvector of A corresponding to λ if and only if **x** is a nontrivial solution of $(\lambda I - A)\mathbf{x} = \mathbf{0}$, in other words:

$$\begin{bmatrix} \lambda-3 & 1 \\ 1 & \lambda-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $\lambda = 4$:

$$\begin{bmatrix} 4-3 & 1 \\ 1 & 4-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow Nul\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = Nul\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix} \therefore \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$ Is an eigenvector for A when $\lambda = 4$ and we shall label it $\mathbf{u}_1$

Let $\lambda = 2$:

$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow Nul\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = Nul\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix} \therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ Is an eigenvector for A when $\lambda = 4$ and we shall label it $\mathbf{u}_2$

Step 3: Because the vectors lie under different eigenvalues and are already orthogonal to each other, we just need to normalize the vectors.

$$\|\mathbf{u}_1\| = \langle \mathbf{u}_1, \mathbf{u}_1 \rangle^{1/2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$\|\mathbf{u}_2\| = \langle \mathbf{u}_2, \mathbf{u}_2 \rangle^{1/2} = \sqrt{(1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$\mathbf{v}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\mathbf{v}2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Step 4: Form P from the $\mathbf{v}_1$and $\mathbf{v}_2$ using these vectors as the column vectors of P.

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Step 5: Solve for $P^{-1}AP$. Because P is an orthogonal matrix, $P^{-1} = P^T = P$. Therefore, $P^{-1}AP = P^TAP = PAP$.

$$PAP = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -4/\sqrt{2} & 4/\sqrt{2} \\ 2/\sqrt{2} & 2/\sqrt{2} \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

Problem 1

Given vectors **u** = (u_1, u_2 ,u_3, u_4) and **v** = (v_1, v_2, v_3, v_4), determine which of the following are inner products on R^4.

a) <**u**, **v**> = $u_1v_3 + u_2v_4$

b) <**u**, **v**> = $-u_1v_1+u_2v_2-2u_3v_3+3u_3v_3$ $= -u_1v_1+u_2v_2+u_3v_3$?

Solution:

a) NOT an inner product space.

Axiom 1: <u, v> = <v, u> This does not hold for (a), $u_1v_3 + u_2v_4$ is not equal to $v_1u_3 + v_2u_4$

Axiom 2: <u + v, z> = <u, z> + <v, z> This axiom does hold for (a).

Axiom 3: <ku, v> = k<u, v> This axiom does hold for (a).

Axiom 4: <v,v> ≥ 0 and <v,v> = 0 if and only if v=0. This does not hold, for example, if v=(1,1,0,0) <v, v> would be equal to 0 but v is not 0.

b) Inner product space

Axiom 1: <u, v> = <v, u> This axiom holds for (b).

Axiom 2: <u + v, z> = <u, z> + <v, z> This axiom holds for (b).

Axiom 3: <ku, v> = k<u, v> This axiom holds for (b).

Axiom 4: <v,v> ≥ 0 and <v,v> = 0 if and only if v=0. This axiom holds for (b). no way!

$\langle v,v\rangle = -v_1^2+v_2^2+v_3^2 < 0$ for $\underline{v} = (1,0,0)$.

Problem 2

Find the cosine of the angle θ between the vectors **u** = (1, 5, 2, 3) and **v** = (-2, 4, 7, -9), using the Euclidean inner product for R^4.

Solution:

The angle between two vectors in a vector space is defined as:

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

Therefore, since

$$\langle \mathbf{u}, \mathbf{v}\rangle = (1,5,2,3) \bullet (-2,4,7,-9) = 1(-2)+5(4)+2(7)+3(-9) = -2+20+14-27 = 5$$

and

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u}\rangle^{1/2} = \sqrt{1^2+5^2+2^2+3^2} = \sqrt{1+25+4+9} = \sqrt{39}$$

$$\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v}\rangle^{1/2} = \sqrt{(-2)^2+4^2+7^2+(-9)^2} = \sqrt{4+16+49+81} = \sqrt{150} = 5\sqrt{6}$$

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{5}{\sqrt{39}(5\sqrt{6})} = \frac{5}{15\sqrt{26}} = \frac{1}{3\sqrt{26}}$$

Problem 3

Consider the vector space R^3 with the Euclidean inner product. Transform the basis vectors $\underline{u}_1$ = (-2, 2, -4), $\underline{u}_2$ = (2, 0, 2), $\underline{u}_3$= (0, 1, 1) into an orthogonal basis {$\underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$}.

Solution:
We will use the Gram-Schmidt process to accomplish this transformation. Thus:

$\underline{v}_1 = \underline{u}_1 = (-2, 2, -4)$ – choose (-1,1,-2) or (1,-1,2)

$\underline{v}_2 = \underline{u}_2 - \text{proj}_{w1}\underline{u}_2 = (2, 0, 2) - \frac{-12}{24}(-2, 2, 4) = (1, 1, 4)$

$\underline{v}_3 = \underline{u}_3 - \text{proj}_{w2}\underline{u}_3 = (0, 1, 1) - \frac{-2}{24}(-2, 2, -4) - \frac{4}{18}(1, 1, 4) = (\frac{-7}{18}, \frac{17}{18}, \frac{4}{9})$

Thus :
$\underline{v}_1 = (-2, 2, -4)$
$\underline{v}_2 = (1, 1, 4)$
$\underline{v}_3 = (\frac{-7}{18}, \frac{17}{18}, \frac{4}{9})$ – choose (-7,17,8) or (7,-17,-8)

is an orthogonal basis for the vector space.

Problem 4

If possible, find the QR-Decomposition of the matrix: – good problem type.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Solution:
The column vectors of A are

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Applying the Gram-Schmidt process:

$$\mathbf{v}_1 = \mathbf{u}_1 = (1,0,1)$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (0,1,2) - \frac{2}{2}(1,0,1) = (-1,1,1)$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = (-1,1,2) - \frac{1}{2}(1,0,1) - \frac{4}{3}(-1,1,1) = (-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6})$$

– choose (-1,-2,1) or (1,2,-1)

Now we normalize the new vectors and get:

$$\mathbf{q}_1 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}), \quad \mathbf{q}_2 = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), \quad \mathbf{q}_3 = (-\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6})$$

Written another way,

$$\mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} -\sqrt{6}/6 \\ -\sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix}$$

So, $Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & -\sqrt{6}/6 \\ 0 & 1/\sqrt{3} & -\sqrt{6}/3 \\ 1/\sqrt{2} & 1/\sqrt{3} & \sqrt{6}/6 \end{bmatrix}$

good

Now, R is given by:

$$R = \begin{bmatrix} <\mathbf{u}_1,\mathbf{q}_1> & <\mathbf{u}_2,\mathbf{q}_1> & <\mathbf{u}_3,\mathbf{q}_1> \\ 0 & <\mathbf{u}_2,\mathbf{q}_2> & <\mathbf{u}_3,\mathbf{q}_2> \\ 0 & 0 & <\mathbf{u}_3,\mathbf{q}_3> \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{3} & 4/\sqrt{3} \\ 0 & 0 & \sqrt{6}/6 \end{bmatrix}$$

Thus the QR-decomposition of A = $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & -\sqrt{6}/6 \\ 0 & 1/\sqrt{3} & -\sqrt{6}/3 \\ 1/\sqrt{2} & 1/\sqrt{3} & \sqrt{6}/6 \end{bmatrix} \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{3} & 4/\sqrt{3} \\ 0 & 0 & \sqrt{6}/6 \end{bmatrix}$

Problem 5

Consider the bases B = {$\underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$} and B' = {$\underline{q}_1$, $\underline{q}_2$, $\underline{q}_3$} for R^3, where

$$\underline{v}_1 = \begin{bmatrix} -4 \\ 0 \\ -5 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \ \underline{q}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{q}_2 = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}, \ \underline{q}_3 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

Find the transition matrix $P_{B'B}$ from B to B' and compute the coordinate vector $[w]_{B'}$ of the vector w = $\begin{bmatrix} -3 \\ 2 \\ 9 \end{bmatrix}$ using the transition matrix.

this is a great explanation

Solution:
There is some matrix B and some matrix B' such that

$B(v)_B = B'(v)_{B'}$ = V

where $(v)_B$ and $(v)_{B'}$ represent the coordinate vectors with respect to B and B', respectively.

Thus to solve for the matrix P which transitions from B to B', we multiply the previous equation by the inverse of B'. Thus:

$(B')^{-1}B(v)_B = (B')^{-1}B'(v)_{B'} \Rightarrow (B')^{-1}B(v)_B = (v)_{B'}$

So, the matrix $(B')^{-1}B$ is the transition matrix for a coordinate vector from B to B'. Thus:

$(B')^{-1} \neq$

$$\left[\begin{array}{ccc|ccc} -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 5 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{-1}{2} & \frac{1}{5} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & \frac{-1}{5} \\ 0 & 1 & 0 & 0 & \frac{-1}{2} & \frac{1}{5} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \end{array}\right] \text{ so } B' = \begin{bmatrix} -1 & 0 & \frac{-1}{5} \\ 0 & \frac{-1}{2} & \frac{1}{5} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \text{ and}$$

$$(B')^{-1}B = \begin{bmatrix} -1 & 0 & \frac{-1}{5} \\ 0 & \frac{-1}{2} & \frac{1}{5} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} * \begin{bmatrix} -4 & 0 & 5 \\ 0 & 2 & 0 \\ -5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & \frac{-26}{5} \\ -1 & -1 & \frac{1}{5} \\ 0 & 1 & 0 \end{bmatrix}$$

Just row reduce $[B' \mid B]$ to $[I \mid P_{B'B}]$

The coordinate vector $[w]_B$ can be putting the following augmented matrix into reduced-row echelon form:

$$\begin{bmatrix} -4 & 0 & 5 & -3 \\ 0 & 2 & 0 & 2 \\ -5 & 0 & 1 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{-5}{4} & \frac{3}{4} \\ 0 & 2 & 0 & 2 \\ 0 & 0 & \frac{-21}{4} & \frac{-21}{4} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{-5}{4} & \frac{3}{4} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

so $[w]_B = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $[w]_{B'}$ is given by $P\,[w]_B = \begin{bmatrix} 5 & 0 & \frac{-26}{5} \\ -1 & -1 & \frac{1}{5} \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{24}{5} \\ \frac{-14}{5} \\ 1 \end{bmatrix}$

Checking is good.

We can check this observing that $B'[w]_{B'}$ is indeed w: $\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 2 \\ 0 & 5 & 5 \end{bmatrix} * \begin{bmatrix} \frac{24}{5} \\ -\frac{14}{5} \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 9 \end{bmatrix}$

Problem 6

Prove the following: If A is orthogonal, then det(A) = 1 *or* det(A) = -1.

Solution:
We know det(I) = 1.
We further know by definition that if A is orthogonal, $A^{-1} = A^T$ which means, $I = A^TA$.
So, $1 = \det(I) = \det(A^TA) = \det(A^T)\det(A)$. (Theorem 2.3.4)
By theorem, $\det(A^T) = \det(A)$ (Theorem 2.2.2)
Thus, $\det(A^T)\det(A) = (\det(A))^2$
Thus, $1 = (\det(A))^2$
$\sqrt{1} = \sqrt{(\det(A))2}$
$\det(A) = \pm 1$

Problem 7

Find the orthogonal projection of the vector **v** = (6, 1, 9, 4) on the subspace of R^4 spanned by the vectors :

$\mathbf{v}_1 = (1, 2, 0, 3)$, $\mathbf{v}_2 = (-3, 0, 3, 0)$, $\mathbf{v}_3 = (1, -1, -2, -2)$

Solution:
The subspace of R^4 spanned by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ (which we'll call W) is the same as Col A (the column space of A) where

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 0 & -1 \\ 0 & 3 & -2 \\ 3 & 0 & -2 \end{bmatrix}$$

great explanation

The question is asking us to find $\text{proj}_W \mathbf{v}$. We can find $\text{proj}_W \mathbf{v}$ using theorem 6.4.2, since $\text{proj}_W \mathbf{v} = A\mathbf{x}$ where $\mathbf{x}$ is the least squares solution of $A\mathbf{x} = \mathbf{v}$ and Col $A = W$. So, first we find the least squares solution of $A\mathbf{x} = \mathbf{v}$, which corresponds to the system below.

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & 0 & -1 \\ 0 & 3 & -2 \\ 3 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 9 \\ 4 \end{bmatrix}$$

~~Which means~~ that

$$A^T A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ -3 & 0 & 3 & 0 \\ 1 & -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 1 \\ 2 & 0 & -1 \\ 0 & 3 & -2 \\ 3 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 14 & -3 & -7 \\ -3 & 18 & -9 \\ -7 & -9 & 10 \end{bmatrix}$$

$$A^T \mathbf{v} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ -3 & 0 & 3 & 0 \\ 1 & -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ 9 \\ -21 \end{bmatrix}$$

We can use these to form a normal system of the form $A^T A\mathbf{x} = A^T\mathbf{v}$, which can also be expressed as an augmented matrix to give us the least squares solution of $\mathbf{x}$.

$$\begin{bmatrix} 14 & -3 & -7 \\ -3 & 18 & -9 \\ -7 & -9 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 9 \\ -21 \end{bmatrix}$$

$$\begin{bmatrix} 14 & -3 & -7 & 20 \\ -3 & 18 & -9 & 9 \\ -7 & -9 & 10 & -21 \end{bmatrix} \sim \begin{bmatrix} 7 & 9 & -10 & 21 \\ 21 & -126 & 63 & -63 \\ 14 & -3 & -7 & 20 \end{bmatrix} \sim \begin{bmatrix} 7 & 9 & -10 & 21 \\ 0 & -153 & 93 & -126 \\ 0 & -21 & 13 & -22 \end{bmatrix} \sim$$

$$\begin{bmatrix} 7 & 9 & -10 & 21 \\ 0 & 51 & -31 & 42 \\ 0 & 357 & -221 & 374 \end{bmatrix} \sim \begin{bmatrix} 7 & 9 & -10 & 21 \\ 0 & 51 & -31 & 42 \\ 0 & 0 & -4 & 80 \end{bmatrix} \sim \begin{bmatrix} 7 & 9 & 0 & -179 \\ 0 & 51 & 0 & -578 \\ 0 & 0 & 1 & -20 \end{bmatrix} \sim \begin{bmatrix} 7 & 0 & 0 & -77 \\ 0 & 3 & 0 & -34 \\ 0 & 0 & 1 & -20 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & -\frac{34}{3} \\ 0 & 0 & 1 & -20 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 \\ -\frac{34}{3} \\ -20 \end{bmatrix}$$

By Theorem 6.4.2, given $\mathbf{x}$ as the least squares solution of $A\mathbf{x} = \mathbf{v}$ and

$$\text{proj}_w \mathbf{v} = A\mathbf{x} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 0 & -1 \\ 0 & 3 & -2 \\ 3 & 0 & -2 \end{bmatrix} \begin{bmatrix} -11 \\ -\frac{34}{3} \\ -20 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 6 \\ 7 \end{bmatrix}$$

Or, using the original notation, $\text{proj}_w \mathbf{v} = (3, -2, 6, 7)$.

Problem 8:

$0 = (\lambda-1)(\lambda-2)+4$

$= \lambda^2 - 3\lambda + 6$

$\Rightarrow \lambda = \frac{3 \pm \sqrt{9-4\cdot 6}}{2} = \frac{3 \pm \sqrt{-15}}{2}$

$= \frac{3}{2} \pm i\frac{\sqrt{15}}{2}$

Find the eigenvalues of A^6 for $A = \begin{bmatrix} 1 & 2 & 9 & 5 \\ -2 & 2 & -10 & -9 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

Solution:

We can reduce A to an upper triangular matrix through the following elementary row operation:

"this block" has complex eigenvalues, which is relevant; these don't necessarily have same eigenvalues. — This is important to understand.

$$A=\begin{bmatrix}1 & 2 & 9 & 5\\ -2 & 2 & -10 & -9\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 7\end{bmatrix}\neq\begin{bmatrix}1 & 2 & 9 & 5\\ 0 & 6 & 8 & 1\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 7\end{bmatrix}.$$

By inspection, the eigenvalues will be 1, 6, 0, and 7.

To determine, the eigenvalues of A^6, we can use Theorem 7.1.3 to determine that the eigenvalues of $A^k = \lambda^k = 1^6, 6^6, 0^6, 7^6 = 1, 46656, 0, 117649$

Problem 9:

Find a matrix P (if it exists) that diagonalizes

$$A=\begin{bmatrix}2 & 0 & -2\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}$$

If P does exist, verify your answer by finding $P^{-1}AP$.

Solution:

First, we need to find the eigenvalues and bases for the eigenspaces for A.
The eigenvalues of A are the values of λ for which $\det(\lambda I - A) = 0$ (by theorem). From theorem 7.1.1, we know that since this matrix is upper triangular, its eigenvalues are the entries along its diagonal, namely:

$$\lambda = 2 \quad \text{or} \quad \lambda = 3$$

(If the matrix were not upper triangular, we would need to solve the characteristic equation for the matrix, i.e.: $\det(\lambda I - A) = 0$.)

Now for the eigenspaces. First we must find the eigenvectors for A. By definition, $\mathbf{x}$ is an eigenvector of A corresponding to λ iff $(\lambda I - A)\mathbf{x} = 0$ has a non-trivial solution for $\mathbf{x}$. In other words, we need to find the non-trivial solutions of

$$\begin{bmatrix}\lambda-2 & 0 & 2\\ 0 & \lambda-3 & 0\\ 0 & 0 & \lambda-3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

(x is a non-trivial solution of $(\lambda I - A)\mathbf{x} = 0$)

First, let's let $\lambda = 2$. We plug 2 in for λ and solve the resulting system.

$$\begin{bmatrix}0 & 0 & 2\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\Rightarrow\begin{bmatrix}0 & 0 & 2\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}\sim\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

Which yields $x_1 = t$, $x_2 = 0$, $x_3 = 0$. The eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors where

$$\mathbf{x}=\begin{bmatrix}t\\0\\0\end{bmatrix}=t\begin{bmatrix}1\\0\\0\end{bmatrix}$$ The vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda=2$.

Now let $\lambda = 3$.

$$\begin{bmatrix}1&0&2\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Which yields $x_1 = -2t$, $x_2 = s$, $x_3 = t$. The eigenvectors of A corresponding to $\lambda = 3$ are the nonzero vectors where

$$\mathbf{x}=\begin{bmatrix}-2t\\s\\t\end{bmatrix}=s\begin{bmatrix}0\\1\\0\end{bmatrix}+t\begin{bmatrix}-2\\0\\1\end{bmatrix}$$ The vectors $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-2\\0\\1\end{bmatrix}$ form a basis for the eigenspace corresponding to $\lambda = 3$ since the vectors are linearly independent.

Since there are 3 basis vectors total for the eigenspaces of A, we know by theorem 7.2.1 that A is diagonalizable. The column vectors for matrix P are the basis vectors for the eigenspaces of A:

$$P=\begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix}$$

Problem 10:

Find an orthonormal set of eigenvectors which span the eigenspaces of A, and determine if A is diagonalizable.

may not be possible since ✱ A is not symmetric.

$$A=\begin{bmatrix}1&-1&4\\-1&1&-2\\2&-2&-2\end{bmatrix}$$

Solution:

The characteristic equation of A is by definition

det(λI – A) = 0 where λ represents the eigenvalues of the matrix A. Thus:

$$\det(\lambda I - A)=\det\begin{bmatrix}\lambda-1&1&-4\\1&\lambda-1&2\\-2&2&\lambda+2\end{bmatrix}\sim\det\begin{bmatrix}\lambda-1&1&-4\\1&\lambda-1&2\\0&2\lambda&\lambda+6\end{bmatrix}\sim$$

Cofactor expansion along the first column yields:

$$\det\begin{bmatrix} \lambda-1 & 1 & -4 \\ 1 & \lambda-1 & 2 \\ 0 & 2\lambda & \lambda+6 \end{bmatrix} = (\lambda-1)\,[\,(\lambda-1)(\lambda+6)-(4\lambda)\,] - [(\lambda+6)+(8\lambda)\,]$$

We can expand and collect like terms :

$(\lambda-1)\,[\,(\lambda-1)(\lambda+6)-(4\lambda)\,] - [(\lambda+6)+(8\lambda)\,] = \lambda^3 - 16\lambda = \lambda(\lambda - 4)(\lambda + 4)$

So the eigenvalues associated with A are 0(~~trivial~~) 4 and -4. —nothing trivial about it

If $\lambda = 4$ then $(\lambda I - A)x=0$ for a nontrivial eigenvector x. thus

$$(\lambda I - A)x=0 \Rightarrow \begin{bmatrix} \lambda-1 & 1 & -4 \\ 1 & \lambda-1 & 2 \\ -2 & 2 & \lambda+2 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 1 & 3 & 2 \\ -2 & 2 & 6 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so we can solve this system using Gaussian elimination

$$\begin{bmatrix} 3 & 1 & -4 \\ 1 & 3 & 2 \\ -2 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -8 & -10 \\ 0 & 8 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{-11}{4} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 \end{bmatrix}$$ which is code for

$x_1 - \frac{11}{4}x_3 = 0$ and

$x_2 + \frac{5}{4}x_3 = 0$

so we can parameterize the system:

$x_3 = t$

$x_2 = \frac{-5}{4}t$

$x_1 = \frac{11}{4}t$

so that

$$x = \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} \frac{11}{4}t \\ \frac{-5}{4}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{11}{4} \\ \frac{-5}{4} \\ 1 \end{bmatrix} \text{ so } \begin{bmatrix} 11/4 \\ -5/4 \\ 1 \end{bmatrix}$$ is a basis for the eigenspace corresponding to $\lambda = 4$

—change to (11, -5, 4)

Similarly

If $\lambda = -4$ then $(\lambda I - A)x=0$ for a nontrivial eigenvector x. thus

$(\lambda I - A)x=0 \Rightarrow \begin{bmatrix} \lambda-1 & 1 & -4 \\ 1 & \lambda-1 & 2 \\ -2 & 2 & \lambda+2 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -5 & 1 & -4 \\ 1 & -5 & 2 \\ -2 & 2 & -2 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

so we can solve this system using Gaussian elimination:

$\begin{bmatrix} -5 & 1 & -4 \\ 1 & -5 & 2 \\ -2 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 2 \\ 0 & -24 & 6 \\ 0 & -8 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & \frac{-1}{4} \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & \frac{-1}{4} \\ 0 & 0 & 0 \end{bmatrix}$ which is code for

$x_1 + \frac{3}{4}x_3 = 0$ and

$x_2 + \frac{-1}{4}x_3 = 0$

so we can parameterize the system:

$x_3 = t$

$x_2 = \frac{1}{4}t$

$x_1 = \frac{-3}{4}t$

so that

$x = \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} \frac{-3}{4}t \\ \frac{1}{4}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{-3}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix}$ so $\begin{bmatrix} -3/4 \\ 1/4 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = -4$

/ change to (-3,1,4)

We can normalize both of these vectors to obtain orthonormal eigenvectors :

$\begin{bmatrix} 11/4 \\ -5/4 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 11\sqrt{8}/36 \\ -5\sqrt{8}/36 \\ \sqrt{8}/9 \end{bmatrix}$ and $\begin{bmatrix} -3/4 \\ 1/4 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -3\sqrt{8}/4\sqrt{13} \\ \sqrt{8}/4\sqrt{13} \\ \sqrt{8}/\sqrt{13} \end{bmatrix}$ so

not so!! see *

$(11,-5,4)\cdot(-3,1,4) = -33-5+16 \neq 0$

$$\begin{bmatrix} \frac{11\sqrt{8}}{36} \\ -\frac{5\sqrt{8}}{36} \\ \frac{\sqrt{8}}{9} \end{bmatrix} \text{ and } \begin{bmatrix} \frac{-3\sqrt{8}}{4\sqrt{13}} \\ \frac{\sqrt{8}}{4\sqrt{13}} \\ \frac{\sqrt{8}}{\sqrt{13}} \end{bmatrix}$$

are not an orthonormal set of eigenvectors which span the eigenspaces corresponding to the eigenvalues 4 and -4 of the Matrix A

Since we can see that there is only a set of two orthonormal eigenvectors (which we could have, and should have, seen earlier) there is, by theorem, NO matrix P which diagonalizes A.

Problem 1:

Determine which of the following are orthogonal with respect to their defined inner product.

transpose

a) <U,V>=tr(UTV), where

$$U=\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \quad V=\begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

$$\text{Let } U=\begin{bmatrix} 4 & 1 \\ -1 & 5 \end{bmatrix} \quad V=\begin{bmatrix} -5 & 0 \\ 0 & 4 \end{bmatrix}$$

Solution:

<U,V>=tr(U^TV)= u_1v_1+ u_3v_3+ u_2v_2+u_4v_4

=5•4+0•1+0•-1+4•5= -20 +20 =0

Thus, V is orthogonal to U.

b) Using the same inner product as in a,

$$\text{Let } U=\begin{bmatrix} 4 & 1 \\ -1 & 5 \end{bmatrix} \quad V=\begin{bmatrix} 2 & 1 \\ -19 & 2 \end{bmatrix}$$

Solution:

<U,V>=tr(U^TV)= u_1v_1+ u_3v_3+ u_2v_2+u_4v_4

2•4+1•1+-19•-1+2•5= 8+1+19+10=38≠0

Thus, V is **not orthogonal** to U with respect to the given inner product.

c) **<p,q>**=$a_0b_0+a_1b_1+a_2b_2+a_3b_3$, where

p= $a_0+a_1x+a_2x^2+a_3x^3$, and **q**= $b_0+b_1x+b_2x^2+b_3x^3$

Let **p**= $2+-x^2$, and **q**= $x+-x^3$

Solution:

<p,q>=2•0+0•1+-1•0+0•-1=0

Thus **p** is orthogonal to **q** with respect to the given inner product.

d) **<p,q>**=$\int_{-1}^{1} p(x)q(x)dx$

Let **p**=x^2+1 and **q**=x^3+x

Solution:

$$\int_{-1}^{1} (x^2+1)(x^3+x)dx=\int_{-1}^{1} x^5+2x^3+xdx = \frac{1}{6}x^6+\frac{1}{2}x^4+\frac{1}{2}x^2 \Big|_{-1}^{1}$$

$= (\frac{1}{6}+\frac{1}{2}+\frac{1}{2})-(\frac{1}{6}+\frac{1}{2}+\frac{1}{2})=0$

Thus, **p** is orthogonal to **q** with respect to the given inner product.

Problem 2

Determine if the following inner product is a real inner product space by:

a) First, listing the 4 axioms that need to be satisfied to determine an inner product space

b) Then, does the inner product defined by
$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_3 + u_2 v_2 + u_3 v_1$
satisfy the 4 inner product axioms? (Show all work for full credit)

Solution 2

a) 1) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmerty axiom]
2) $\langle \mathbf{u} + \mathbf{v}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{z} \rangle + \langle \mathbf{v}, \mathbf{z} \rangle$ [Additivity axiom]
3) $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$ [Homogeneity zxiom]
4) $\langle \mathbf{u}, \mathbf{v} \rangle \geq 0$ [Positivity axiom]
and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$
if and only if $\mathbf{v} = \mathbf{0}$

b) We will determine this by checking all the axioms one at a time
1) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$?
$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_3 + u_2 v_2 + u_3 v_1$ and $\langle \mathbf{v}, \mathbf{u} \rangle = v_1 u_3 + v_2 u_2 + v_3 u_1$
$\langle \mathbf{v}, \mathbf{u} \rangle$ can be rearranged to be $\langle \mathbf{v}, \mathbf{u} \rangle = u_1 v_3 + u_2 v_2 + u_3 v_1$
so Axiom 1 holds
2) $\langle \mathbf{u} + \mathbf{v}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{z} \rangle + \langle \mathbf{v}, \mathbf{z} \rangle$?

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{z} \rangle = (u_1 + v_1)z_3 + (u_2 + v_2)z_2 + (u_3 + v_3)z_1$$
$$= u_1 z_3 + v_1 z_3 + u_2 z_2 + v_2 z_2 + u_3 z_1 + v_3 z_1$$
$$\langle \mathbf{u}, \mathbf{z} \rangle + \langle \mathbf{v}, \mathbf{z} \rangle = (u_1 z_3 + u_2 z_2 + u_3 z_1) + (v_1 z_3 + v_2 z_2 + v_3 z_1)$$
$$= u_1 z_3 + v_1 z_3 + u_2 z_2 + v_2 z_2 + u_3 z_1 + v_3 z_1$$

so Axiom 2 holds
3) $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{v}, \mathbf{u} \rangle$?

$$\langle k\mathbf{u}, \mathbf{v} \rangle = ku_1 v_3 + ku_2 v_2 + ku_3 v_1 = k(u_1 v_3 + u_2 v_2 + u_3 v_1) = k\langle \mathbf{u}, \mathbf{v} \rangle$$

so Axiom 3 holds
4) We must check both parts of Axiom 4
$\langle \mathbf{u}, \mathbf{v} \rangle \geq 0$?

$$\langle \mathbf{v}, \mathbf{v} \rangle = v_1 v_3 + v_2 v_2 + v_3 v_1 = v_1 v_3 + (v_2)^2 + v_3 v_1$$

The $(v_2)^2$ term in this result must be positive but the $v_1 v_3$ and the $v_3 v_1$ need not necessarily be positive so the expression $\langle \mathbf{v}, \mathbf{v} \rangle = v_1 v_3 + (v_2)^2 + v_3 v_1$ could be negative and $\langle \mathbf{u}, \mathbf{v} \rangle \ngeq 0$
Also does $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$
This is clearly not the case because if $v_1 v_3$ and the $v_3 v_1$ can be negative, then $\langle \mathbf{v}, \mathbf{v} \rangle = v_1 v_3 + (v_2)^2 + v_3 v_1$ could equal zero if $(v_2)^2 = -(v_1 v_3 + v_1 v_3)$
so Axiom 4 fails on both counts

So this is not a real inner product space.

Problem 3

Given $S=\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1=(0,1,0)$ $\mathbf{v}_2=(-1,0,1)$ and $\mathbf{v}_3=(1,0,1)$

a) Find S′, where S′ is an orthonormal basis of S with the Euclidean inner product. In other words, "normalize" S. *since S is already orthogonal – say that*

Solution:

$$S'=\left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}\right\}$$

$$\|v_1\|=\sqrt{0^2+1^2+0^2}=1 \quad \|v_2\|=\sqrt{(-1)^2+0^2+1^2}=\sqrt{2}$$

$$\|v_3\|=\sqrt{1^2+0^2+1^2}=\sqrt{2}$$

$$S'=\left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}\right\}\{(0,1,0), (\frac{-1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})$$

post cedent?, yuk.

b) Find the coordinate vector of **u** relative to S′, $(u)_{S'}$. Then express the vector $\mathbf{u}=(2,3,1)$ as a linear combination of the vectors of the orthonormal basis found in part a.

Solution:

In order to express **u** as a linear combination of the vectors in S′, we need to find a coordinate vector relative to the basis S′. We can find a coordinate vector in either of two ways. The first way is to use the formula

$$(u)_{S'}=\langle\mathbf{u},\mathbf{v}_1\rangle,\langle\mathbf{u},\mathbf{v}_2\rangle,\langle\mathbf{u},\mathbf{v}_3\rangle$$

$$(u)_{S'}=(2\cdot0+3\cdot1+1\cdot0), (2\cdot\frac{-1}{\sqrt{2}}+3\cdot0+1\cdot\frac{1}{\sqrt{2}}), (2\cdot\frac{1}{\sqrt{2}}+3\cdot0+1\cdot\frac{1}{\sqrt{2}})$$

$$(u)_{S'}=(3, \frac{-1}{\sqrt{2}}, \frac{3}{\sqrt{2}})$$

To express **u** as a linear combination of the vectors in S′ we use this formula

$$u=\langle\mathbf{u},\mathbf{v}_1\rangle\mathbf{v}_1+\langle\mathbf{u},\mathbf{v}_2\rangle\mathbf{v}_2+\langle\mathbf{u},\mathbf{v}_3\rangle\mathbf{v}_3.$$

$$u=(2\cdot0+3\cdot1+1\cdot0)\mathbf{v}_1+(2\cdot-\frac{1}{\sqrt{2}}+3\cdot0+1\cdot\frac{1}{\sqrt{2}})\mathbf{v}_2+(2\cdot\frac{1}{\sqrt{2}}+3\cdot0+1\cdot\frac{1}{\sqrt{2}})\mathbf{v}_3$$

Thus $\mathbf{u}=3\mathbf{v}_1+\frac{-1}{\sqrt{2}}\mathbf{v}_2+\frac{3}{\sqrt{2}}\mathbf{v}_3$

The second is by writing

$k_1(0,1,0)+k_2(\frac{-1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})+k_3(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})=(2,3,1)$ and solving for each scalar.

To do this, we insert the vectors into the columns of a matrix:

$$\begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ 1 & 0 & 0 & 3 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}$$

Swap R1 with R2

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}$$

Add R2+R3 and replace R3 with the result

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ 0 & 0 & \frac{2}{\sqrt{2}} & 3 \end{bmatrix}$$

Multiply R2 by –2, and add to R3 replace R2 with the result.

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & \frac{2}{\sqrt{2}} & 0 & -1 \\ 0 & 0 & \frac{2}{\sqrt{2}} & 3 \end{bmatrix}$$

Multiply R2 and R3 by √2/2

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{2}{\sqrt{2}} & 3\frac{\sqrt{2}}{2} \end{bmatrix}$$

Thus $k_1=3$ $k_2= -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}}$ $k_3=\frac{3\sqrt{2}}{2} = \frac{3}{\sqrt{2}}$

$(u)\ S' = (3, -\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}) = (3, -\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}})$

We write **u** a linear combination of S′ as

u = 3v1 + -√2/2v2 + 3√2/2v3 or u = 3v1 + -1/√2v2 + 3/√2v3

$\mathbf{u} = 3\mathbf{v}_1 + -\frac{\sqrt{2}}{2}\mathbf{v}_2 + \frac{3\sqrt{2}}{2}\mathbf{v}_3$ **or** $\mathbf{u} = 3\mathbf{v}_1 + \frac{-1}{\sqrt{2}}\mathbf{v}_2 + \frac{3}{\sqrt{2}}\mathbf{v}_3$

Problem 4

Let W be the subspace of R^4 spanned by the vectors

$\mathbf{v}_1 = (1,4,5,2)$, $\mathbf{v}_2 = (2,1,3,0)$ and $\mathbf{v}_3 = (-1,3,2,2)$

a) Find a basis for the orthogonal complement of W

b) As a check, verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ and $\mathbf{v}_3$ are indeed orthogonal to the basis for the orthogonal complement of W

Solution 4

a) To solve this, we use Theorem 6.2.6 which states that the nullspace of A and the rowspace of A are orthogonal complements in R^n. Therefore, we will find the basis for the nullspace of A. We will make the the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ as the row space of the matrix A.

$\mathbf{v}_1 = (1,4,5,2)$, $\mathbf{v}_2 = (2,1,3,0)$ and $\mathbf{v}_3 = (-1,3,2,2)$

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

And the nullspace of A is the solution to the homogeneous system

$$\begin{aligned} x_1 + 4x_2 + 5x_3 + 2x_4 &= 0 \\ 2x_1 + x_2 + 3x_3 \qquad &= 0 \\ -x_1 + 3x_2 + 2x_3 + 2x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 3 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 7 & 7 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & 7 & 7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 7 & 28 & 35 & 14 & 0 \\ 0 & 28 & 28 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 7 & 0 & 7 & -2 & 0 \\ 0 & 28 & 28 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & -\frac{2}{7} & 0 \\ 0 & 1 & 1 & \frac{4}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t + \frac{2}{7}s \\ -t - \frac{4}{7}s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$$

The basis of the nullspace of A are $\boldsymbol{w}_1 = (-1, -1, 1, 0)$ and $\boldsymbol{w}_2 = \left(\frac{2}{7}, \frac{-4}{7}, 0, 1\right)$ – change to $(2, -4, 0, 7)$

And the nullspace of A and the rowspace of A are orthogonal complements thus ?

And the rowspace of A was the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ that spanned W

The basis of the orthogonal complement of W are

$\boldsymbol{w}_1 = (-1, -1, 1, 0)$ and $\boldsymbol{w}_2 = \left(\frac{2}{7}, \frac{-4}{7}, 0, 1\right)$

b) We can check that they are orthogonal by taking the euclidean inner product. An inner product of zero will mean that the vectors are orthogonal.

$\langle \mathbf{v}_1, \boldsymbol{w}_1 \rangle = (1, 4, 5, 2) \cdot (-1, -1, 1, 0) = -1 - 4 + 5 + 0 = 0$

$\langle \mathbf{v}_1, \boldsymbol{w}_2 \rangle = (1, 4, 5, 2) \cdot \left(\frac{2}{7}, \frac{-4}{7}, 0, 1\right) = \frac{2}{7} - \frac{16}{7} + 0 + \frac{14}{7} = 0$

$\langle \mathbf{v}_2, \boldsymbol{w}_1 \rangle = (2, 1, 3, 0) \cdot (-1, -1, 1, 0) = -2 - 1 + 3 + 0 = 0$

$\langle \mathbf{v}_1, \boldsymbol{w}_2 \rangle = (2, 1, 3, 0) \cdot \left(\frac{2}{7}, \frac{-4}{7}, 0, 1\right) = \frac{4}{7} - \frac{4}{7} + 0 + 0 = 0$

$\langle \mathbf{v}_3, \boldsymbol{w}_1 \rangle = (-1, 3, 2, 2) \cdot (-1, -1, 1, 0) = 1 - 3 + 2 + 0 = 0$

$\langle \mathbf{v}_3, \boldsymbol{w}_2 \rangle = (-1, 3, 2, 2) \cdot \left(\frac{2}{7}, \frac{-4}{7}, 0, 1\right) = -\frac{2}{7} - \frac{12}{7} + 0 + \frac{14}{7} = 0$

Problem 5
Gram-Schmidt Problem.

a.) Given the following basis $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ express the general form of Gram-Schmidt process for converting this basis into an othogonal basis in terms of the inner product.

b.) Let $\mathbb{R}^3$ have the inner product $<u,v> = \mathbf{u}_1\mathbf{v}_1 + 2\mathbf{u}_2\mathbf{v}_2 + 3\mathbf{u}_3\mathbf{v}_3$. Use the Gram-Schmidt process to transform $\{u_1,u_2,u_3\}$ into an othogonal basis.

$$\mathbf{u}_1 = (1,1,1) \qquad \mathbf{u}_2 = (1,1,0) \qquad \mathbf{u}_3 = (1,0,0)$$

c.) Using the othogonal basis from part b, transform the orthogonal basis into an orthonormal basis.

Solution 5:

a.) General Form

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$\mathbf{v}_2 = u_2 - \frac{< u_2, v_1 >}{\| v_1 \|^2} v_1$$

$$\mathbf{v}_3 = u_3 - \frac{< u_2, v_1 >}{\| v_1 \|^2} v_1 - \frac{< u_3, v_2 >}{\| v_2 \|^2} v_2$$

b.) Find orthogonal basis.

$$\mathbf{v}_1 = \mathbf{u}_1 = (1,1,1)$$

$$\mathbf{v}_2 = (1,1,0) - \frac{(1,1,0)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1) = (1,1,0) - \left(\frac{3}{6}, \frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right)$$

O.K.   change to $(1,1,-1)$

$$\mathbf{v}_3 = (1,0,0) - \frac{(1,0,0)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1) - \frac{(1,0,0)\cdot\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)}{\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)\cdot\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)}\cdot\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$$

$$= (1,0,0) - \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) - \left(\frac{1}{8}, \frac{1}{8}, -\frac{1}{2}\right) = \left(\frac{17}{24}, \frac{-7}{24}, \frac{-1}{24}\right)$$

$\mathbf{v}_1 = (1,1,1)$

$\mathbf{v}_2 = \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right)$

$\mathbf{v}_3 = \left(\frac{17}{24}, -\frac{7}{24}, -\frac{1}{24}\right)$ – change to $(17, -7, -1)$

c.) Find orthonormal basis. v_1

$$\|\mathbf{v}_1\| = \sqrt{(1,1,1)\cdot(1,1,1)} = \sqrt{1+1+1} = \sqrt{3}$$

length is $(\underline{v}_1 \cdot \underline{v}_1)^{1/2}$ $= (1\cdot 1 + 2\cdot 1\cdot 1 + 3\cdot 1\cdot 1)^{1/2} = \sqrt{6}$, – you have to be consistent between $\langle , \rangle$ and $\| \ \|$.

$$\|\mathbf{v}_2\| = \sqrt{\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)\cdot\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)} = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \frac{\sqrt{3}}{2}$$

$$\|\mathbf{v}_3\| = \sqrt{\left(\frac{17}{24}, -\frac{7}{24}, -\frac{1}{24}\right)\cdot\left(\frac{17}{24}, -\frac{7}{24}, -\frac{1}{24}\right)} = \sqrt{\frac{289}{576}+\frac{49}{576}+\frac{1}{576}} =$$
$$= \sqrt{\frac{113}{192}} = \frac{\sqrt{339}}{24}$$

$$\frac{\boldsymbol{v_1}}{\|\boldsymbol{v_1}\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\frac{\boldsymbol{v_2}}{\|\boldsymbol{v_2}\|} = \frac{\left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right)}{\frac{\sqrt{3}}{2}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

$$\frac{\boldsymbol{v_1}}{\|\boldsymbol{v_1}\|} = \frac{\left(\frac{17}{24}, -\frac{7}{24}, -\frac{1}{24}\right)}{\frac{\sqrt{339}}{24}} = \left(\frac{17}{\sqrt{339}}, -\frac{7}{\sqrt{339}}, -\frac{1}{\sqrt{339}}\right)$$

Problem 6

Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$ *given by*

$$\begin{aligned} 7x_1 - x_2 &= 5 \\ x_1 - 3x_2 &= -1 \\ -2x_1 + 4x_2 &= 3 \end{aligned}$$

and find the orthogonal projection of $\mathbf{b}$ *on the column space* A.

Solution 6

$$A = \begin{bmatrix} 7 & -1 \\ 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$$

Since A *has linearly independent column vectors, we know that there is a unique least squares solution.*

$$A^T A = \begin{bmatrix} 7 & 1 & -2 \\ -1 & -3 & 4 \end{bmatrix} \begin{bmatrix} 7 & -1 \\ 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 54 & -18 \\ -18 & 26 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 7 & 1 & -2 \\ -1 & -3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 28 \\ 10 \end{bmatrix}$$

Because $A^T A\mathbf{x} = A^T\mathbf{b}$,

$$\begin{bmatrix} 54 & -18 \\ -18 & 26 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 28 \\ 10 \end{bmatrix}. \quad (A^T A)^{-1}(A^T A)\mathbf{x} = (A^T A)^{-1} A^T\mathbf{b}$$

$$\mathbf{x_1} = \frac{\mathbf{227}}{\mathbf{270}} \quad \mathbf{x_2} = \frac{\mathbf{29}}{\mathbf{30}}$$

— re-engineer to get integer solutions.

proj$_w$ $\mathbf{b}$, *when* W *is the column space of* A, *is* $A\mathbf{x}$

$$A\mathbf{x} = \begin{bmatrix} 7 & -1 \\ 1 & -3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \frac{227}{270} \\ \frac{29}{30} \end{bmatrix} = \begin{bmatrix} \frac{664}{135} \\ -\frac{278}{135} \\ \frac{59}{27} \end{bmatrix}$$

Problem 7 *If P is the transition matrix from a basis B' to a basis B, and Q is the transition matrix from B to a basis C, what is the transition matrix from B' to C? What is the transition matrix from C to B'?*

Solution 7

Let bases $B = \{u_1, u_2, u_3\}$, $B' = \{v_1, v_2, v_3\}$, $C = \{w_1, w_2, w_3\}$, so that

$[u_1 \mid u_2 \mid u_3](u)_B = [v_1 \mid v_2 \mid v_3](v)_{B'}$. Then

$u = [u_1 \mid u_2 \mid u_3]^{-1}[v_1 \mid v_2 \mid v_3](v)_{B'}$ *Therefore* $P = [u_1 \mid u_2 \mid u_3]^{-1}[v_1 \mid v_2 \mid v_3]$.

~~*and*~~ Similarly

$[w_1 \mid w_2 \mid w_3](w)_C = [u_1 \mid u_2 \mid u_3](u)_B$

$w_C = [w_1 \mid w_2 \mid w_3]^{-1}[u_1 \mid u_2 \mid u_3](u)_B$, $Q = [w_1 \mid w_2 \mid w_3]^{-1}[u_1 \mid u_2 \mid u_3]$

The transition matrix from B' to C $= [w_1 \mid w_2 \mid w_3]^{-1}[v_1 \mid v_2 \mid v_3] = QP$

$\left(\textit{Because } QP = [w_1 \mid w_2 \mid w_3]^{-1}[u_1 \mid u_2 \mid u_3][u_1 \mid u_2 \mid u_3]^{-1}[v_1 \mid v_2 \mid v_3]\right)$

The transition matrix from C to B' $= (QP)^{-1}$

by the theorem 6.5.1 (*p.*344).

Problem 8
Orhtogonal Matrix Problem.

What must the values of **c** be so that the following matrix is orthogonal?

$$\begin{bmatrix} \frac{(1+c)}{2} & \frac{c}{2} \\ \frac{c}{2} & \frac{(1+c)}{2} \end{bmatrix}$$

Solution 8

By Theorem 6.2.2 we know that if A is orthogonal, then det(A) =1 or det(A) = -1.

$$\text{Therefore det}\left(\begin{bmatrix} \frac{(1+c)}{2} & \frac{c}{2} \\ \frac{c}{2} & \frac{(1+c)}{2} \end{bmatrix}\right) = 1 \text{ or } -1$$

$$\det = \left(\frac{(1+c)}{2}\frac{(1+c)}{2}\right) - \left(\frac{c}{2}\frac{c}{2}\right) = 1 \text{ or } -1$$

$$\det = \left(\frac{1}{4} + \frac{1}{2}c + c^2 - c^2\right) = 1 \text{ or } -1$$

$$\det = \left(\frac{1}{4} + \frac{1}{2}c\right) = 1 \text{ or } -1$$

$$\frac{1}{2}c = \frac{3}{4} \qquad c = \frac{3}{2} \qquad \text{or} \qquad \frac{1}{2}c = -\frac{5}{4} \qquad c = -\frac{5}{2}$$

$$c = \frac{3}{2}, -\frac{5}{2}$$

not generally sufficient,
but might be here.
At any rate, this
is dangerous thinking.

Problem 9

a) Find the eigenvalues of $\begin{bmatrix} -1 & -4 \\ 2 & 5 \end{bmatrix}$

b) Find the right eigenvector

c) Find the left eigenvector

Solution 9

a) Eigenvalues (λ) can be obtained by the expression

det [λI-A] = 0

$$\det\begin{bmatrix} \lambda+1 & -4 \\ 2 & \lambda-5 \end{bmatrix} = 0 \Rightarrow (\lambda+1)(\lambda-5)+8=0 \Rightarrow \lambda^2 - 4\lambda + 3 = 0 \Rightarrow$$

$$(\lambda-1)(\lambda-3)=0 \qquad \lambda=1 \ \ and \ \ \lambda=3$$

check your work

b) We find the right eigenvector(s) **x** by realizing that $x \in Nul(\lambda I - A)$

$$Nul(\lambda I - A)\Big|_{\lambda=1} = Nul\begin{bmatrix} \lambda+1 & +4 \\ -2 & \lambda-5 \end{bmatrix}\Big|_{\lambda=1} = Nul\begin{bmatrix} 2 & +4 \\ -2 & -4 \end{bmatrix} = Nul\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} = span\left\{\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)\right\}$$

$$Nul(\lambda I - A)\Big|_{\lambda=3} = Nul\begin{bmatrix} \lambda+1 & +4 \\ -2 & \lambda-5 \end{bmatrix}\Big|_{\lambda=3} = Nul\begin{bmatrix} 4 & +4 \\ -2 & -2 \end{bmatrix} = Nul\begin{bmatrix} 1 & +1 \\ 0 & 0 \end{bmatrix} = span\left\{\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right\}$$ So

right eigenvectors are $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ when λ=1 or $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ when λ=3or any scalar multiple of these vectors

c) We find the left eigenvector(s) **y** by realizing that $y \in Nul(\lambda I - A^T)$

$$Nul(\lambda I - A^T)\Big|_{\lambda=1} = Nul\begin{bmatrix} \lambda+1 & -2 \\ +4 & \lambda-5 \end{bmatrix}\Big|_{\lambda=1} = Nul\begin{bmatrix} 2 & -2 \\ +4 & -4 \end{bmatrix} = Nul\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = span\left\{\left(\begin{bmatrix} 1 \\ +1 \end{bmatrix}\right)\right\}$$

$$Nul(\lambda I - A^T)\Big|_{\lambda=3} = Nul\begin{bmatrix} \lambda+1 & -2 \\ +4 & \lambda-5 \end{bmatrix}\Big|_{\lambda=3} = Nul\begin{bmatrix} 4 & -2 \\ +4 & -2 \end{bmatrix} = Nul\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = span\left\{\left(\begin{bmatrix} 1 \\ +2 \end{bmatrix}\right)\right\}$$

So left eigenvectors are $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ when λ=1or $\mathbf{y} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ when λ=3 or any scalar multiple of these vectors

Problem 1 (18 points total)

a) 4 points
b) 4 points
c) 5 points
d) 5 points

Problem 2 (15 points total)

a) 5 points
b) 10 points (2 points for each Axiom check 1-3 and 4 points for Axiom 4 check)

Problem 3 (15 points total)

a) 2 points
b) 12 points

Problem 4 (15 points total)

a) 10 points
b) 5 points

Problem 5 (15 points total)

a) 3 points
b) 6 points
c) 6 points

Problem 6 (15 points total)

(If final answer is wrong, partial points can be awarded for correct steps)

Problem 7 (15 points total)

(If final answer is wrong, partial points can be awarded for correct steps)

Problem 8 (15 points total)

(If final answer is wrong, partial points can be awarded for correct steps)

Problem 9 (15 points total)

a) 5 points
b) 5 points
c) 6 points

Problem 10 (12 points total)

a) 5 points
b) 5 points
c) 2 points