Exam Key Midterm 2 Fall Semester 2005

Question 1:
Determine if the following transformations are one to one:

A. The transformation defined by:

$$w_1 = x_1 + 2x_2$$
$$w_2 = 2x_1 + 5x_2$$

B. The transformation: $T = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Solution 1:

According to Theorem 4.3.1 the follow two statements are interchangeable:
T is one-to-one.
T is invertible.
det (T) does not equal zero.

A. Yes, the transformation is one to one because T is invertible

$$\Rightarrow \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

B. No, the transformation is not one to one because det (T) equals zero, making it not invertible.

Question 2:
Determine if the following are subspaces of R^4:

a) all vectors of the form (a,b,0,0)
b) all vectors of the form (a,b,c,d) when a + b + c + d = 0
c) all matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $d = b\text{-}a$ and $c = a\text{-}b$

Solution 2:

Any subspace S must satisfy the following two requirements:
(a) For any **u** and **v** within S, **u**+**v** must also exist within S.
(b) For any **u** within S and k any real number scalar, k**u** must also exist within S.

a) Yes, because
(a) (a,b,0,0)+(a',b',0,0)=(a+a',b+b',0,0), which exists in S
(b) k(a,b,0,0) = (ka,kb,0,0), which exists in S

b) Yes, because
(a) (a,b,c,d)+(a',b',c',d') = (a+a',b+b',c+c',d+d'), which exists in S - why?
(b) k(a,b,c,d) = (ka,kb,kc,kd) $\neq$ k(0) = 0 ?

c) No, because a matrix is not a subspace of R^4 tricky!

Question 3:
Find the dimension of and a basis for the solution space for the system:

$$0 = x_1 + x_2 + 4x_3$$
$$0 = 2x_2 - 1x_3 + x_4$$

Solution 3:

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{9}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \begin{matrix} x_1 = -\frac{9}{2}x_3 - \frac{1}{2}x_4 \\ x_2 = \frac{1}{2}x_3 - \frac{1}{2}x_4 \end{matrix} \Rightarrow \cancel{x} = \begin{bmatrix} -\frac{9}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} x_4$$

$$\not\Rightarrow \left\{ \begin{matrix} v_1 = (-\frac{9}{2}, \frac{1}{2}, 1, 0), \\ v_2 = (-\frac{1}{2}, -\frac{1}{2}, 0, 1) \end{matrix} \right\}$$ is the basis.

Dim(S) = 2 .

Question 4:
Are the vectors $u = (4,2,1)$ and $v = (-1,3,-2)$ orthogonal?

Solution 4: If the Euclidian inner product of two vectors in R^n is equal to zero the vectors are orthogonal. Since $u \bullet v = \cancel{0}(4,2,1) \bullet (-1,3,-2) = 0)$ u and v are orthogonal.
O.K.

Question 5:
Use the Wronskian to show that the sets of vectors $f_1 = 1$, $f_2 = x$, $f_3 = x^2e^x$ are linearly independent.

"a bunch of things"

Solution 5: The Wronskian is formed by finding ~~f''~~ and then finding the determinant of the matrix that is formed.

$$W(x) = \begin{vmatrix} 1 & x & x^2e^x \\ 0 & 1 & x^2e^x + 2xe^x \\ 0 & 0 & x^2e^x + 2xe^x + 2xe^x + 2e^x \end{vmatrix} = x^2e^x + 2xe^x + 2xe^x + 2e^x$$. Since the Wronskian is not equal to zero somewhere the set of vectors are linearly independent.

Question 6:
Is v a subspace of R^3 when $v = \{(a,b,c)$ ~~and~~ $\mid b = a + c\}$?

Solution 6: v is a subspace because v is closed under addition and scalar multiplication.

Closed under addition: if $v = (a, a+c, c)$ and $u = (d, d+f, f)$ then $u + v$ $= (a+d, a+c+d+f, c+f)$ which follows the form (a,b,c) when $b = a + c$, therefore v is ~~closed under addition.~~ make more obvious by rearranging

Closed under scalar multiplication: if $v = (a, a+c, c)$ then $kv =$ $(ka, ka+kc, kc)$ which is of the form (a,b,c) when $b = a + c$, therefore v is closed under scalar multiplication. o.k.

Question 7:
Find the standard matrix for stated composition of operators on R^3, a reflection about the x-y plane followed by a rotation of 30° about the x-axis followed by an orthogonal projection to the x-z plane.

Solution 7:
The standard matrices that make up this composition are as follows:

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & \sin 30 & \cos 30 \end{bmatrix}, T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The linear transformation can be expressed as the composition

$T = T_3 \circ T_2 \circ T_1 \overset{?}{=} [T_3]\ [T_2]\ [T_1]$,

$$\text{Thus, } T \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & \sin 30 & \cos 30 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & -\sin 30 & -\cos 30 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\sin 30 \\ 0 & 0 & -\cos 30 \end{bmatrix}.$$

Question 8:

Find the standard matrix for the operator T: $R^3 \rightarrow R^3$; given by

$\mathbf{w}_1 = x_1 + 2x_2 + x_3$

$\mathbf{w}_2 = x_2 + x_3$

$\mathbf{w}_2 = x_3$

Then calculate T(-1, 2, 4) by matrix multiplication and check by substituting back in to the original equations.

what is this? these are not vectors

Solution 8:

? ?

$T(-1,2,4) = (k_1\ \mathbf{w}_1,\ k_2\,\mathbf{w}_2,\ k_3\,\mathbf{w}_2)$ which has a system of equations

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} -1+4+4 \\ 2+8 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 4 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Check:

$\mathbf{w}_1 = -1 + (2(2)) + 4 = 7$

$\mathbf{w}_2 = 2 + 8 = 10$

$\mathbf{w}_3 = 4$

Question 9:

The vectors **u**=(2, 4, -2), **v**=(3,2,1) are in R^3. Show that **w**=(9,2,7) is a linear combination of **u** and **v**.

Solution 9: For **w** to be a linear combination of **u** and **v** there must be scalars k_1 and k_2 such that **w**= k_1**u**+ k_2**v**.

That is, $(9,2,7)= k_1(2,4,-2)+ k_2(3,2,1)$

$$\begin{aligned} 2k_1+3k_2&=9 \\ 4k_1+2k_2&=2 \\ -2k_1+k_2&=7 \end{aligned} \Rightarrow \begin{bmatrix} 2 & 3 & 9 \\ 4 & 2 & 2 \\ -2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 9 \\ 0 & -4 & -16 \\ 0 & 4 & 16 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 9 \\ 0 & -4 & -16 \\ 0 & 0 & 0 \end{bmatrix}$$

which yields; $k_2=4$ and $k_1=-3/2$. Thus, the linear combination of **u** and **v** that make up **w** is **w**=(-3/2)**u**+4**v**.

Question 10:

Show that if **v** is a nonzero vector in R^n, then $(1/\|\mathbf{v}\|)\mathbf{v}$ has Euclidean norm 1.

Solution 10:

To find the Euclidean norm, we simply ~~solve~~ find $\|(1/\|\mathbf{v}\|)\mathbf{v}\|$. $(1/\|\mathbf{v}\|)$ is a scalar. By theorem 4.1.4.c, we know that for a scalar k and a vector **v** which is in R^n, then $\|k\mathbf{v}\|=|k|\,\|\mathbf{v}\|$. So, $\|(1/\|\mathbf{v}\|)\mathbf{v}\|=|(1/\|\mathbf{v}\|)|\,\|\mathbf{v}\|$, ~~where $(1/\|\mathbf{v}\|)=k$,~~ which equals 1. antecedent?

Section 4.1 (21 pts)

1. (9 pts)
U = (1,4,3)   V=(0,2,7)   W=(2,4,3)
Calculate the following: (3pts Each)

a) 2U · 3W
b) ||u+v||
c) (3u – 4w) · 2V

<u>Answers</u>
a) 2U · 3W
=2(1,4,3) · 3(2,4,3)
=(2,8,6) ·(6,12,9)
=2(6) + 8(12) + 6(9)
= 12+96+54= **<u>162</u>**
b) ||u+v||
=||(1,4,3)+(0,2,7)||
=||(1,6,10)||
$= \sqrt{1^2 + 6^2 + 10^2}$
$= \sqrt{137}$
c)(3u - 4w) · 2V
= (3(1,4,3) – 4(2,4,3)) · 2(0,2,7)
= ((3,12,9) – (8,16,12)) · (0,4,14)
=(-5,-4,-3) · (0,4,14)
=0(-5) + -4(4) + -3(14)
= 0 – 16 – 42 = **<u>-58</u>**

2. (12 Pts)
Show that the following are True (3pts Each)

a) If $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$, then **u** and **v** are orthogonal.
b) If **u** is orthogonal to **v** and **w**, then **u** is orthogonal to **v + w.**
c) If $||\mathbf{u}-\mathbf{v}||^2 = 0$, then **u = v.**
d) If $||k\mathbf{u}|| = k||\mathbf{u}||$, then $k \geq 0$.

<u>Answers</u>
a) If $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$, then **u** and **v** are orthogonal.

$(u_1 + v_1)^2 + \ldots.(u_n + v_n)^2 = u_1^{\ 2} + \ldots u_n^{\ 2} + v_1^{\ 2} + \ldots v_n^{\ 2}$

$= u_1^{\ 2} + 2u_1v_1 + v_1^{\ 2} + \ldots \cancel{+}u_n^{\ 2} + 2u_nv_n + v_n^{\ 2} = u_1^{\ 2} + v_1^{\ 2} + \ldots..u_n^{\ 2} + v_n^{\ 2}$

$= 2u_1v_1 + \ldots..2u_nv_n = 0$

$= u_1v_1 + \ldots..u_nv_n = 0$

True!

b) If **u** is orthogonal to **v** and **w,** then **u** is orthogonal to **v** + **w.**

$u_1v_1 + \dots u_nv_n = 0$

$u_1w_1 + \dots u_nw_n = 0$

$u_1(v_1 + w_1) + \dots u_n(v_n + w_n) = \cancel{0}$ - don't assume result - show it,

$= u_1v_1 + u_1w_1 + \dots u_nv_n + u_nw_n = \cancel{0}$

$= (u_1v_1 + \dots + u_n v_n) + (u_1w_1 + \dots + u_nw_n) \cancel{=0}$ $= 0 + 0 = 0$

(Sub) $u_1v_1 + \dots u_nv_n = 0$

(Sub) $u_1w_1 + \dots u_nw_n = 0$

0+0=0

By using our premise and Substitution this Validates the statement.

True!

c) If $||\mathbf{u}\text{-}\mathbf{v}||^2 = 0$, then **u = v.**

$\sqrt{(u_1 - v_1)^2 + \dots (u_n - v_n)^2} = 0$

This will always be 0 or Positive,

In order to be true, it must be 0

Thus $u_1 - v_1 = 0$

So $u_1 = v_1$ other components?

d) If $||k\mathbf{u}|| = k||\mathbf{u}||$, then $k \geq 0$.

$\sqrt{k^2u_1^2 + \dots k^2u_n^2}$

$= \sqrt{k^2(u_1^2 + \dots u_n^2)}$

$= k\sqrt{(u_1^2 + \dots u_n^2)}$?

=k||u||

Ture

Assuming result? At any rate, somewhat trivial.

4.2 Linear Tansformations from R^n to R^m

1) $T(X_1,X_2) = (X_1+X_2, 4X_1X_2, X_1^3-X_2)$ (2 Points each)

a. (2,3) b. (1,5) c. (0,1)

what is the question?

Solution

a. $(2,3) = (2+3, 4(2)(3), 2^3-3) = (4,24,5)$

b. $(1,5) = (1+5, 4(1)(5), 1^3-5) = (6,20,-4)$

c. $(0,1) = (0+1, 4(0)(1), 0^3-1) = (1,0,-1)$

2) Find the standard matrix for the linear transformation T defined by the formula. (2 Points each)

a. $T(X_1,X_2, X_3,X_4) = (7X_1 + 2X_2 - X_3 + X_4, X_2 + X_3, -X_1)$

b. $T(X_1,X_2, X_3) = (2X_2 - X_3, -X_1 + 2X_2, 3X_1-2X_3)$

Solution

a. $\begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 2 & -1 \\ -1 & 2 & 0 \\ 3 & 0 & -2 \end{bmatrix}$

3) (2 Points each) Use matrix multiplication to find the reflection of

a. (-1,2) about the y-axis b. (2,-5,3) about the xy-plane

Solution

a. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow (-1,-2)$ b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow (2,-5,-3)$

where's the matrix multiplication?
At any rate, if you know how to obtain matrix, you know how to obtain image w/o it.

4) Find the standard matrix for the stated composition of linear operators on R^3.
(3 Points each)

a. A reflection about the xy-plane, followed by a reflection about the xz-plane, followed by an orthogonal projection on the yz-plane.

b. A rotation of 30° about the x-axis, followed by a rotation of 30° about the z-axis, followed by a contraction by a factor of $K=\frac{1}{4}$

<u>Solution</u>

a. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$?

b. $\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{8} & -\frac{1}{2} & 0 \\ \frac{1}{8} & \frac{\sqrt{3}}{8} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$?, ~~Missing a factor?~~

$\begin{bmatrix} \frac{\sqrt{3}}{8} & -\frac{1}{2} & 0 \\ \frac{1}{8} & \frac{\sqrt{3}}{8} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{16} & \frac{1}{16} \\ \frac{1}{8} & \frac{3}{16} & -\frac{3}{16} \\ 0 & \frac{1}{8} & \frac{\sqrt{3}}{8} \end{bmatrix}$

disorganized. Also, show sine's and cossines so people know what you're doing.

(4.3)

matrices are not in $\mathbb{R}^n$,

1. What makes a $T:R^n \rightarrow R^m$ linear? (5 points)

A: $T:R^n \rightarrow R^m$ is linear if and only if any two matrices in R^n have the following to be true:

a. $T(\underline{\mathbf{u}} + \underline{\mathbf{v}}) = T(\underline{\mathbf{u}}) + T(\underline{\mathbf{v}})$
b. $T(c\underline{\mathbf{u}}) = cT(\underline{\mathbf{u}})$

2. Find the inverse operators of the following one-to-one linear operators. (15 points)

a. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b. $\begin{aligned} w_1 &= x_1 + x_3 \\ w_2 &= x_2 \\ w_3 &= x_3 \end{aligned}$

c. $\begin{aligned} w_1 &= 2x_1 + x_2 + 3x_3 \\ w_2 &= x_1 + 2x_2 + x_3 \\ w_3 &= x_2 \end{aligned}$

Answers for Question 2

Find the inverse of the matrices T^{-1}.

a. the inverse of this matrix is the matrix again- $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R1-R3 \rightarrow R1} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

b.

$$T^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c.

$$\begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{2R3-R1} \begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & -1 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{R3-3R2} \begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow{R1-3R3} \begin{bmatrix} 2 & 1 & 0 & -2 & 6 & -3 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 2 & -1 \end{bmatrix} \xrightarrow{R1-R2} \begin{bmatrix} 2 & 0 & 0 & -2 & 6 & -4 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow{1/2R1} \xrightarrow{-R3} \begin{bmatrix} 1 & 0 & 0 & -1 & 3 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} -1 & 3 & -2 \\ 0 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

5.1 (9pts)

List 3 of the 10 axioms of Vector Space (3pts Each)

1. If **u** and **v** are objects in V, then **u + v** is in V.
2. **u + v = v + u**
3. **u + (v + w) = (u + v) + w**
4. There is an Object **0** in V, called a zero vector for V, such that **0 + u = u + 0 = u**
5. For each **u** in V, there is an object –**u** in V, called a negative of **u,** such that **u + (-u) = (-u) + u = 0.**
6. If k is any scalar and **u** is any object in V, then k**u** is in V.
7. k(**u+v**) = ku+k**v**
8. $(k + m)$**u** = k**u** + m**u**
9. k(m**u**) = (km)**u**
10. 1**u=u**

5.2 Subspaces

1) Determine whether the solution space of the system Ax=0 is a line through the origin, a plane through the origin, or the origin only. If it is a plane, find an equation for it. If it is a line, find parametric equations for it. (2 points each)

a. $A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix}$ b. $A = \begin{bmatrix} 1 & 2 & -6 \\ 1 & 4 & 4 \\ 3 & 10 & 6 \end{bmatrix}$ c. $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{bmatrix}$

<u>Solution</u>

a.

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0$$

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & -2 & -3 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0$$

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0$$

Line

$\mathbf{X_3 = t}$

$X_2 + \frac{3}{2} X_3 = 0$

$\mathbf{X_2 = -\frac{3}{2} t}$

$-X_1 - \frac{3}{2} t + t = 0$

$\mathbf{X_1 = -\frac{1}{2} t}$

b.

$$A = \begin{bmatrix} 1 & 2 & -6 \\ 1 & 4 & 4 \\ 3 & 10 & 6 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0$$

$$A = \begin{bmatrix} 1 & 2 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0$$

Origin Only

c.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0$$

Line

$X_1 = -2t + t$

$\mathbf{X_1 = -t}$

$\mathbf{X_2 = -2t}$

$\mathbf{X_3 = t}$

2) Express the following as linear combinations of u = (3,2,5), v = (1,4,9), and w = (3,1,4). (3 points each)

a. (6,2,5) b. (2,3,5)

- can do both at same time - saves time.

Solution

a. $\begin{bmatrix} 3 & 1 & 3 & 6 \\ 2 & 4 & 1 & 2 \\ 5 & 9 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{11}{4} \\ 0 & 1 & 0 & \frac{-3}{4} \\ 0 & 0 & 1 & \frac{-1}{2} \end{bmatrix}$

b. $\begin{bmatrix} 3 & 1 & 3 & 2 \\ 2 & 4 & 1 & 3 \\ 5 & 9 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{7}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{-5}{3} \end{bmatrix}$

11/4(3,2,5) – ¾(1,4,9) – 1/2(3,1,4) = (6,2,5)

(2,3,5) = 7/3(3,2,5) + 0(1,4,9) – 5/3(3,1,4)

3) Determine whether the given vectors span R^3. (2 points each)

a. $V_1 = (2,-1,3)$ $V_2 = (4,1,2)$ $V_3 = (8,-1,8)$

b. $V_1 = (6,2,1)$ $V_2 = (3,4,2)$ $V_3 = (9,2,6)$

c. $V_1 = (1,1,1)$ $V_2 = (3,2,5)$ $V_3 = (6,2,4)$

d. $V_1 = (3,2,1)$ $V_2 = (0,0,0)$ $V_3 = (1,2,3)$

Solution

using some "equivalent statements"

a. $\begin{bmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{bmatrix}$ det = 16 – 12 – 16 – 24 + 32 + 4 = 0 → no span

b. $\begin{bmatrix} 6 & 3 & 9 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$ det = 144 + 6 + 36 - 36 - 36 - 24 = 90 $\neq 0$ → spans

c. $\begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 2 \\ 1 & 5 & 4 \end{bmatrix}$ det = 8 + 6 + 30 – 12 – 10 – 12 = 10 $\neq 0$ → spans

d. $\begin{bmatrix} 3 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ det = 0 + 0 + 0 – 0 – 0 – 0 = 0 → no span

(5.3)

1. Determine whether the following sets are linearly independent or dependant and show why. (10pts.)

a. (0,1,0,1), (2,3,1,0), (3,2,1,4) in R^4.

b. (1,2), (5,3), (4,8) in R^2.

c. (1,2,3), (2,5,7), (3,4,2), in R^3.

d. $\mathbf{p_1}=2-x+2x^2$, $\mathbf{p_2}=4+x-3x^2$, $\mathbf{p_3}=1+x-x^2$ in P_2.

e. (0,1,1,1,3), (4,2,3,4,4), (1,1,3,2,1), (6,5,7,4,2), (0,3,3,3,9) in R^5.

Answers:

a. Lin. Independence is shown by demonstrating that any one vector cannot be expressed as a linear multiple of any combination of the rest.

$(0,1,0,1)=c_1(2,3,1,0)+c_2(3,2,1,4)$

$$\begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 0 \\ 0 & -5 & 2 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 0 \\ 0 & -5 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & 13 \end{bmatrix}$$

at this point we cannot get a definite answer for c_2 and therefore it is not a linear multiple and the set is linearly independent.

b. r>n which means that it will always be dependant. — r? n?

c. $(1,2,3)= c_1(2,5,7)+c_2(3,4,2)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 7 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -5 \end{bmatrix}$$

why c_2? just $0=-5$.

c_2 remains undefined, so the set is independent.

d. $\mathbf{p_1}=2-x+2x^2$, $\mathbf{p_2}=4+x-3x^2$, $\mathbf{p_3}=1+x-x^2$ in P_2.

$2-x+2x^2=c_1(4+x-3x^2)+c_2(1+x-x^2)$

$$\begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & 1 \\ 2 & -3 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 1 \\ 0 & 6 & 3 \\ 0 & -7 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 1 \\ 0 & 6 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

where does the matrix come from? Discuss basis of P_2, etc.

once again c_2 is undetermined, so the set is independent.

e. (0,1,1,1,3) and (0,3,3,3,9) ~~and~~ are scalar multiples of each other which makes the set dependant.

2. Come up with 3 functions that are linearly independent and show that they are linearly independent as a set of vectors in $C^2(-\infty,\infty)$. (10 points)

Answer:

$\mathbf{f_1}=x$ $\mathbf{f_2}=e^x$ $\mathbf{f_3}=2x^2$

Calculate the Wronskian to determine independence or dependence.

$$W(x) = \begin{vmatrix} x & e^x & 2x^2 \\ 1 & e^x & 4x \\ 0 & e^x & 4 \end{vmatrix} = (4xe^x + 0 + 2x^2e^x - 0 - 4x^2e^x - 4e^x) \neq 0 \textit{ for all values}$$

? I'm not so sure.

of x. — really?

Therefore this set of vectors are linearly independent in $C^2(-\infty, \infty)$.

Section 5.4

1. If S is any set of vectors in a vector space, V, what conditions on S allow that set to be a basis for V? (2 points)

The two conditions are that S is linearly independent and that S spans V.

2. Find the coordinate vector of $\mathbf{w}$ relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$: (4 points total)
a. $\mathbf{u}_1 = (2, -1)$, $\mathbf{u}_2 = (-4, 3)$, $\mathbf{w} = (1,4)$
b. $\mathbf{u}_1 = (-4, 7)$, $\mathbf{u}_2 = (2,0)$, $\mathbf{w} = (0,-3)$

a. $\mathbf{w}_s = (7,6)$ — show how.
b. $\mathbf{w}_s = (-3/2,-6/7)$

3. Determine ~~the~~ a basis and the dimension for the solution space of the following homogeneous system: (3 points)
$x + y + z = 0$
$3x + 2y - 2z = 0$
$4x + 3y - z = 0$
$6x + 5y + z = 0$

the augmented matrix set up and is reduced to RRE form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{array}\right] \xrightarrow{\textit{reduces}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right] \xrightarrow{\textit{reduces}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The basis is given by assigning a parameter and solving for the unknowns.
$x = 4t$
$y = -5t$
$z = t$
the basis is $\{(4,-5,1)\}$ *therefore the dimension is 1.*

4. a. Determine a basis given that: (2 points) — I don't understand the question.
$\mathbf{u}_1 = (3,-1,2,1)$, $\mathbf{u}_2 = (1,1,0,-1)$, $\mathbf{u}_3 = (4,0,2,0)$

Put the vectors into an expanded or coefficient form and set up the augmented matrix, adding a row of zeros to complete it, then solve the system

$$\begin{bmatrix} 1 & 1 & 0 & -1 & \vert & 0 \\ 3 & -1 & 2 & 1 & \vert & 0 \\ 4 & 0 & 2 & 0 & \vert & 0 \\ 0 & 0 & 0 & 0 & \vert & 0 \end{bmatrix} \xrightarrow{reduces} \begin{bmatrix} 1 & 1 & 0 & -1 & \vert & 0 \\ 0 & 4 & -2 & -4 & \vert & 0 \\ 0 & 4 & -2 & -4 & \vert & 0 \\ 0 & 0 & 0 & 0 & \vert & 0 \end{bmatrix} \xrightarrow{reduces} \begin{bmatrix} 1 & 1 & 0 & -1 & \vert & 0 \\ 0 & 2 & -1 & -2 & \vert & 0 \\ 0 & 0 & 0 & 0 & \vert & 0 \\ 0 & 0 & 0 & 0 & \vert & 0 \end{bmatrix}$$

which leaves us with the solution:

x = – t – s
y = ½ t + s
z = t
w = s

or a basis of
(-1, ½, 1, 0); (-1, 1, 0 1)

I still don't get the question, even after seeing the "solution." – this is a well-defined question.

b. Determine dim[span{$\mathbf{u}_1$,$\mathbf{u}_2$,$\mathbf{u}_3$}] (1 point)
2, given the basis has 2 vectors.

How do you know this?

5.5 (10pts)

1. Express the product A**x** as a linear combo of the column vectors of A.(3pts)

$$A=\begin{bmatrix} 8 & -1 & 7 & 2 \\ 3 & 5 & 8 & 1 \\ 1 & 0 & 4 & 3 \\ 2 & -2 & 8 & 10 \end{bmatrix} \quad \mathbf{x}=\begin{bmatrix} 1 \\ -2 \\ 5 \\ 0 \end{bmatrix}$$

Answer

$$1\begin{bmatrix} 8 \\ 3 \\ 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 5 \\ 0 \\ -2 \end{bmatrix} + 5\begin{bmatrix} 7 \\ 8 \\ 4 \\ 8 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 1 \\ 3 \\ 10 \end{bmatrix}$$

2. Determine if B is in Column space of A. If so, express B as a linear Combination of A (3pts)

$$A=\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 3 & 2 \end{bmatrix} \quad B=\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

Answer — some more explanation is in order

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 1 & 2 & 2 \\ 2 & 3 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 3 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & \cancel{0} & -1/2 \end{bmatrix}$$

B is Column Space

$$4\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} -1/2\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

3. Find a basis for the nullspace of A

$$A=\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 0 \\ 7 & -6 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 1 & -19 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -16 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 16t$

$x_2 = 19t$

$x_3 = t$

$t\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$ or $v_1 = (16,19,1)$ is ~~the~~ a basis?.

Section 5.6

1. Find the rank of the following matrices(4 points):

$$A\begin{bmatrix}5\\6\\0\end{bmatrix}\qquad B\begin{bmatrix}5 & -2\\6 & 3\\0 & 1\end{bmatrix}\qquad C\begin{bmatrix}5 & -2 & 0\\6 & 3 & 0\\0 & 1 & 0\end{bmatrix}\qquad D\begin{bmatrix}3 & 0 & 1 & 0\\0 & 0 & 1 & 0\\9 & 0 & 3 & 0\end{bmatrix}$$

Solution

rank(A) = 1

rank(B) = 2

rank(C) = 2

rank(D) = 2

Remind of the definition.

2. Find the nullity of the matrices in 1 (4 points):

Solution

nullity(A) = 0

nullity(B) = 0

nullity(C) = 1

nullity(D) = 2

on a,b,c,d,e?

3. Give the conditions that would make the following system of equations consistent (4 points):

x + y = a

2x + y = b

4x + 3y = c

-x + 5y = d

3x – 2y = e

Solution:

$$\left[\begin{array}{cc|c}1 & 1 & a\\2 & 1 & b\\4 & 3 & c\\-1 & 5 & d\\3 & -2 & e\end{array}\right]\Rightarrow\left[\begin{array}{cc|c}1 & 1 & a\\0 & 1 & 2a-b\\0 & 1 & 4a+c\\0 & 6 & a+d\\0 & 5 & 3a+e\end{array}\right]\Rightarrow\left[\begin{array}{cc|c}1 & 1 & a\\0 & 1 & 2a-b\\0 & 1 & 4a+c\\0 & 1 & \frac{a+d}{6}\\0 & 1 & \frac{3a+e}{5}\end{array}\right]\Rightarrow$$

$$\left[\begin{array}{cc|c}1 & 0 & -a+b\\0 & 1 & 2a-b\\0 & 0 & -2a-b+c\\0 & 0 & \frac{11a-b+d}{6}\\0 & 0 & \frac{7a-b+e}{5}\end{array}\right]$$

in order for the system to be consistent, the last three rows in the augmented matrix must be equivalent. ~~Or:~~ and what else?

$-2a - b + c = 0$

$1/6(11a - b + d) = 0$

$1/5(7a - b + e) = 0$

solving and using parameters gives:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & \frac{30}{13} & \frac{-30}{13} & 0 \\ 0 & 1 & 0 & \frac{42}{13} & \frac{-55}{13} & 0 \\ 0 & 0 & 1 & \frac{102}{13} & \frac{-115}{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$a = 1/13(-30s + 30t)$

$b = 1/13(-42s + 55t)$

$c = 1/13(-102s + 115t)$

$d = s$

$e = t$

4. Prove that rank(A) = rank(A^T) (4 points):

solution:

rank(A) = dim(row space (A)) = dim(column space(A^T) = rank (A^T)

1. Solve the following system for $x_1\ x_2\ x_3$. (10 pts)

$$(1,-1,4)\bullet(x_1\ x_2\ x_3)=10$$
$$(6,4,0)\bullet(x_1\ x_2\ x_3)=2$$
$$(4,-5,-1)\bullet(x_1\ x_2\ x_3)=7$$

Solution: First, we need to convert the system into a more useable form:

$$x_1 - x_2 + 4x_3 = 10$$
$$6x_1 + 4x_2 = 2 \quad ,$$
$$5x_1 - 5x_2 - x_3) = 7$$

then, augment the matrix to $\left[\begin{array}{ccc|c} 1 & -1 & 4 & 10 \\ 6 & 4 & 0 & 2 \\ 4 & -5 & -1 & 7 \end{array}\right]$, then row reduce to echelon from.

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 10 \\ 3 & 2 & 0 & 1 \\ 4 & -5 & -1 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 4 & 10 \\ 0 & 5 & -12 & -29 \\ 0 & -1 & -17 & -33 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 4 & 10 \\ 0 & 1 & -80 & -161 \\ 0 & 0 & -97 & -194 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 4 & 10 \\ 0 & 1 & -80 & -161 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

This yields $x_1=1$, $x_2=-1$, *and* $x3=2$.

---

2. Calculate the standard matrix for the following compositions of linear operations of R^3.
   a) A rotation of 30° about the x-axis, followed by a rotation of 30° about the z-axis, followed by a contraction of $k=\frac{1}{2}$ (5 pts)
   b) A reflection about the xy-plane, followed by a reflection about the xz-plane, followed by an orthogonal projection on the yz-plane. (5 pts)
   c) A rotation of 270° about the x-axis, , followed by a rotation of 90° about the y-axis, followed by a rotation of 180° about the z-axis. (5 pts)

Solution: To find the standard matrix, use "Composition of Three Transformations."
Remember $[T_3 \circ T_2 \circ T_1] = [T_1][T_2][T_3]$

a) $T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & \sin 30 & \cos 30 \end{bmatrix}$ $\quad T_2 = \begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad T_3 = \frac{1}{2}$

Because T_3 is a scalar, we can multiply that through last. T_2 x T_1 yields:
(Remember $\cos 30 = \sqrt{3}/2$, and $\sin 30 = 1/2$

$$\begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & \sin 30 & \cos 30 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & \sqrt{3}/4 & 1/4 \\ 1/2 & 3/4 & \sqrt{3}/4 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix}$$, and contracting by $k = ½$

results in $\begin{bmatrix} \sqrt{3}/4 & \sqrt{3}/8 & 1/8 \\ 1/4 & 3/8 & \sqrt{3}/8 \\ 0 & 1/4 & \sqrt{3}/4 \end{bmatrix}$

b) $T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$[T_3][T_2][T_1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

use explicit sines & cosines so one can check more easily!

c) $T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad T_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad T_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$[T_3][T_2][T_1] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

---

3. Which of the following are linear combinations of $A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$?

a) $= \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ b) $= \begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$ c) $= \begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$

(5 pts.) (5 pts.) (5 pts.)

Solution: To find the linear combinations, we must "transpose A, B and C into

$4\alpha + \beta + 0 = 6$ ‖ — where does this come from? Show.

$-\beta + 2\delta = 8$?

$-2\alpha + 2 + \delta = -1$ for a. Then we can form an augmented matrix. By "adding" parts b and c,

$-2\alpha + 3\beta + 4\delta = -8$

we can solve all three simultaneously.

$$\begin{bmatrix} 4 & 1 & 0 & | & 6 & | & 6 & | & -1 \\ 0 & -1 & 2 & | & -8 & | & 0 & | & 5 \\ -2 & 2 & 1 & | & -1 & | & 3 & | & 7 \\ -2 & 3 & 4 & | & -8 & | & 8 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1/2 & | & 1/2 & | & 3/2 & | & -7/2 \\ 0 & 1 & -2 & | & 8 & | & 0 & | & -5 \\ 4 & 1 & 0 & | & 6 & | & 6 & | & -1 \\ -2 & 3 & 4 & | & -8 & | & 8 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1/2 & | & 1/2 & | & 3/2 & | & -7/2 \\ 0 & 1 & -2 & | & 8 & | & 0 & | & -5 \\ 0 & 7 & 8 & | & -10 & | & 22 & | & 1 \\ 0 & 1 & 3 & | & -7 & | & 5 & | & -6 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1 & -1/2 & | & 1/2 & | & 3/2 & | & -7/2 \\ 0 & 1 & -2 & | & 8 & | & 0 & | & -5 \\ 0 & 0 & 22 & | & -66 & | & 22 & | & 36 \\ 0 & 0 & -5 & | & 15 & | & 5 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1/2 & | & 1/2 & | & 3/2 & | & -7/2 \\ 0 & 1 & -2 & | & 8 & | & 0 & | & -5 \\ 0 & 0 & 1 & | & -3 & | & 1 & | & 18/11 \\ 0 & 0 & 0 & | & 0 & | & 0 & | & 1 \end{bmatrix}$$

You could stop here, or finish reducing it to $\begin{bmatrix} 1 & 0 & 0 & | & 1 & | & 1 & | & 0 \\ 0 & 1 & 0 & | & 2 & | & 2 & | & 0 \\ 0 & 0 & 1 & | & -3 & | & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 & | & 0 & | & 1 \end{bmatrix}$.This shows that a and b are consistent, but not c. Thus a and b are linear combinations of A, B and C.

---

4. Which of the following sets of vectors in P_2 are linearly dependant?

a) $2 - x + 4x^2, 3 + 6x + 2x^2, 2 + 10x - 4x^2$ (5 pts.)

b) $6 - x^2, 1 + x + 4x^2$ (5 pts.)

c) $1 + x2 + 3x^2, -1 - x + x^2, 1 + 4x + 11x^2$ (5 pts.)

d) $1 + 3 + 3x^2, x + 4x^2, 5 + 6x + 3x^2, 7 + 2x - x^2$ (5 pts.)

Solution: If a nonempty set of vectors has only one trivial answer, then it is linearly independent. To find which are linearly dependant, we need to transform these into augmented matrices and row reduce. where does this matrix come from? Explain.

a) $\begin{bmatrix} 2 & 3 & 2 \\ -1 & 0 & 10 \\ 4 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & -1 \\ 0 & 1 & 18/13 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$ Not linearly dependant.

b) $\begin{bmatrix} 6 & 1 \\ 0 & 1 \\ -1 & 4 \end{bmatrix} \sim \begin{bmatrix} -1 & -4 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow$ Linearly dependant. (More rows then columns Theorem 5.3.3) NA

c) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 2 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 & 11/3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Linearly dependant.

d) $\begin{bmatrix} 1 & 0 & 5 & 7 \\ 3 & 1 & 6 & 2 \\ 3 & 4 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 & 2 & 2/3 \\ 0 & 1 & -1 & -1/3 \\ 0 & 0 & 1 & 7/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -14/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 7/3 \end{bmatrix} \Rightarrow$ ~~Not~~ linearly dependant.

5. Let **u** = (3,12,4,7,5) and **v** = (1,9,3,2,6). Find the following:

a) **u** • **v**.

b) ||u||

c) d(**u**,**v**)

Solution: This problems are from 4.1 on Euclidean n-Space.

a) $\mathbf{u} \bullet \mathbf{v} = 3\times1+12\times9+4\times3+7\times2+5\times6 = 3+108+12+14+30 = 167$ (2 pts)

b) $||u|| = (\mathbf{u} \times \mathbf{v}) = \sqrt{3^2+12^2+4^2+7^2+5^2} = \sqrt{9+144+16+49+25} = \sqrt{243} = 9\sqrt{3}$ (3 pts)

c) $d(\mathbf{u},\mathbf{v}) = ||\ \mathbf{u} - \mathbf{v}\ || = \sqrt{(3-1)^2+(12-9)^2+(4-3)^2+(7-2)^2+(5-6)^2}$ (5 pts)

$= \sqrt{2^2+3^2+1^2+5^2+-1^2} = \sqrt{4+9+1+25+1} = \sqrt{40} = 2\sqrt{10}$

---

6. Do $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ form a linearly independent set of vectors in $C^2(-\infty, \infty)$? (15 pts)

$\mathbf{f}_1 = 1+x^2$

$\mathbf{f}_2 = 3 - x$

$\mathbf{f}_3 = 2$

Solution: We can use the Wronskian to show that this set is linear in dependant.

$$W|\mathbf{f1}\quad \mathbf{f2}\quad \mathbf{f3}| = \begin{bmatrix} 1+x^2 & 3-x & 2 \\ 2x & -1 & 0 \\ 2 & 0 & 0 \end{bmatrix} = 2\times -1\times 2 = -4 \neq 0$$ somewhere so it does form a linearly independent set of vectors.

---

7. Let **x** = (1, 3, 8), **y** = (2, 4, 1), and **z** = (0, 4, 7).

a. Show that the set S = {**x**, **y**, **z**} is a basis for R^3. (5 pts)

b. Find the coordinate vector of **v** = (3, 11, 7) with respect to S. (5 pts)

c. Find the vector **w** in R^3 whose coordinate vector with respect to the basis S is (5 pts) $(\mathbf{w})_S = (1, -2, 4)$.

Solution: ? ||

a. If S is a basis for R^3, then ul the vectors can be represented as a homogenous system.

$c_1\mathbf{x} + c_2\mathbf{y} + c_3\mathbf{z} = \mathbf{0}$ which can be represented as

$c_1(1, 3, 8) + c_2(2, 4, 1) + c_3(0, 4, 7) = (0, 0, 0)$

which can be encoded as $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 4 \\ 8 & 1 & 7 \end{bmatrix} = A$. If $\det A \neq 0$ then the set S must be linearly independent and it must span R^3, by theorem and thus it would be a basis for R^3.

$\det A = 28 + 64 + 0 - 0 - 42 - 4 = 46 \neq 0$, thus S is a basis for R^3.

b. $c_1 \quad + \quad 2c_2 = 3$
$3c_1 + 4c_2 + 4c_3 = 11$
$8c_1 + c_2 + 7c_3 = 7$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 3 & 4 & 4 & 11 \\ 8 & 1 & 7 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & -2 & 4 & 2 \\ 0 & -15 & 7 & -17 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 30 & -60 & -30 \\ 0 & -30 & 14 & -34 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 30 & -60 & -30 \\ 0 & 0 & -46 & -64 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 32/23 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 41/23 \\ 0 & 0 & 1 & 32/23 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -12/23 \\ 0 & 1 & 0 & 41/23 \\ 0 & 0 & 1 & 32/23 \end{array}\right].$$ Thus the coordinate vector is $(-12/23, 41/23, 32/23)$

c. $\mathbf{w} = 1\mathbf{x} - 2\mathbf{y} + 4\mathbf{z}$
$\mathbf{w} = 1(1, 3, 8) - 2(2, 4, 1) + 4(0, 4, 7)$
$\mathbf{w} = (1, 3, 8) - (4, 8, 2) + (0, 16, 28)$
$\mathbf{w} = (-3, 11, 34)$

---

8. . Determine whether the following set is a vector space under the given operations:
The set of all pairs of real numbers (x, y) with the operations
$(x, y) + (x', y') = (xx', yy')$ and $k(x, y) = (kx, ky)$.
Show how each axiom holds or fails. (10 pts.)

Solution
Let $\mathbf{u} = (x, y)$, $\mathbf{v} = (x', y')$, $\mathbf{w} = (x'', y'')$, and k be any scalar

1. $\mathbf{u} + \mathbf{v} = (x, y) + (x', y') = (xx', yy')$ which is in V **Holds**
2. $\mathbf{u} + \mathbf{v} = (x, y) + (x', y') = (xx', yy') = (x'x, y'y) = (x', y') + (x, y) = \mathbf{v} + \mathbf{u}$ **Holds**
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (x, y) + (x'x'', y'y'') = (xx'x'', yy'y'') = (xx', yy') + (x'', y'') =$
$(\mathbf{u} + \mathbf{v}) + \mathbf{w}$ **Holds**
4. $0\mathbf{u} = 0(x, y) = (0x, 0y) = (0, 0) = \mathbf{0}$. $\mathbf{0} + \mathbf{u} = (0, 0) + (x, y) = (0x, 0y) =$
$(0, 0) \neq \mathbf{u}$ — this is not axiom 4. Note $\underline{0} = (1, 1)$. **Fails**
5. Because Axiom 4 fails, then there is no zero vector. Thus Axiom 5 fails as well. **Fails**
6. $k\mathbf{u} = k(x, y) = (kx, ky)$ which is in V **Holds**
7. $k(\mathbf{u}+\mathbf{v}) = k(xx', yy') = (kxx', kyy') \neq (k^2\ xx', k^2\ yy') = (kx, ky) + (kx', ky') = k\mathbf{u} + k\mathbf{v}$ **Fails**
8. $(k + m)\mathbf{u} = (k + m)(x, y) = (kx + mx, ky + my) \neq (kmx^2, kmy^2) =$
$(kx, ky) + (mx, my) = k\mathbf{u} + m\mathbf{u}$ **Fails**
9. $k(m\mathbf{u}) = k(m(x, y)) = k(mx, my) = (kmx, kmy) = (km)(x, y) = (km)\mathbf{u}$ **Holds**
10. $1\mathbf{u} = 1(x, y) = (x, y) = \mathbf{u}$ **Holds**

It is not a vector space because Axioms ~~4~~, 5, 7, and 8 fail for the set.

---

9. Using Theorem 5.2.1 which states:

If W is a set of one or more vectors from vector space V, then W is a subspace of V if and only if the following conditions hold.
(a) If **u** and **v** are vectors in W, then **u** + **v** is in W.
(b) If k is any scalar and **u** is any vector in W, then k**u** is in W.

Determine which of the following are subspaces, and show why they are or aren't subspaces of:
(a) R^3 for all vectors of the form (a, 0, 0) (5 pts.)
(b) R^3 for all vectors of the form (a, 1, 1) (5 pts.)
(c) P_3 for all polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$ (5 pts.)
(d) P_3 for all polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ for which a_0, a_1, a_2, and a_3 are integers (5 pts.)
(e) M_{22} for all n x n matrices A such that tr(A) = 0 (5 pts.)

Solution:

(a) Yes, vectors in the form of (a,0,0) are a subspace in R^3.

Using part (a) of Theorem 5.2.1: (a,0,0) + (a,0,0) = (2a,0,0) which is still in the form of (a,0,0)

Using part (b) of Theorem 5.2.1: k(a,0,0) = (ka,0,0) which is still in the form of (a,0,0)

Therefore both parts of Theorem 5.2.1 pass and all vectors of the form (a,0,0) are a subspace.

(b) No, vectors in the form of (a,1,1) are not a subspace in R^3.

Using part (a) of Theorem 5.2.1: (a,1,1) + (a,1,1) = (2a,2,2) which is not in the form of (a,1,1)

Using part (b) of Theorem 5.2.1: k(a,1,1) = (ka,k,k) which is not in the form of (a,1,1) (If k=1, then it is in the correct form, but all other cases fail)

Therefore both parts of Theorem 5.2.1 fail and all vectors of the form (a,1,1) are not a subspace.

(c) Yes, polynomials in the form of $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$ are a subspace in P_3.

Using part (a) of Theorem 5.2.1: $a_0 + a_1x + a_2x^2 + a_3x^3 + a'_0 + a'_1x + a'_2x^2 + a'_3x^3 = a_0 + a'_0 + (a_1 + a'_1)x + (a_2 + a'_2)x^2 + (a_3 + a'_3)x^3$ which is still in the form of $a_0 + a_1x + a_2x^2 + a_3x^3$ where $a_0 + a_1 + a_2 + a_3 = 0$.

didn't show this yet.

Using part (b) of Theorem 5.2.1: $k(a_0 + a_1x + a_2x^2 + a_3x^3) = ka_0 + ka_1x + ka_2x^2 + ka_3x^3$ which is still in the form of $a_0 + a_1x + a_2x^2 + a_3x^3$ where $a_0 + a_1 + a_2 + a_3 = 0$.
didn't show this yet.

Therefore both parts of Theorem 5.2.1 pass and all vectors of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$ are a subspace in P_3.

(d) No, polynomials in the form of $a_0 + a_1x + a_2x^2 + a_3x^3$ for which a_0, a_1, a_2, and a_3 are integers are not a subspace in P_3.

Using part (a) of Theorem 5.2.1: $a_0 + a_1x + a_2x^2 + a_3x^3 + a'_0 + a'_1x + a'_2x^2 + a'_3x^3 = a_0 + a'_0 + (a_1 + a'_1)x + (a_2 + a'_2)x^2 + (a_3 + a'_3)x^3$ which is still in the form of $a_0 + a_1x + a_2x^2 + a_3x^3$ where $a_0 + a_1 + a_2 + a_3 = 0$.

Using part (b) of Theorem 5.2.1: $k(a_0 + a_1x + a_2x^2 + a_3x^3) = ka_0 + ka_1x + ka_2x^2 + ka_3x^3$ which is not in the form of $a_0 + a_1x + a_2x^2 + a_3x^3$ where $a_0 + a_1 + a_2 + a_3 = 0$ if k isn't an integer.
?

Therefore one parts of Theorem 5.2.1 fails and the other passes. Therefore all vectors of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ for which a_0, a_1, a_2, and a_3 are integers are not a subspace in P_3.

(e) Yes, Matrices in the form where all n x n matrices A such that tr(A) = 0 are a subspace in M_{22}.

Using part (a) of Theorem 5.2.1: $A_{11} + A_{22} + \ldots + A_{nn} + A_{11} + A_{22} + \ldots + A_{nn} = 2(A_{11} + A_{22} + \ldots + A_{nn}) = 2(0) = 0$ which is still in the form of tr(A) = 0.
Not sufficient to use same matrix twice.

Using part (b) of Theorem 5.2.1: $k(A_{11} + A_{22} + \ldots + A_{nn}) = k(0) = 0$ which is still of the form of tr(A) = 0.

Therefore both parts of Theorem 5.2.1 pass and all matrices of the form of n x n matrices A such that tr(A) = 0 are a subspace in M_{22}.

---

10. State whether the set spans:
(a) R^3 for $\mathbf{v}_1 = (2,2,2)$ $\mathbf{v}_2 = (0,0,3)$ $\mathbf{v}_3 = (0,1,1)$ (5 pts.)
(b) R^3 for $\mathbf{v}_1 = (2,-1,2)$ $\mathbf{v}_2 = (4,1,2)$ $\mathbf{v}_3 = (8,-1,8)$ (5 pts.)
(c) P_2 for all $\mathbf{p}_1 = 1 - x + 2x^2$, $\mathbf{p}_2 = 3 + x$, $\mathbf{p}_3 = 5 - x + 4x^2$, $\mathbf{p}_4 = -2 - 2x + 2x^2$ (5 pts)

Solution:

(a) Setup the matrix to take the determinant: $\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

Take the determinant: $2 \times 1 \times 3 = 6 \neq 0$ $\therefore$ It is linearly independent and in the span of R^3

(b) (b) Setup the matrix to take the determinant: $\begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & 2 \\ 8 & -1 & 8 \end{bmatrix}$

(c) $\left[\begin{array}{ccc|cc} 2 & -1 & 3 & 2 & -1 \\ 4 & 1 & 2 & 4 & 1 \\ 8 & -1 & 8 & 8 & -1 \end{array}\right] \Rightarrow -(24-4-32)+(16-16-12)=12+(-12)=0$

$\therefore$ With the det(A) = 0, we know that it is linearly dependent and does not ~~see the~~ span ~~of~~ R^3

Exam Key #2

1. Given vectors $\underline{u}$ and $\underline{v}$ where $\underline{u} = (-1,3,6,4)$ and $\underline{v} = (2,4,6,8)$, find $\|\underline{u}+\underline{v}\|^2$.

2. Find the standard matrix for a vector in $\mathbb{R}^3$ dilated by a factor of 3, followed by an orthogonal projection on the yz-plane, and then rotated θ degrees about the y-axis.

3. Use elementary matrices to find the standard matrix for $T\colon R^3 \to R^3$, when $T\colon R^3 \to R^3$ reflects a vector across the xy-plane, contracts that vector by a factor of 1/3, and then projects that vector orthogonally onto the yz-plane.

4. Determine whether the following is a vector space: the set of all triples of real numbers (x,y,z) where $(x,y,z)"+"(x',y',z') = (x\cdot z', y\cdot y', z\cdot x')$ and $k(x,y,z) = (x,y,z)$. Justify your answer.

5. Determine whether the following are linear combinations of $\underline{v}_1 = (1,2,3)$, $\underline{v}_2 = (2,5,1)$, and $\underline{v}_3 = (0,0,3)$ and express each as a linear combination if one exists.

   a) (6,5,8)
   b) (3,2,12)

6. If $\underline{v}_1 = (1,-3,-2)$, $\underline{v}_2 = (-2,7,4)$, and $\underline{v}_3 = (3,-8,-6)$,

   a) Is this set of vectors linearly dependent?
   b) Define the solution space for $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$. – what does that mean?,
   c) What type of space is this? – ditto?

7. Are these sets of vectors linearly dependent (solve by inspection)?

   a) $\{(1,1,0),(0,0,0)\}$
   b) $\{(1,0,1),(0,1,0)\}$
   c) $\{(1,0,1),(0,1,0),(4,4,4)\}$
   d) $\{(1,1,1),(2,2,3)\}$

8. Let $f_1(x) = x^2$ and $f_2(x) = \sin^2(x)$.

   a) What is a Wronskian (no, a type of sausage is incorrect)?
   b) Find the Wronskian of these functions.
   c) What does this tell you about the set?

9. Determine the dimension of and a basis for the solution space of the systems:

a) $$\begin{aligned} x_1 - 2x_2 + 7x_3 &= 0 \\ -3x_1 + 6x_2 - 21x_3 &= 0 \\ 2x_1 + 2x_2 - 5x_3 &= 0 \end{aligned}.$$

b) $$\begin{aligned} x_1 - 3x_2 + x_3 + 5x_4 &= 0 \\ -2x_1 + x_2 + 7x_3 - 2x_4 &= 0 \end{aligned}.$$

10. By inspection, find bases for the row and column spaces of the following matrices.

a) $$\begin{bmatrix} 1 & -2 & 1 & 5 & 0 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

b) $$\begin{bmatrix} 1 & -7 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c) $$\begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution to 1 (5 points):

$$\underline{u}+\underline{v}=(1,7,12,12) \Rightarrow \|\underline{u}+\underline{v}\| = \sqrt{(1)^2+(7)^2+(12)^2+(12)^2} = \sqrt{338}.$$

Solution to 2 (10 points):

Put solutions next to questions next time so I can remember what question was.

$$\begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot 3 = \begin{bmatrix} 0 & 1 & \sin(\theta) \\ 0 & 1 & 0 \\ 0 & 0 & \cos(\theta) \end{bmatrix} \cdot 3 = \begin{bmatrix} 0 & 0 & 3\sin(\theta) \\ 0 & 3 & 0 \\ 0 & 0 & 3\cos(\theta) \end{bmatrix}$$

Solution to 3 (10 points):

$$\begin{aligned} e_1 &= (1,0,0) \to (1,0,0) \to (1/3,0,0) \to (0,0,0) \\ e_2 &= (0,1,0) \to (0,1,0) \to (0,1/3,0) \to (0,1/3,0) \\ e_3 &= (0,0,1) \to (0,0,-1) \to (0,0,-1/3) \to (0,0,-1/3) \end{aligned} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}$$

Solution to 4 (10 points):

It is not a vector space.
It fails axiom 2: $(x,y,z)+(x',y',z')=(xz',yy',zx')$, but
$(x',y',z')+(x,y,z)=(x'z,y'y,z'x)$.
It fails axiom 3: $\left[(x,y,z)+(x',y',z')\right]+(x'',y'',z'')=(xz'z'',yy'y'',zx'x'')$, but
$(x,y,z)+\left[(x',y',z')+(x'',y'',z'')\right]=(xz'x'',yy'y'',zx'z'')$.

Solution to 5 (10 points each):

a) $k_1\underline{v}_1+k_2\underline{v}_2+k_3\underline{v}_3=(6,5,8)$, so $\begin{bmatrix}\underline{v}_1 & \underline{v}_2 & \underline{v}_3\end{bmatrix}\cdot\begin{bmatrix}\underline{k}\end{bmatrix}=\begin{bmatrix}6\\5\\8\end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 2 & 5 & 0 & 5 \\ 3 & 1 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 1 & 0 & -7 \\ 0 & -5 & 3 & -10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 3 & -45 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -15 \end{array}\right],$$

so $20\underline{v}_1+(-7)\underline{v}_2+(-15)\underline{v}_3=(6,5,8)$.

b) $k_1\underline{v}_1 + k_2\underline{v}_2 + k_3\underline{v}_3 = (3,2,12)$, so $\begin{bmatrix}\underline{v}_1 & \underline{v}_2 & \underline{v}_3\end{bmatrix}\cdot\begin{bmatrix}\underline{k}\end{bmatrix} = \begin{bmatrix}3\\2\\12\end{bmatrix}$.

$$\left[\begin{array}{ccc|c}1 & 2 & 0 & 3\\ 2 & 5 & 0 & 2\\ 3 & 1 & 3 & 12\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 2 & 0 & 3\\ 0 & 1 & 0 & -4\\ 0 & -5 & 3 & 3\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 2 & 0 & 3\\ 0 & 1 & 0 & -4\\ 0 & 0 & 3 & -17\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & 0 & 11\\ 0 & 1 & 0 & -4\\ 0 & 0 & 1 & -17/3\end{array}\right],$$

so $11\underline{v}_1 + (-4)\underline{v}_2 + (-17/3)\underline{v}_3 = (3,2,12)$.

Solution to 6 (10 points part a, 5 points each b and c):

a) Express these vectors as $A\underline{k} = \underline{0}$: $\begin{bmatrix}1 & -2 & 3\\ -3 & 7 & -8\\ -2 & 4 & -6\end{bmatrix}\cdot\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

$\begin{bmatrix}1 & -2 & 3\\ -3 & 7 & -8\\ -2 & 4 & -6\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 3\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$, which is code for $\begin{array}{l}k_1 - 2k_2 + 3k_3 = 0\\ k_2 + k_3 = 0\end{array}$.

So, set a parameter $k_3 = t$. Thus, $\begin{array}{l}k_1 = -5t\\ k_2 = -t\\ k_3 = t\end{array}$, so $(-5t)\underline{v}_1 + (-t)\underline{v}_2 + (t)\underline{v}_3 = 0$ for all t in the real numbers and $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is linearly dependent.

b) The solution space of the system $A\underline{k} = \underline{0}$ for $A = \begin{bmatrix}1 & -2 & 3\\ -3 & 7 & -8\\ -2 & 4 & -6\end{bmatrix}$ is

$\underline{k} = (-5,-1,1)$.

c) This subspace is a line with the equation $f(x,y,z) = -5x - y + z$.

this is not the eqn for any geometrical object, but perhaps a definition of a function f.

Solution to 7 (5 points each):

a) Yes; any set that contains the zero vector is linearly dependent.
b) No; neither vector is a scalar multiple of the other.
c) Yes; $4\underline{v}_1 + 4\underline{v}_2 = \underline{v}_3$.

Solution to 8 (10 points a and b, 5 points c):

w antecedent

a) A Wronskian is a function of a set of $n-1$ times differentiable functions that can prove linear independence of the set. It's written $W(x) = \begin{vmatrix} f_1(x) & \cdots & f_n(x) \\ \vdots & & \vdots \\ f_1^{n-1}(x) & \cdots & f_n^{n-1}(x) \end{vmatrix}$.

b) $W(x) = \begin{vmatrix} x^2 & \sin^2(x) \\ 2x & 2\sin(x)\cos(x) \end{vmatrix} = 2x^2 \sin(x)\cos(x) - 2x\sin^2(x) \neq 0$ somewhere?

c) The system is linearly independent.

Solution to 9 (10 points each):

a) To find a basis for the solution space of the system, first find the solutions:

$$\begin{bmatrix} 1 & -2 & 7 \\ -3 & 6 & -21 \\ 2 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 7 \\ 0 & 6 & -19 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & -19/6 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } \begin{aligned} x_1 &= -2/3t \\ x_2 &= 19/6t \\ x_3 &= t \end{aligned}.$$

Thus, the solution vector is $\begin{bmatrix} -2/3t \\ 19/6t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} -2/3 \\ 19/6 \\ 1 \end{bmatrix}$. Since the vector is linearly independent, this is the basis and the solution space is one-dimensional.

b) To find a basis for the solution space of the system, first find the solutions:

$$\begin{bmatrix} 1 & -3 & 1 & 5 \\ -2 & 1 & 7 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 & 5 \\ 0 & -5 & 9 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -22/5 & 1/5 \\ 0 & 1 & -9/5 & -8/5 \end{bmatrix},$$

so $\begin{aligned} x_1 &= 22/5s - 1/5t \\ x_2 &= 9/5s + 8/5t \\ x_3 &= s \\ x_4 &= t \end{aligned}$. Then $\begin{bmatrix} 22/5s \\ 9/5s \\ 1s \\ 0s \end{bmatrix} + \begin{bmatrix} -1/5t \\ 8/5t \\ 0t \\ 1t \end{bmatrix} = s \cdot \begin{bmatrix} 22/5 \\ 9/5 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -1/5 \\ 8/5 \\ 0 \\ 1 \end{bmatrix}$, so these are the solution vectors. Since the vectors are linearly independent (not scalar multiples of each other), this is the basis and the solution space is two-dimensional.

Solution to 10 (5 points each):

a) The vectors

$$\begin{aligned}\underline{r}_1 &= \begin{bmatrix}1 & -2 & 1 & 5 & 0\end{bmatrix}\\ \underline{r}_2 &= \begin{bmatrix}0 & 1 & 0 & 2 & 2\end{bmatrix}\\ \underline{r}_3 &= \begin{bmatrix}0 & 0 & 1 & 4 & 3\end{bmatrix}\\ \underline{r}_4 &= \begin{bmatrix}0 & 0 & 0 & 1 & 0\end{bmatrix}\end{aligned}$$

form a basis for the row space, and the vectors

$$\underline{c}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \underline{c}_2 = \begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \underline{c}_3 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \underline{c}_4 = \begin{bmatrix}5\\2\\4\\1\end{bmatrix}$$

form a basis for the column space.

b) The vectors

$$\begin{aligned}\underline{r}_1 &= \begin{bmatrix}1 & -7 & 0 & 3\end{bmatrix}\\ \underline{r}_2 &= \begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}\\ \underline{r}_3 &= \begin{bmatrix}0 & 0 & 1 & 2\end{bmatrix}\\ \underline{r}_4 &= \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}\end{aligned}$$

form a basis for the row space, and the vectors

$$\underline{c}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \underline{c}_2 = \begin{bmatrix}-7\\1\\0\\0\end{bmatrix}, \underline{c}_3 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \underline{c}_4 = \begin{bmatrix}3\\0\\2\\1\end{bmatrix}$$

form a basis for the column space.

c) The vectors

$$\begin{aligned}\underline{r}_1 &= \begin{bmatrix}1 & 2 & -4\end{bmatrix}\\ \underline{r}_2 &= \begin{bmatrix}0 & 1 & 5\end{bmatrix}\\ \underline{r}_3 &= \begin{bmatrix}0 & 0 & 1\end{bmatrix}\end{aligned}$$

form a basis for the row space, and the vectors

$$\underline{c}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \underline{c}_2 = \begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \underline{c}_3 = \begin{bmatrix}-4\\5\\1\\0\end{bmatrix}$$

form a basis for the column space.

1. a) ~~Define~~ Write the Cauchy-Schwarz inequality. (5 pts.)

   b) Verify that the Cauchy-Schwarz inequality holds for

$$\mathbf{u}=(2,3,-1),\ \mathbf{v}=(0,2,5)\quad (5\text{ pts.})$$

   c) Determine whether the following vectors are orthogonal

$$\mathbf{u}=(4,-1,2),\ \mathbf{v}=(3,2,1)\quad (5\text{ pts.})$$

Solution: a) The Cauchy-Schwarz inequality is defined to be $|u \cdot v| \le \|u\|\|v\|$

b) Plugging the values of the vectors into the Cauchy-Schwarz inequality yields

$$|2(0)+3(2)-1(5)| \le \sqrt{2^2+3^2+(-1)^2}\sqrt{0^2+2^2+5^2}$$

or

$$|1| \le \sqrt{14}\sqrt{29}$$

c) Since two vectors are orthogonal when their dot product equals zero,

$$(4)(3)+(-1)(2)+(2)(1)=12\neq 0\ ,\quad ?$$

Therefore **u** and **v** are not orthogonal.

2. Determine if (3, 5, 11), (7, 8, 22), and (4, -1, 7) is a linearly independent set of vectors. (15 pts.)

Solution: By definition, the set of vectors is linearly independent iff $k_1(3,5,11)+k_2(7,8,22)+k_3(4,-1,7)=\underline{0}$ has only the trivial solution. This equation can be

rewritten as

$$\begin{aligned} 3k_1+7k_2+4k_3&=0\\ 5k_1+8k_2-k_3&=0\\ 11k_1+22k_2+7k_3&=0 \end{aligned}$$

by distributing the k's.

So

$$\begin{bmatrix} 3 & 7 & 4 \\ 5 & 8 & -1 \\ 11 & 22 & 7 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing the coefficient matrix, we obtain

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $k_3 = t$ $k_2 = -t$ $k_1 = -2t$ and this system of equations has more than just the trivial solution so the set is not linearly independent.

3. Show that the transformation from R^n to R^m by multiplication of a $n \times n$ matrix A is one to one, where

$$A = \begin{bmatrix} 5 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{(15 pts.)}$$

Solution: We show that T_A is one to one by determining the invertibility of matrix A. Since invertibility and the existence of a determinant are equivalent statements, we find the determinant of A.

$$\det A = \begin{vmatrix} 5 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 10 \neq 0$$

which shows that A is invertible, and therefore the transformation from R^n to R^m is one to one.

4. Find the dimension and the basis of the solution space of the system of equations. (15 pts.)

$$\begin{gathered} 4x + 14y + 10z = 0 \\ x + y + z = 0 \\ 2x + 7y + 5z = 0 \end{gathered}$$

Solution: Any set of vectors that is linearly independent and spans the solution space is a basis for the solution space.

The augmented matrix to the system of equations is

$$\begin{bmatrix} 4 & 14 & 10 \\ 1 & 1 & 1 \\ 2 & 7 & 5 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

Reduce the matrix to reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 2/5 \\ 0 & 1 & 3/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is code for z =t, y = -3/5t and x= -2/5t.

?

The solution vectors can be written $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2/5t \\ -3/5t \\ t \end{bmatrix} = t\begin{bmatrix} -2/5 \\ -3/5 \\ 1 \end{bmatrix}$

Which shows the vector $v = (-2/5, -3/5, 1)$ spans the solution space, and because there is only one vector in the set it is independent by necessity. The solution space is therefore one dimensional.

unless it is $\underline{0}$.

5. Determine whether the given vectors span R^3

$$\mathbf{x} = (2,1,3),\ \mathbf{y} = (3,-2,0),\ \mathbf{z} = (-2,5,1) \quad \text{(15 pts.)}$$

Solution: For the vectors to span R^3, there must be some vector **b** where

$$\mathbf{b} = k_1\mathbf{x} + k_2\mathbf{y} + k_3\mathbf{z}$$

or

$$\begin{aligned} 2k_1 + k_2 + 3k_3 &= b_1 \\ 3k_1 - 2k_2 + 0k_3 &= b_2 \\ -2k_1 + 5k_2 + k_3 &= b_3 \end{aligned}$$

This sentence is "information free."

Since the coefficient matrix A of the system has to be consistent for the system to span R_3, we take the determinant of A

?

matrices are not consistent or inconsistent.

$$\det A = \begin{vmatrix} 2 & 1 & 3 \\ 3 & -2 & 0 \\ -2 & 5 & 1 \end{vmatrix} = (2)(-2)(1) + (1)(0)(-2) + (3)(3)(5) - (1)(3)(1) - (2)(0)(5) - (3)(-2)(-2)$$

$$= 26 \neq 0$$

Therefore, the vectors do span R^3.

6. Determine if the set of all vectors of the form (a,b,c) with addition being defined as (a,b,c)+(x,y,z) = (a+x,0,c+z), and scalar multiplication as usual, is a vector space. (15 pts.)

Solution: This is a vector space if it comply's with all axioms defining a vector space on p.222 of the textbook.

a) (a,b,c)+(x,y,z) = (a+x,0,c+z), which is of the form (a,b,c)

b) (a,b,c)+(x,y,z) = (a+x,0,c+z), (x,y,z)+(a,b,c) = (x+a,0,c+z) =(a+x,0,c+z). $\overset{?}{=}$

c) (a,b,c)+((x,y,z)+(g,e,f)) = (a,b,c)+(x+g,0,z+f) = (a+(x+g),0,c+(z+f)) = ((a+x)+g,0,(c+z)+f) = (a+x,0,c+z)+(g,e,f) = ((a,b,c)+(x,y,z))+(g,e,f).

d) $\underline{0}$+(a,b,c)=(a+b+c)+$\underline{0} \neq$ (a,b,c) – but what is/should be $\underline{0}$? That is the real question.

There is no additive identity for this space so it is not a vector space.

I'm not so sure yet.

7. Show that the vectors **a**=(2,1,0), **b**=(1,1,3), and **c**=(4,2,3) make up a basis for R_3. (15 pts.)

Solution: A set of vectors is defined as a basis when it is linearly independent and spans the vector space it is in. The conditions can be checked by seeing if the coefficient matrix A of the systems

$$\begin{aligned} 2k_1 + k_2 + 0k_3 &= b_1 \\ k_1 + k_2 + 3k_3 &= b_2 \\ 4k_1 + 2k_2 + 3k_3 &= b_3 \end{aligned}$$

– why the b's?

has a non-zero determinant. Since

$$\det A = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 4 & 2 & 3 \end{vmatrix} = 3 \neq 0$$

the vectors are a basis for R_3.

8. Show that the composition of two linear transformations is not always commutative. (15 pts.)

Solution: note: There are many solutions to this problem. This is just one of them.

Let $T_1 : R^2 \to R^2$ be the orthogonal projection onto the x-axis.
Let $T_2 : R^2 \to R^2$ be the rotation through the angle θ.

So the standard matrix for T_1 is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and the standard matrix for T_2 is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$T_1 \circ T_2 = [T_1][T_2] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{bmatrix}$$

$$T_2 \circ T_1 = [T_2][T_1] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix} \neq ?$$

9. Use the Wronskian of f_1, f_2 and f_3 to determine if the system is linearly independent, where

$$f_1 = x, f_2 = \sin x, f_3 = \cos x \text{ (15 pts.)}$$

Solution: The Wronskian of the set is

$$W(x) = \begin{vmatrix} x & \sin x & \cos x \\ 1 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = -x$$ – why?

Since -x could be non zero in the interval $(-\infty, \infty)$, the system is ~~not~~ linearly independent.

10. Prove the uniqueness of basis representation. (15 pts.)

Solution: Let $S = \{v_1, v_2, ..., v_n\}$ be a basis for vector space V. By definition of a basis, S must span V. So for every vector $v \in V$, $v = c_1 v_1 + c_2 v_2 + ... + c_n v_n$. Suppose that there is more than one way to express v as a linear combination of $\{v_1, v_2, ..., v_n\}$. Then there would be another equation such that $v = k_1 v_1 + k_2 v_2 + ... + k_n v_n$.

So

$v - v = \underline{0} = c_1 v_1 + c_2 v_2 + ... + c_n v_n - k_1 v_1 + k_2 v_2 + ... + k_n v_n$

$= (c_1 - k_1)v_1 + (c_2 - k_2)v_2 + ... + (c_n - k_n)v_n$ – I like this approach.

Because S is a basis it is also linearly independent. So

$$(c_1 - k_1)v_1 + (c_2 - k_2)v_2 + ... + (c_n - k_n)v_n = \mathbf{0} \Rightarrow (c_1 - k_1) = (c_2 - k_2) = ... = (c_n - k_n) = 0$$

Therefore $c_1 = k_1,\ c_2 = k_2,\ ...,\ c_n = k_n$.

Problem 1 (22 points total)

a) 8 points total
    i. 2 points
    ii. 2 points
    iii. 2 points
    iv. 2 points
b) 6 points total
    i. 3 points
    ii. 3 points
c) 8 points total
    i. 4 points
    ii. 4 points

Problem 2 (15 points total)

a) 5 points
b) 5 points
c) 5 points

Problem 3 (16 points total)

(if final answer is wrong, partial points can be awarded for correct steps in either of the 3 possible solutions. Either solution is fine.)

Problem 4 (10 points total)

(1 point per Axiom – Answers can be given in any order)

Problem 5 (20 points total)

a) 5 points
b) 5 points
c) 5 points
d) 5 points

Problem 6 (16 points total)

a) 8points
b) 8points

Problem 7 (16 points total)

a) 8 points
b) 8 points

Problem 8 (15 points total)

a) 5 points
b) 5 points
c) 5 points

Problem 9 (10 points total)

Problem 10 (10 points total)

Problem 1

The following are vectors in Euclidean-n space. Perform the requested operations on these vectors.

$\mathbf{u} = (4,-1,-3,1)$ $\mathbf{v} = (5,3,-2,0)$ and $\mathbf{w} = (-2, 7, 1, -4)$,

a) Use addition or scalar multiplication to evaluate the result of the operations performed.
   i. $\mathbf{w} - \mathbf{v}$
   ii. $2\mathbf{u} + 3\mathbf{v}$
   iii. $4(3\mathbf{v}+2\mathbf{w})$
   iv. $2(5\mathbf{u}+\mathbf{w})-(\mathbf{v}-\mathbf{w})$
b) Find the dot product
   i. $\mathbf{v} \bullet \mathbf{w}$
   ii. $\mathbf{u} \bullet 4\mathbf{v}$
c) Find the following
   i. Norm of $\mathbf{u}$
   ii. Distance between $\mathbf{u}$ and $\mathbf{w}$

Solution 1

a) Use addition or scalar multiplication to evaluate the result of the operations performed.
   i. $\mathbf{w} - \mathbf{v} = (-2,7,1,-4) - (5,3,-2,0) = (-2-5,7-3,1-(-2),-4-0) = (-7,4,3,-4)$
   ii. $2\mathbf{u} + 3\mathbf{v} = 2(4,-1,-3,1) + 3(5,3,-2,0) = (8,-2,-6,2) + (15,9,-6,0)$ $=(23,7,-12,2)$
   iii. $4(3\mathbf{v}+2\mathbf{w})$ This problem can be solved two main ways. Either to distribute the "4" first and creating $12\mathbf{v}+8w$ and then evaluating or you can evaluate the inside first and then distribute the "4". Our solution will show the latter.
   $4(3\mathbf{v}+2\mathbf{w}) = 4(\ 3(5,3,-2,0) + 2(-2, 7, 1, -4)\ )$
   $= 4(\ (15,9,-6,0) + (-2, 7, 1, -4)\ ) = 4(13,16,-5,-4) = (52,64,-20,-16)$
   iv. $2(5\mathbf{u}+\mathbf{w})-(\mathbf{v}-\mathbf{w})$ There are a number of ways to solve this one using techniques in this solution key. We will use a new method utilizing the commuticative and associative properties of vectors. $2(5\mathbf{u}+\mathbf{w})-(\mathbf{v}-\mathbf{w}) = 10u+2w-v+w = 10u-v+3w$, now this becomes just standard scalar multiplication and vector addition.
   $10u-v+3w = 10(4,-1,-3,1)-(5,3,-2,0)+3(-2, 7, 1, -4)$
   $=(40,-10,-30,10)-(5,3,-2,0)+(-6,21,3,-12) = (29,8,-25,-2)$
b) Find the dot product
   i. $\mathbf{v} \bullet \mathbf{w} = (v_1 \bullet w_1 + v_2 \bullet w_2 + v_3 \bullet w_3 + v_4 \bullet w_4) = (5,3,-2,0) \bullet (-2, 7, 1, -4)$ $= 5\bullet-2+ 3\bullet 7+ -2\bullet 1+ 0\bullet-4 = -10 + 21 + -2 + 0 = 9$, notice that the answer is a scalar value.
   ii. $\mathbf{u} \bullet 4\mathbf{v}$ There are couple ways to do this cross. One is multiply the "4" into the vector and then take the dot product of the two. We will show that we can leave the "4" at the left because it is a scalar and multiply it later.
   $\mathbf{u} \bullet 4\mathbf{v} = 4\mathbf{u} \bullet \mathbf{v} = 4(\mathbf{u} \bullet \mathbf{v}) = 4(\ (4,-1,-3,1) \bullet (5,3,-2,0)\ ) =$
   $=4(20+-3+6+0) = 4(23) = 92$

c) Find the following

i. Norm of **u**

$\sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2} = \sqrt{4^2 + (-1)^2 + (-3)^2 + 1^2}$

$= \sqrt{16 + 1 + 9 + 1} = \sqrt{27} = 3\sqrt{3}$

ii. Distance between **u** and **w**

$\sqrt{(w_1 - u_1)^2 + (w_2 - u_2)^2 + (w_3 - u_3)^2 + (w_4 - u_4)^2}$

$=\sqrt{(-2-4)^2 + (7+1)^2 + (1+3)^2 + (-4-1)^2}$

$=\sqrt{(-6)^2 + (8)^2 + (4)^2 + (5)^2} = \sqrt{36+64+16+25} = \sqrt{141}$

Problem 2

a) Use matrix multiplication to find the reflection of (1,2) about the y-axis.

Solution:

The standard matrix for reflecting a vector about the y-axis is

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now we multiply the standard matrix [T] by the vector (1, 2).

$$[w] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

a bit trivial.

which you would have known w/o matrix.

Therefore the reflection of (1,2) about the y-axis is (-1, 2).

b) Use matrix multiplication to find the image of the vector (-2, 1, 2) if it is rotated 30° about the x-axis.

Solution:

The standard matrix for rotating a vector about the x-axis is

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Now, we multiply the standard matrix [T] by the vector being rotated, which is in this case (-2, 1, 2).

$$[w] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ \dfrac{\sqrt{3}-2}{2} \\ \dfrac{1+2\sqrt{3}}{2} \end{bmatrix}$$

Therefore the image of (-2, 1 ,2) rotated 30° about the x-axis is $\left(-2, \dfrac{\sqrt{3}-2}{2}, \dfrac{1+2\sqrt{3}}{2}\right)$

c) Find the standard matrix, [T], for a rotation of 60º about the x-axis in R^2, followed by a reflection about the line y = x.

Solution:

$$[T] \neq \begin{bmatrix} \cos 60^o & -\sin 60^o \\ \sin 60^o & \cos 60^o \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{-\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Problem 3.

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear map. Find $T\begin{pmatrix}1\\5\end{pmatrix}$.

Given that...

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$$

I like this problem.

Solution:
Solution a:

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\begin{pmatrix}1\\-1\end{pmatrix} = T\begin{pmatrix}1\\0\end{pmatrix} + T\begin{pmatrix}0\\-1\end{pmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$$

$$T\begin{pmatrix}1\\-1\end{pmatrix} = T\begin{pmatrix}1\\0\end{pmatrix} - T\begin{pmatrix}0\\1\end{pmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$$

Find $T\begin{pmatrix}0\\1\end{pmatrix}$ by subtracting:

$$T\begin{pmatrix}1\\0\end{pmatrix} - T\begin{pmatrix}1\\0\end{pmatrix} + T\begin{pmatrix}0\\1\end{pmatrix} = \begin{bmatrix}3\\2\end{bmatrix} - \begin{bmatrix}4\\5\end{bmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{bmatrix}-1\\-3\end{bmatrix}$$

By ~~definition~~ theorem $[T] = \left[T\begin{pmatrix}1\\0\end{pmatrix} \middle| T\begin{pmatrix}0\\1\end{pmatrix}\right]$

$$T = \begin{bmatrix}3 & -1\\2 & -3\end{bmatrix}$$

Find $T\begin{pmatrix}1\\5\end{pmatrix} = \begin{bmatrix}3 & -1\\2 & -3\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix}-2\\-13\end{bmatrix}$.

Solution b:

$$\begin{pmatrix}1\\5\end{pmatrix} = k_1\begin{pmatrix}1\\0\end{pmatrix} + k_2\begin{pmatrix}1\\-1\end{pmatrix}$$

$$T\begin{pmatrix}1\\5\end{pmatrix} = T\left(k_1\begin{pmatrix}1\\0\end{pmatrix} + k_2\begin{pmatrix}1\\-1\end{pmatrix} \right)$$

$$k_1\begin{pmatrix}1\\0\end{pmatrix} + k_2\begin{pmatrix}1\\-1\end{pmatrix}$$

$$T\begin{pmatrix}1\\5\end{pmatrix} = k_1 T\begin{pmatrix}1\\0\end{pmatrix} + k_2 T\begin{pmatrix}1\\-1\end{pmatrix}$$

$$k_1 + k_2 = 1 \qquad k_1 = 6$$
$$-k_2 = 5 \qquad k_2 = -5$$

$$T\begin{pmatrix}1\\5\end{pmatrix} = k_1\begin{bmatrix}3\\2\end{bmatrix} + k_2\begin{bmatrix}4\\5\end{bmatrix}$$

$$T\begin{pmatrix}1\\5\end{pmatrix} = 6\begin{bmatrix}3\\2\end{bmatrix} + -5\begin{bmatrix}4\\5\end{bmatrix}$$

$$T\begin{pmatrix}1\\5\end{pmatrix} = \begin{bmatrix}18\\12\end{bmatrix} + \begin{bmatrix}-20\\-25\end{bmatrix} = \begin{bmatrix}-2\\-13\end{bmatrix}$$

Solution c:

$$T\begin{pmatrix}1\\5\end{pmatrix} = \begin{bmatrix}3 & 4\\2 & 5\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}3 & 4\\2 & 5\end{bmatrix}\begin{bmatrix}1 & 1\\0 & -1\end{bmatrix}^{-1}\begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix}-2\\-13\end{bmatrix}$$

Problem 4

6. List 10 Vector Space Axioms.

Solution 4

6.

1) If **u** and **v** are objects in V, then **u** + **v** is in V.

2) **u** + **v** = **v** + **u**

3) **u** + (**v** + **w**) = (**u** + **v**) + **w**

4) There is an object 0 in V, called a zero vector for V, such that **0** + **u** = **u** + **0** = **u** for all **u** in V.

5) For each u in V, there is an object –**u** in V, called a negative of u, such that **u** + (-**u**) = (- **u**) + **u** = 0.

6) If k is any scalar and u is any object in V, then ku is in V.

7) k(**u** + **v**) = k**u** + k**v**

8) (k + m)**u** = k**u** + m**u**

9) k(m**u**) = (km)(**u**)

10) 1**u** = **u**

Problem 5

The linear operator T : R3->R3 is defined by the equations

$$w_1 = 2x_1 + x_2 - x_3$$
$$w_2 = 4x_1 - 3x_2 + x_3$$
$$w_3 = 2x_2 - x_3$$

a) Find the standard matrix [T]

b) Show that this linear operator T is one to one (Hint: Maybe by an equivelent statement)

c) Given **x** = (1,2,3), meaning that $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$,

use the standard matrix [T] to transform $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

d) Show that $[T^{-1}]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ returns **x**

Solution 5

a) Standard matrix is formed from the coefficients of x_1, x_2 and x_3. The equations can be rewritten in matrix form.

$$[T] = \begin{bmatrix} 2 & 1 & -1 \\ 4 & -3 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

b) If we can show that the matrix is invertible, then that, by equivelent statement, means that T is one to one.

Using methods from chapter one, We find the inverse of [T] to be

$$[T^{-1}] = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ -2 & 1 & 3 \\ -4 & 2 & 5 \end{bmatrix}$$ Thus [T] is one to one

c) **w** = [T] **x**

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 4 & -3 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 4 & -3 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ so } \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

d) $[T^{-1}]$ **w** = **x**

$$\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ -2 & 1 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \underset{\text{o.k.}}{\rightarrow} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ -2 & 1 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{x}$$

Problem 6

Theorem 4.3.3 in the book states that

If T: $R^n \to R^m$ is a linear transformation, and $\mathbf{e}_1$, $\mathbf{e}_2$, ...$\mathbf{e}_n$ are the standard basis vectors for R^n, then the standard matrix for T is

$[T] = [T(\mathbf{e}_1)|T(\mathbf{e}_2)|...|T(\mathbf{e}_n)]$

a) Use this theorem to find the standard matrix for the T: $R^2 \to R^2$ that
First reflects a vector about the line y = x
Second reflects that vector about the x-axis and
Finally, projects the vector onto the y-axis.

b) Using the standard matrix, take the vector v = (1,2) through the full transformation.

Solution 6

We will take this problem into a few parts. But first, a brief explanation. $\mathbf{e}_1$ and $\mathbf{e}_2$ are the standard basis vectors for R^2. Thus $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

When we are using this theorem, we need to ask, "What does T do to the vector?" An easy way to see this is to transform the standard basis vectors with T. Then, as per Theorem 4.3.3, these transformed standard basis vectors will combine to form the standard matrix in the form of the equation above.

We will first find the standard matrices for each operation that we are performing. We will call the operations

T_1 = Reflects a vector about the line y = x

T_2 = Reflects that vector about the x-axis

T_3 = Projects the vector onto the y-axis.

$[T_1] = [T_1(\mathbf{e}_1)|T_1(\mathbf{e}_2)]$

First we must determine what T_1 does to $\mathbf{e}_1$. To reflect a vector about the line x = y will transform x values into y values and y values into x values of the resulting vector. Thus $T_1(\mathbf{e}_1) = T_1\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ You can visually see this on a x,y graph. A vector (1,0) reflected about y = x would indeed be (0,1). It follows that $T_1(\mathbf{e}_2) = T_1\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So the form $[T_1] = [T_1(\mathbf{e}_1)|T_1(\mathbf{e}_2)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$[T_2] = [T_2(\mathbf{e}_1)|T_2(\mathbf{e}_2)]$ All values of y will transform to -y. so

$T_2(\mathbf{e}_1) = T_2\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (meaning a vector lying on the x-axis reflected about the x-axis is just that vector)

$$T_2(\mathbf{e}_2) = T_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \; so$$

So the form $[T_2] = [T_2(\mathbf{e}_1)|T_2(\mathbf{e}_2)] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Finally $[T_3] = [T_3(\mathbf{e}_1)|T_3(\mathbf{e}_2)]$ The vector transforms into its y component.

$T_3(\mathbf{e}_1) = T_3\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (meaning a vector lying on the x-axis projected onto the y-axis is just the **0** vector)

$T_3(\mathbf{e}_2) = T_3\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (meaning the y component projected onto the y axis is just the y component)

So the form $[T_3] = [T_3(\mathbf{e}_1)|T_3(\mathbf{e}_2)] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

We have all of the Transformations, we must multiply them in the right order to get the standard matrix T.

Thinking about it $T(\mathbf{x}) = [\mathrm{T}]\mathbf{x}$. We must multiply the **x** vector by the first operation first so $[T_1]$ will be closest to **x** (meaning farthest to the right close to **x**) followed by $[T_2]$, then $[T_3]$

So $[T] = [T_3][T_2][T_1] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

b) $T(\mathbf{x}) = [\mathrm{T}]\mathbf{x} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ This is graphically and intuitively what we would expect.

Shown another way below.

$$[T]\mathbf{x} = [T_3][T_2][T_1]\mathbf{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Problem 7 :

In each part of this problem, a set of objects is given, together with the operations of addition and scalar multiplication. Determine which sets are vector spaces under the given operations. For those that are not vector spaces, list all axioms that fail to hold.

a)The set of all pairs of real numbers (x, y) with the operations

(x, y) + (x', y') = (x + x', y + y') and k (x, y) = (2kx, 2ky)

Solution:
k(m (x, y)) = k(2mx, 2my) = (2•2mx, 2•2my) = (4mx, 4my)

(km)(x,y)

k ~~(mx, my)~~ = (2kmx, 2kmy) Axiom 9 fails.

1 (x,y) = (2x, 2y) Axiom 10 fails.

Therefore, this is not a vector space.

b)The set of all triples of real numbers (x, y, z,) with the operations

(x, y, z) + (x', y', z') = (x + x', y + y', z + z') and k (x, y, z) = (kx ,ky ,kz)

Solution:
This is a vector space under the given operations. Axioms 1 and 6 are satisfied.

c)The set of all 2 x 2 matrices of the form

$$\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix}$$

with standard matrix addition and scalar multiplication.

Solution:

$$\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} + \begin{bmatrix} a' & a'+b' \\ a'+b' & b' \end{bmatrix} = \begin{bmatrix} a+a' & (a+a')+(b+b') \\ (a+a')+(b+b') & b+b' \end{bmatrix}$$

$$k\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} = \begin{bmatrix} ka & ka+kb \\ ka+kb & kb \end{bmatrix}$$

This is a vector space under the given operations. Axioms 1 and 6 are satisfied.

Problem 8

8. Determine whether the set of vectors S = {V_1, V_2, V_3} is linearly dependent or independent.

a. ,where V_1 = (2, -1, 0, 3), V_2 = (1, 2, 5, -1), V_3 = (7, -1, 5, 8) $\in \mathbb{R}^4$.

b. ,where V_1 = (0, 0, a, a), V_2 = (b, b, 0, 0), V_3 = (c, c, 0, d) <a, b, c are not zero.>

c. ,where V_1 = (0, 3, 1, -1), V_2 = (6, 0, 5, 1), V_3 = (4, -7, 1, 3)

Solution 8

8. a. According to the theorem 5.3.3,

Let S = {$v_1, v_2, \ldots, v_r$} be a set of vectors in R^n. If $\overset{3}{\overset{\parallel}{r}} > \overset{4}{\overset{\parallel}{n}}$, then S is linearly dependent.

The set of vectors is linearly dependent. – why? I don't get it.

b. $k_1v_1 + k_2v_2 + k_3v_3 = 0$

what is a?

k_1*a = 0. Since a is not zero, k_1 is zero. When k_3 is zero, we know , k_2*b is zero. b is not zero thus k_2 is zero. Therefore, k_3*d is zero, and d becomes zero ? what is b?

c. 6b + 4c = 0 .... b = (3/2)*c – what are you doing?
3a – 7c = 0 .... a = (7/3)*c
a + 5b + c = 0
-a + b +3c = 0

There are more solutions than a=b=c=0 therefore, the set of vectors is linearly Dependent.

Problem 9

9.

If S = {v_1, v_2, ….., v_r} and S\` = {$w_1$, $w_2$,…….. ,$w_k$} are two sets of vectors in a vector space V, then  Span{$v_1$, $v_2$, ….., $v_r$} = Span{$w_1$, $w_2$,…….. ,$w_k$}  If and only if each vector in S is a linear combination of those in S\` and each vector in S\` is a linear combination of those in S.

Use the theorem above to show that v_1 = (1, 6, 4) v_2 = (2, 4, -1) v_3 = (2, 4, -1), and w_1 = (1, -2, -5), w_2 = (0, 8, 9) span the same subspace of R^3 .

Solution 9

Explain what you're doing.

9. a + 2b + 2c = 1 , 0
   6a + 4b +4c = -2 , 8
   4a –b –c = -5 , 9

(a, b, c) = (-1, b, 1-b), (2, b, -1-b)

a = 1, 2, 4
-2a + 8b = 6, 4, 4
-5a + 9b = 4, -1, -1

(a, b) = (1,1), (2, 1), (3/2, 19/9)

Each vector in the first set of vectors is a linear combination of those in the second, and each vector in the second set of vectors is a linear combination of those in the first. Therefore, these two sets of vectors span the same subspace of $R^{3.}$

Problem 10: The Wronski Problem

Show whether the following set of vectors in F(-∞, ∞) is linearly dependent or linearly independent.

$f_1 = \pi$
$f_2 = \sin(x)$
$f_3 = \cos(x)$
$f_4 = e^x$

Solution 10

$$W(x) = \begin{bmatrix} \pi & \sin(x) & \cos(x) & e^x \\ 0 & \cos(x) & -\sin(x) & e^x \\ 0 & -\sin(x) & -\cos(x) & e^x \\ 0 & -\cos(x) & \sin(x) & e^x \end{bmatrix} = \pi \; det \begin{bmatrix} \cos(x) & -\sin(x) & e^x \\ -\sin(x) & -\cos(x) & e^x \\ -\cos(x) & \sin(x) & e^x \end{bmatrix} =$$

expansion on column 1

$$= \pi \; det \begin{bmatrix} \cos(x) & -\sin(x) & e^x \\ -\sin(x) & -\cos(x) & e^x \\ 0 & 0 & 2e^x \end{bmatrix} = \pi \; 2e^x \left(-\cos^2(x) - \sin^2(x)\right)$$

add 1 times row 1 to 3 expansion on row 3

$$= \pi \; 2e^x \; -(1) = -2\pi e^x$$

det W(x) = $-2\pi e^x \neq 0$ therefore linearly independent. make better sentences.

1) Answer the following questions (3 points each):

a) What is the rank of a matrix?
b) What is the nullity of a matrix?
c) How are the rank and nullity related to the columns of a matrix?

Solution:

a) The rank is the common dimension of the row space and column space of a matrix.
b) The nullity is the dimension of the nullspace of a matrix.
c) The nullity and the rank added together equal the number of columns in a matrix (rank + nullity = n).

2) For matrix A, find bases for the row and column spaces (20 points):

$$A = \begin{pmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 13 & 5 \\ 2 & 4 & 6 & 8 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Solution:

"R1" denotes "Row 1" / Obtain the row-echelon form.

$$\begin{pmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 13 & 5 \\ 2 & 4 & 6 & 8 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$ (R1) – (R4), (R2) – 2 times (R4)

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 1 & 13 & 5 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$ Swap (R3) and (R4)

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 1 & 13 & 5 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ Swap (R3) and (R1)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 13 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{Row-Echelon form}$$

Basis for Row Space (the rows with a leading one):

R1 = (1 2 3 4)

R2 = (0 1 13 5)

R3 = (0 0 1 3)

— I don't think this works always – use transpose and talk about columns.

Basis for Column Space (the original columns corresponding to the columns in the row-echelon form with a leading one):

$$C1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \quad C2 = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \end{pmatrix} \quad C3 = \begin{pmatrix} 4 \\ 13 \\ 6 \\ 3 \end{pmatrix}$$

3) Determine the rank and nullity of matrix B (20 points):

$$B = \begin{pmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 9 & 27 \\ 2 & 6 & 18 & 0 \\ 4 & 8 & 12 & 36 \end{pmatrix}$$

Solution:

"R1" denotes "Row 1" / Obtain the reduced row-echelon form.

$$\begin{pmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 9 & 27 \\ 2 & 6 & 18 & 0 \\ 4 & 8 & 12 & 36 \end{pmatrix} \qquad \text{(R3) – 2 times (R1), (R4) – 4 times (R1)}$$

$$\begin{pmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 9 & 27 \\ 0 & 2 & 12 & -18 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(R3) – 2 times (R2)}$$

$$\begin{pmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 9 & 27 \\ 0 & 0 & -6 & -72 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(R3) times (-1/6)}$$

$$\begin{pmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 9 & 27 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ (R1) – 3 times (R3), (R2) – 9 times (R3)

$$\begin{pmatrix} 1 & 2 & 0 & -27 \\ 0 & 1 & 0 & -81 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ (R1) – 2 times (R2)

$$\begin{pmatrix} 1 & 0 & 0 & 135 \\ 0 & 1 & 0 & -81 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ Reduced Row-Echelon Form

Doing these operations is the same as solving the equation Bx = 0
The rank is the number of leading variables (3) and the nullity is the number of parameters (1) in the general solution.
Rank = 3
Nullity = 1

4) What conditions need to hold for a set of vectors J = {v1, v2, . . .,vn} to be a basis in a vector space V (8 points)?

Solution:

By definition, if V is a vector space and J = {v1, v2, . . . vn} is a set of vectors in V, it follows that J is a basis for V if two conditions are true:

1) J is linearly independent
2) J spans V

5) a. What is the definition of the Euclidean Length of Norm? (10 pts)

b. Let v = (1, -3, 0, 2), and u = (2, 0, 1, 1). Find 2v + u / v ?

Solution:

a) ||v|| := (v1² + v2² + . . . + vn²)^½ V = ||v|| ?

b) 2v + u / v = (2v + u) / v = [[(2² + (-6)² + 0² + 4²)^ ½ + (2² + 0² + 1² + 1²)^ ½] / (1² + (-3)² + 0² + 2²)^ ½] = 2.65

what are you doing?

6) Find the standard matrix for the stated composition of Linear operators on R^3. (15 pts)
A rotation 30° about the y-axis, followed by a rotation 60° about the z-axis, followed by a dilation with factor k = 2.

Solution:

(y-axis)

$$\begin{bmatrix} \cos 30^\circ & 0 & \sin 30^\circ \\ 0 & 1 & 0 \\ -\sin 30^\circ & 0 & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \sqrt{(3)}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{(3)}/2 \end{bmatrix}$$

(z-axis)

$$\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 \\ \sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{(3)}/2 & 0 \\ \sqrt{(3)}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Take the product of the two matrices to get:

$$\begin{bmatrix} \sqrt{(3)}/4 & -3/4 & 1/2 \\ \sqrt{(3)}/2 & 1/2 & 0 \\ -1/4 & \sqrt{(3)}/4 & \sqrt{(3)}/2 \end{bmatrix}$$

which order! very important!

Then multiply the matrix by the dilation factor k = 2.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The final answer is:

$$\begin{bmatrix} \sqrt{(3)}/2 & -3/2 & 1 \\ \sqrt{3} & 1 & 0 \\ -1/2 & \sqrt{(3)}/2 & \sqrt{3} \end{bmatrix}.$$

7) Determine the dimension of and a basis for the solution space of the system: (8 pts)

$$w - 3x + 4y + z = 0$$
$$2w + y - 3z = 0$$
$$2w - 6x + 8y + 2z = 0$$

Solution:

make sentences.

$$\begin{bmatrix} 1 & -3 & 4 & 1 \\ 2 & 0 & 1 & -3 \\ 2 & -6 & 8 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & 1 \\ 0 & 6 & -7 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$w = 3x - 4y - 5z$
$x = 7/6y + 5/6z$
$y = s$
$z = t$

[w]   [-1/2s - 5/2t]   [-1/2]   [-5/2]

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7/6s + 5/6t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 7/6 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 5/6 \\ 0 \\ 1 \end{bmatrix}.$$

These two vectors are the basis, and the dimension is 2.

8) Determine whether or not the following set is a vector space and list the axioms that fail if any exist (2 points per axiom, 10 for correctly stating if it's a vector space):
The set S of all triples of real numbers (x,y,z) with the operations

$u + v = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_2, u_2 + v_3, u_3 + v_1)$ and

$ku = k(u_1, u_2, u_3) = (ku_1, ku_2, ku_3)$

Solution:

Let $u, v, w \in S, k \in \mathbb{R}$

1. $u + v = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_2, u_2 + v_3, u_3 + v_1) \in S$
2. $u + v = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_2, u_2 + v_3, u_3 + v_1)$
$\neq (v_1 + u_2, v_2 + u_3, v_3 + u_1) = (v_1, v_2, v_3) + (u_1, u_2, u_3) = v + u$
Fails!
3. $u + (v + w) = (u_1, u_2, u_3) + ((v_1, v_2, v_3) + (w_1, w_2, w_3))$
$= (u_1, u_2, u_3) + (v_1 + w_2, v_2 + w_3, v_3 + w_1)$
$= (u_1 + (v_2 + w_3), u_2 + (v_3 + w_1), u_3 + (v_1 + w_2))$
$= (u_1 + v_2 + w_3, u_2 + v_3 + w_1, u_3 + v_1 + w_2)$
$= ((u_1 + v_2) + w_3, (u_2 + v_3) + w_1, (u_3 + v_1) + w_2)$
$= (u_1 + v_2, u_2 + v_3, u_3 + v_1) + (w_1, w_2, w_3)$
$= ((u_1, u_2, u_3) + (v_1, v_2, v_3)) + (w_1, w_2, w_3)$
$= (u + v) + w$
4. Suppose $\exists$ $b \in S$ such that $b + u = u + b = u$
$u + b = (u_1, u_2, u_3) + (b_1, b_2, b_3)$
$= (u_1 + b_2, u_2 + b_3, u_3 + b_1)$
$= (b_1 + u_2, b_2 + u_3, b_3 + u_1)$
$= (u_1, u_2, u_3)$
$\Rightarrow u_1 + b_2 = b_1 + u_2 = u_1$
$\Rightarrow b_1 = u_1 + b_2 - u_2,\ b_2 = b_1 + u_2 - u_1 = 0$
$\Rightarrow b_1 = u_1 - u_2,\ b_2 = u_2 - u_1 = 0$
$\Rightarrow u_1 = u_2$, a restriction on u
$\Rightarrow$ our assumption was false
Fails!
5. Failure implied by 4 (zero vector does not exist)
Fails!
6. $ku = k(u_1, u_2, u_3) \neq (ku_1, k^2 u_2, k^3 u_3)$. $ku_1, k^2 u_2, k^3 u_3 \in \mathbb{R} \Rightarrow ku \in S$

7.

$$k(u+v)=k((u_1,u_2,u_3)+(v_1,v_2,v_3))=k(u_1+v_2,u_2+v_3,u_3+v_1)$$
$$\cancel{\neq}(k(u_1+v_2),k^2(u_2+v_3),k^3(u_3+v_1))$$
$$=(ku_1+kv_2,k^2u_2+k^2v_3,k^3u_3+k^3v_1)$$
$$\neq(ku_1+k^2v_2,k^2u_2+k^3v_3,k^3u_3+kv_1)$$
$$=(ku_1,k^2u_2,k^3u_3)+(kv_1,k^2v_2,k^3v_3)$$
$$=k(u_1,ku_2,ku_3)+k(v_1,v_2,v_3)$$
$$=ku+kv$$

Fails!

8.

$$(k+l)u=(k+l)(u_1,u_2,u_3)\cancel{\neq}((k+l)u_1,(k+l)^2u_2,(k+l)^3u_3)$$
$$\neq(ku_1+l^2u_2,k^2u_2+l^3u_3,k^3u_3+lu_1)$$
$$=(ku_1,k^2u_2,k^3u_3)+(lu_1,l^2u_2,l^3u_3)=ku+lu$$

Fails!

9.

$$k(lu)=k(l(u_1,u_2,u_3))=k(lu_1,l^2u_2,l^3u_3)\cancel{\neq}(k(lu_1),k^2(l^2u_2),k^2(l^2u_3))$$
$$=(klu_1,k^2l^2u_2,k^3l^3u_3)=((kl)u_1,(kl)^2u_2,(kl)^3u_3)$$
$$=(kl)(u_1,u_2,u_3)=(kl)u$$

10. $1u=1(u_1,u_2,u_3)=(1u_1,1^2u_2,1^3u_3)=(u_1,u_2,u_3)$

Not a vector space: fails axioms 2, 4, 5, 7, and 8

9) Determine whether or not the following is a subspace of M_{33}. If it is not, explain why. (15 points)

The set S containing all 3x3 matrices A such that det(A)=0.

Solution:

Check closure of addition and scalar multiplication:

Let $A, B \in S$, $k \in \mathbb{R}$

Addition:

$$\det(A+B)\neq\det(A)+\det(B).$$ Fails

show counter example.

Scalar Multiplication:

$$\det(kA)=k^3\det(A)=k^3(0)=0$$

Not a subspace. Fails closure over addition.

10) Determine whether or not the following vectors form a basis for $\mathbb{R}^3$. If not, explain why. (15 points)

(1,3,5), (2,4,6), (3,5,7)

Solution:

Check linear independence or spanning (Both are true if the determinant of the coefficient matrix is nonzero).

Linear Independence:

$$k_1(1,3,5)+k_2(2,4,6)+k_3(3,5,7)$$
$$=(k_{1,}3k_{1,}3k_5)+(2k_{2,}4k_{2,}6k_2)+(3k_{3,}5k_{3,}7k_3)$$
$$=(k_1+2k_2+3k_{3,}3k_1+4k_2+5k_{3,}5k_1+6k_2+7k_3)$$
$$\Rightarrow\begin{vmatrix}1 & 2 & 3\\ 3 & 4 & 5\\ 5 & 6 & 7\end{vmatrix}=0\Rightarrow \text{not a basis}$$

1. Find the standard matrix of the orthogonal projection on the xz-plane followed by a rotation about the z axis through an angle of 90° and followed by a dilation of k=3.

Solution:

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}\cdot 3 = 3\cdot\begin{bmatrix}\cos\theta&-\sin\theta&0\\0&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}3\cos\theta&-3\sin\theta&0\\0&0&0\\0&0&1\end{bmatrix}$$

Now place 90° into θ to get the final standard matrix

$$\begin{bmatrix}0&-3&0\\0&0&0\\0&0&1\end{bmatrix}$$ probably not

2. Which of the following are linear combinations of **u**=(0,8,4), **v**=(-1,9,8)

a) (1,-1,-4)
b) (25,0,1)
c) (3,-11,-16)

combine into one aug. matrix.

Solution:

First we must make a system of linear equations for a b and c.

a) $0k_0 - k_1 = 1$
$8k_0 + 9k_1 = -1$
$4k_0 + 8k_1 = -4$

b) $0k_0 - k_1 = 25$
$8k_0 + 9k_1 = 0$
$4k_0 + 8k_1 = 1$

c) $0k_0 - k_1 = 3$
$8k_0 + 9k_1 = -11$
$4k_0 + 8k_1 = -16$

We then convert them to matrices and solve.

a) $$\begin{bmatrix}0&-1&1\\8&9&-1\\4&8&4\end{bmatrix}\Rightarrow\begin{bmatrix}1&2&-1\\0&-1&1\\8&9&-1\end{bmatrix}\Rightarrow\begin{bmatrix}1&2&-1\\0&-1&1\\0&-7&7\end{bmatrix}\Rightarrow\begin{bmatrix}1&2&-1\\0&-1&1\\0&0&0\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&1\\0&1&-1\\0&0&0\end{bmatrix}$$

Thus $k_0 = 1$ and $k_1 = -1$ and so it is a linear combination

We then convert them to matrices and solve.

b) $$\begin{bmatrix}0&-1&25\\8&9&0\\4&8&1\end{bmatrix}\Rightarrow\begin{bmatrix}1&2&1/4\\0&-1&25\\8&9&0\end{bmatrix}\Rightarrow\begin{bmatrix}1&2&1/4\\0&-1&25\\0&-7&-2\end{bmatrix}\Rightarrow\begin{bmatrix}1&2&1/4\\0&-1&25\\0&0&-177\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&\frac{-199}{4}\\0&-1&-25\\0&0&-177\end{bmatrix}$$

Thus 0 does not equal -177 and so it is NOT a linear combination

We then convert them to matrices and solve.

c) $$\begin{bmatrix}0&-1&3\\8&9&-11\\4&8&-16\end{bmatrix}\Rightarrow\begin{bmatrix}1&2&-4\\0&-1&3\\8&9&-11\end{bmatrix}\Rightarrow\begin{bmatrix}1&2&-4\\0&-1&-3\\0&-7&21\end{bmatrix}\Rightarrow\begin{bmatrix}1&2&-4\\0&-1&3\\0&-1&3\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&2\\0&1&-3\\0&0&0\end{bmatrix}$$

Thus $k_0 = 2$ and $k_1 = -3$ and so it is a linear combination

So a and c are linear combinations of **u** and **v**.

solution set of

3. Determine the dimension and basis of the following system of equations:

$2x_1 + 2x_2 - 2x_3 + 4x_4 = 0$
$6x_2 - 2x_3 + 3x_4 = 0$
$x_1 - 2x_2 - x_3 + 2x_4 = 0$
$-2x_1 + 4x_2 + 2x_3 - 4x_4 = 0$

Solution:

First we must determine the solution of the system of equations, so first place it into matrix form.

$$\begin{bmatrix} 2 & 2 & -2 & 4 & 0 \\ 0 & 6 & -2 & 3 & 0 \\ 1 & -2 & -1 & 2 & 0 \\ -2 & 4 & 2 & -4 & 0 \end{bmatrix} \begin{matrix} R3*2+R4 \\ R1/2 \\ R1*-1/2+R3 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & 2 & 0 \\ 0 & 6 & -2 & 3 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R3*2+R2 \Rightarrow$$

$$\begin{bmatrix} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R2 \leftrightarrow R3 \\ R3/3+R1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & -\frac{2}{3} & 2 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R2/3 \\ R3/3 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & -\frac{2}{3} & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We then solve for x and get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}s \\ \frac{1}{3}s \\ s \\ 0 \end{bmatrix} = s\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{bmatrix}$$ Which is the basis $\left(\frac{2}{3}, \frac{1}{3}, 1, 0\right)$ and since we only have one basis it is R^1.

4. The following set of objects is not a vector space under the given operations of addition and scalar multiplication. Lists all axioms that do not hold for this set.

The set of all pairs of real numbers (x, y) with the operations

$$(x,y)+(x',y') = \left(\tfrac{1}{x}+\tfrac{1}{x'}, \tfrac{1}{y}+\tfrac{1}{y'}\right) \text{ and } k(x,y) = (kx, ky)$$

Solution:

Axiom 3 does not hold because $$(a,b)+((c,d)+(e,f)) = \left(\frac{1}{a}+\frac{1}{\frac{1}{c}+\frac{1}{e}}, \frac{1}{b}+\frac{1}{\frac{1}{d}+\frac{1}{f}}\right)$$

and $$((a,b)+(c,d))+(e,f) = \left(\frac{1}{e}+\frac{1}{\frac{1}{a}+\frac{1}{c}}, \frac{1}{f}+\frac{1}{\frac{1}{b}+\frac{1}{d}}\right)$$

Axiom 4 does not hold because the only object that could be added to any (a,b) to give (a,b) would be (∞,∞) which is not in the set because ∞ is not a real number.

Axiom 7 does not hold because $\quad k((a,b)+(c,d)) = \left(\frac{k}{a}+\frac{k}{c}, \frac{k}{b}+\frac{k}{d}\right)$

but $\quad k(a,b)+k(c,d) = \left(\frac{1}{ka}+\frac{1}{kc}, \frac{1}{kb}+\frac{1}{kd}\right)$

Axiom 8 does not hold because $\quad (k+m)(a,b) = ((k+m)a,(k+m)b)$

but $\quad k(a,b)+m(a,b) = \left(\frac{1}{ka}+\frac{1}{ma}, \frac{1}{kb}+\frac{1}{mb}\right)$

Axioms 1, 2, 5, 6, 9, and 10 hold.

5. Let A**x**=**b** be the linear system

$$\begin{bmatrix} 1 & 2 & 6 \\ 2 & 8 & 4 \\ 3 & 9 & 15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix}$$

Show that **b** is in the column space of A and express **b** as a linear combination of the column vectors of A.

Solution:

First, solve the system by Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 2 & 6 & 4 \\ 0 & 1 & -2 & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{-3}{4} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & \frac{-7}{4} \\ 0 & 0 & 1 & \frac{-3}{4} \end{array}\right]$$

$x_1 = 12, x_2 = \frac{-7}{4}, x_3 = \frac{-3}{4}$

Since the system is consistent, **b** is in the column space of A. From the solution obtained, it follows that

$$\mathbf{b} = 12\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{7}{4}\begin{bmatrix} 2 \\ 8 \\ 9 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 6 \\ 4 \\ 15 \end{bmatrix}$$

6. Use the Wronskian to show that the following set of vectors ~~are~~ is linearly independent:

1, 2x, x^4, e^x

Solution:

$$W(x)=\begin{bmatrix} 1 & 2x & x^4 & e^x \\ 0 & 2 & 4x^3 & e^x \\ 0 & 0 & 12x^2 & e^x \\ 0 & 0 & 24x & e^x \end{bmatrix} \approx \begin{bmatrix} 1 & 2x & x^4 & e^x \\ 0 & 2 & 4x^3 & e^x \\ 0 & 0 & 12x^2 & e^x \\ 0 & 0 & 0 & (1-\frac{2}{x})e^x \end{bmatrix}$$

$$\det W(x)=1(2)(12x^2)(1-(2/x))e^x$$

Let x=1, then detW(x)= -65.2, which is a nonzero value, thus there exist a value of x that causes the set to only have trivial solutions for all values of x, and therefore this set is linearly independent.

7. Find the rank and nullity of matrix A:

$$A=\begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 5 & 5 \\ 3 & -1 & 8 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 5 & 5 \\ 3 & -1 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 7 \\ 3 & -1 & 8 \\ 0 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ since there are

two leading variable rank(A)=2

$x_1+3x_3=0$ $\quad$ $x_1 = -3x_3$ $\quad$ Parameters: $x_1=-3s$

$x_2+x_3=0$ $\quad$ $x_2 = -x_3$ $\quad$ $x_2=-s$

$x_3=s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3s \\ -s \\ s \end{bmatrix} = s\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

because there is only one vector here that forms the basis, the nullity(A)=1

Dimension Theorem for Matrices: rank(a) + nullity(A) =n(number of columns)

2 + 1 = 3

8. Find the Euclidean inner product: u•v

u = (2,4,1,0,8) v = (-5,7,0,-6,3)

Solution:

u•v = (2)(-5) + (4)(7) + (1)(0) + (0)(-6) + (8)(3)
= -10 +28 + 0 + 0 + 24 = 42

9. Use matrix multiplication to find the image of the vector (3,11) when it is rotated through an angle of $\Theta = 225^{\circ}$

Solution:

$$\begin{matrix}(\cos 225^{\circ})(3) - (\sin 225^{\circ})(11) \\ (\sin 225^{\circ})(3) + (\cos 225^{\circ})(11)\end{matrix}$$

$$\begin{matrix}(-\frac{\sqrt{2}}{2})(3) - (-\frac{\sqrt{2}}{2})(11) \\ (-\frac{\sqrt{2}}{2})(3) + (-\frac{\sqrt{2}}{2})(11)\end{matrix}$$

$$\begin{bmatrix} 4\sqrt{2} \\ -7\sqrt{2} \end{bmatrix}$$

10. Determine whether multiplication by A is a one-to-one linear transformation

$$A = \begin{bmatrix} 2 & -2 \\ 4 & 0 \\ 6 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2 & -2 \\ 4 & 0 \\ 6 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 & -2x_2 \\ 4x_1 & 0 \\ 6x_1 & -8x_2 \end{bmatrix}$$

true   but this has little to do with it.

$R^2 \rightarrow R^3$ Linear Transformation, one-to-one (more equations than unknowns)

1. Consider the vectors $\mathbf{u} = (1,2,3), \mathbf{v} = (I, \heartsuit, U)$ and $\mathbf{w} = (7,4,7)$. Evaluate:

   (a) $\mathbf{u} \cdot \mathbf{v}$

   (b) $\|\mathbf{v}\|$

   (c) $\|\mathbf{u} + \mathbf{w}\|$

   (d) $\|\mathbf{w}\|^2 + \|\mathbf{u}\|^2$

   (e) $-2\mathbf{u} + 3\mathbf{w}$.

   Solution

   (a) $\mathbf{u} \cdot \mathbf{v} = 1 * I + 2 * \heartsuit + 3 * U$.

   (b) $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{I^2 + \heartsuit^2 + U^2}$.

   (c) $\|\mathbf{u} + \mathbf{w}\| = \|(1,2,3) + (7,4,7)\| = \|(8,6,10)\| = \sqrt{64 + 36 + 100} = 10\sqrt{2}$.

   (d) $\|\mathbf{w}\|^2 + \|\mathbf{u}\|^2 = \sqrt{7^2 + 4^2 + 7^2}^2 + \sqrt{1^2 + 2^2 + 3^2}^2 = 114 + 14 = 128$.

   (e) $-2\mathbf{u} + 3\mathbf{w} = -2(1,2,3) + 3(7,4,7) = (-2,-4,-6) + (21,12,21) = (19,8,15)$.

2. Consider two vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} \cdot \mathbf{v} = 0$ and $\|\mathbf{u} + \mathbf{v}\| = 4$. Find $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.
   Solution As $\mathbf{u} \cdot \mathbf{v} = 0$, they are orthogonal. Then we know that

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2 = 4^2 = 16.$$

3. Find the eigenvalues and their corresponding eigenvectors of an orthogonal projection on the yz-plane, which has standard matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Solution The formula for the eigenvalues is $\det(\lambda I - T_A) = 0$, or

$$\det \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 1 \end{bmatrix} = 0 \Rightarrow \lambda(\lambda - 1)^2 = 0 \Rightarrow \lambda = 0 \text{ or } 1.$$

   To find the eigenvectors corresponding to $\lambda = 0$, solve the equation

$$\begin{bmatrix} 0-0 & 0 & 0 \\ 0 & 0-1 & 0 \\ 0 & 0 & 0-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   to obtain the result

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}.$$

To find the eigenvectors corresponding to $\lambda = 1$, solve the system

$$\begin{bmatrix} 1-0 & 0 & 0 \\ 0 & 1-1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

to obtain the result

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ s \\ t \end{bmatrix}.$$

4. Consider the set S of matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with addition defined as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa' & bb' \\ cc' & dd' \end{bmatrix};$$

and the usual scalar multiplication. Show whether this set qualifies as a vector space by testing each of the 10 axioms. [*Ed.* Yes, this question is cruel and unusual. I didn't write it.]

Solution

(1)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa' & bb' \\ cc' & dd' \end{bmatrix} \in S.$$

(2)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa' & bb' \\ cc' & dd' \end{bmatrix}$$

$$\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a'a & b'b \\ c'c & d'd \end{bmatrix} = \begin{bmatrix} aa' & bb' \\ cc' & dd' \end{bmatrix}.$$

(3)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} \right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a'a'' & b'b'' \\ c'c'' & d'd'' \end{bmatrix} = \begin{bmatrix} aa'a'' & bb'b'' \\ cc'c'' & dd'd'' \end{bmatrix};$$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \right) + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} = \begin{bmatrix} aa' & bb' \\ cc' & dd' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} = \begin{bmatrix} aa'a'' & bb'b'' \\ cc'c'' & dd'd'' \end{bmatrix}.$$

(4)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(5)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} \frac{1}{a} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{d} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

— still fails since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ but has no "negative"

(6)

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \in S.$$

(7)

$$k\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) = k\begin{bmatrix} aa' & bb' \\ cc' & dd' \end{bmatrix} = \begin{bmatrix} kaa' & kbb' \\ kcc' & kdd' \end{bmatrix};$$

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} + k\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} + \begin{bmatrix} ka' & kb' \\ kc' & kd' \end{bmatrix} = \begin{bmatrix} k^2aa' & k^2bb' \\ k^2cc' & k^2dd' \end{bmatrix} \neq \begin{bmatrix} kaa' & kbb' \\ kcc' & kdd' \end{bmatrix}$$

Therefore this axiom fails.

(8)

$$(k+m)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (k+m)a & (k+m)b \\ (k+m)c & (k+m)d \end{bmatrix};$$

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} + m\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} kma^2 & kmb^2 \\ kmc^2 & kmd^2 \end{bmatrix}.$$

Thus this axiom also fails.

(9)

$$k\left(m\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = k\begin{bmatrix} ma & mb \\ mc & md \end{bmatrix} = \begin{bmatrix} kma & kmb \\ kmc & kmd \end{bmatrix};$$

$$(km)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} kma & kmb \\ kmc & kmd \end{bmatrix}.$$

(10)

$$1\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a & 1b \\ 1c & 1d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

As axioms 7 and 8 fail, this set is not a vector space.

5. (a) Let W be the set of all matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ such that $b = a + c$. For the usual definitions of "+" and "·", and real scalars, show that W is or is not a subspace of M_{22}.

(b) If W' is the set of all matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ such that $b = a + c > 0$, and with usual "+" and "·" and real scalars, is W' a subspace of M_{22}?

Solution

(a) For W to be a subspace of M_{22}, we must show that axioms 1 and 6 hold, *i.e.*, that W is closed under addition and multiplication.

Consider $\mathbf{u} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & a+c \\ 0 & c \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} d & d+f \\ 0 & f \end{bmatrix}$. Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a & a+c \\ 0 & c \end{bmatrix} + \begin{bmatrix} d & d+f \\ 0 & f \end{bmatrix} = \begin{bmatrix} a+d & (a+d)+(c+f) \\ 0 & c+f \end{bmatrix}.$$

This is in the correct form, so $\mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u} + \mathbf{v} \in W$, and W is closed over addition.

$$k\mathbf{u} = k\begin{bmatrix} a & a+c \\ 0 & c \end{bmatrix} = \begin{bmatrix} ka & k(a+c) \\ 0 & kc \end{bmatrix} = \begin{bmatrix} ka & ka+kc \\ 0 & kc \end{bmatrix},$$

which is also of the correct form, so W is also closed over scalar multiplication. Thus, W is a subspace of M_{22}.

(b) If we consider the same $\mathbf{u}$ and $\mathbf{v}$ as in the solution to (a), it is clear that $\mathbf{u} + \mathbf{v}$ is of the correct form, so that W' is closed under addition.

However, if $k \leq 0$ then $ka + kc \leq 0$, so that $k\mathbf{u} \notin W'$ and W' is not closed over multiplication; thus, W' is not a subspace of M_{22}.

6. (a) Determine whether the matrices $\mathbf{m} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{n} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$, and $\mathbf{q} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ span M_{22}.

(b) Do the matrices $\mathbf{r} = \begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix}$, $\mathbf{s} = \begin{bmatrix} 0 & 2 \\ 3 & 3 \end{bmatrix}$, $\mathbf{t} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$ span M_{22}?

Solution

(a) An arbitrary vector $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as a linear combination of the above matrics iff they span M_{22}:

$$k_1\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + k_2\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + k_3\begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix} + k_4\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

This can be represented by the augmented matrix

$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & 1 & a \\ 1 & 0 & 2 & 1 & b \\ 0 & 1 & 2 & 2 & c \\ 1 & 1 & 2 & 1 & d \end{array}\right],$$

and the system will only be consistent if the left side of the partition has a non-zero determinant. Cofactor expansion along the first row provides an~~y~~ efficient way to calculate this determinant:

$$\det\begin{bmatrix} 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix} = 2\det\begin{bmatrix} 0 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix} - 0 + 0 - \det\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} = 2*2-(-2) = 6.$$

As this determinant is not equal to 0, the system is consistent and the matrices span M_{22}.

(b) The matrices **r**, **s**, **t**, and **u** do span M_{22}: Each one can be expressed as a linear combination of **m**, **n**, **p**, and **q**, and vice versa, so the spans of the two sets of vectors are the same.

$$\begin{aligned} \mathbf{r} &= 2\mathbf{m} \quad \Rightarrow \mathbf{m} = \tfrac{1}{2}\mathbf{r}; & \mathbf{s} &= \mathbf{n} + \mathbf{p} \quad \Rightarrow \mathbf{n} = \mathbf{s} - \mathbf{p} = \mathbf{s} - 2\mathbf{t} \\ \mathbf{t} &= \tfrac{1}{2}\mathbf{p} \quad \Rightarrow \mathbf{p} = 2\mathbf{t}; & \mathbf{u} &= \mathbf{q} - \mathbf{m} \quad \Rightarrow \mathbf{q} = \mathbf{m} + \mathbf{u} = \tfrac{1}{2}\mathbf{r} + \mathbf{u}. \end{aligned}$$

7. Use the Wronskian to determine whether $f_1 = 4, f_2 = -4x, f_3 = 2x^2$ are linearly independent.

Solution Set up the matrix

$$W = \begin{bmatrix} 4 & -4x & 2x^2 \\ 0 & -4 & 4x \\ 0 & 0 & 4 \end{bmatrix}$$

and take it's determinant by multiplying entries along the main diagonal, as it is upper-triangular. Then $\det W = (4)(-4)(4) = -64 \neq 0$. Becase the determinant of the Wronskian is not zero, we know that f_1, f_2, and f_3 are linearly independent.

8. In each of the following, select the set which is not a basis of R^n for some integer n.

(a) $\{(1,0),(0,3)\}$; $\{(5,2),(2,5)\}$; $\{(7,2),(-7,-2)\}$

solution?

(b) $\{(1,2,7),(4,7,2),(3,5,0)\}$; $\{(0,2,0),(3,-4,6),(7,1,2)\}$; $\{(1,6,3),(2,-6,-1),(3,1,-7),(0,0,2)\}$

(c) $\{1\}$; $\{0\}$; $\{\}$

9. (a) By inspection, determine whether the vector $\mathbf{b} = (13,2,5,0)$ (in column vector form) is in the column space of

$$A = \begin{bmatrix} 4 & 2 & 3 & 5 \\ -1 & 3 & -5 & 4 \\ 2 & 6 & 3 & 1 \\ 0 & 2 & 4 & 0 \end{bmatrix},$$

and say whether $A\mathbf{x} = \mathbf{b}$ is consistent.

(b) Consider the matrix

$$B = \begin{bmatrix} 1 & 4 & 1 & -3 \\ 2 & 6 & 1 & -1 \\ 0 & 2 & 5 & -7 \\ 1 & 2 & 1 & -7 \end{bmatrix}.$$

Are all the bases vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$ in the row space of B? *Hint: determinants are your friends.*

Solution

(a) As **b** is equal to two times the first colume plus the fourth column, it is in the column space of A. Further, because **b** is in the column space of A, $A\mathbf{x} = \mathbf{b}$ is consistent.

(b) As $\det B \neq 0$, B's row reduced form is the identity matrix. However, as elementary row operations do not change the row space of a matrix, the row space of B is the same as that of I_4, which clearly contains $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$.

10. Given the matrix

$$A = \begin{bmatrix} 1 & 2 & 7 & 1 \\ 2 & 3 & 12 & 1 \\ 0 & 0 & 1 & 4 \\ 2 & 2 & 11 & 4 \end{bmatrix},$$

(a) Find the rank and nullity of A, and show that their sum is equal to the number of columns in A;

(b) Determine whether the row vectors of A form a basis for R^4.

Solution

(a) To determine the rank and nullity of A, we first row-reduce:

$$A = \begin{bmatrix} 1 & 2 & 7 & 1 \\ 2 & 3 & 12 & 1 \\ 0 & 0 & 1 & 4 \\ 2 & 2 & 11 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & -2 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As the reduced form has 3 leading ones and one parameter, the rank is 3 and and the nullity is 1. $3 + 1 = 4$, as expected.

(b) As the row-reduced form of A contains a row of zeros, it is clear that a linear combination of the rows of A exists which is equal to 0, without every coefficient being 0. Thus, the rows of A are linearly dependent, and so they do not form a basis for R^4.

Question 1 (10 points total, 4 points for part a, 3 points for each part of b)

a. Find values for k such that **u** and **v** are orthogonal when **u** = (2, 5, 7) and **v** = (3, k, 2).
b. Verify that the Cauchy-Schwarz inequality holds for the following vector pairs:
   i. **u** = (2, 5, 4) and **v** = (1, 3, 2)
   ii. **u** = (1, 4) and **v** = (3, 2)

SOLUTION

1.a. When **u** and **v** are orthogonal, their dot product will equal zero. Therefore,

$\mathbf{u} \cdot \mathbf{v} = (2, 5, 7) \cdot (3, k, 2) = (2)*(3) + (5)*(k) + (7)*(2) = 6 + 5k + 14 = 20 + 5k$

which equals zero, so

$0 = 20 + 5k$
$20 = -5k$

k = -4

1. b. The Cauchy-Schwarz inequality states: $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$

i. $|\mathbf{u} \cdot \mathbf{v}| = (2)*(1) + (5)*(3) + (4)*(2) = 2 + 15 + 8 = 25$
$\|\mathbf{u}\| \|\mathbf{v}\| = (\sqrt{2^2 + 5^2 + 4^2}) * (\sqrt{1^2 + 3^2 + 2^2}) =$
$(\sqrt{4 + 25 + 16}) * (\sqrt{1 + 9 + 4}) = (\sqrt{45}) * (\sqrt{14}) = \sqrt{630} = 25.0998$

Therefore, $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$ and the Cauchy-Schwarz inequality holds.

ii. $|\mathbf{u} \cdot \mathbf{v}| = (1)*(3) + (4)*(2) = 3 + 8 = 11$
$\|\mathbf{u}\| \|\mathbf{v}\| = (\sqrt{1^2 + 4^2}) * (\sqrt{3^2 + 2^2}) =$
$(\sqrt{1 + 16}) * (\sqrt{9 + 4}) = (\sqrt{17}) * (\sqrt{13}) = \sqrt{221} = 14.866$

Therefore, $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$ and the Cauchy-Schwarz inequality holds.

Question 2 (15 points)
Given the following system of linear equations

$$
\begin{aligned}
w_1 &= 3x_1 + 5x_2 - x_3 \\
w_2 &= \quad\;\; -7x_2 + 3x_3 \\
w_3 &= -x_1 - 7x_2 + 4x_3 \\
w_4 &= 2x_1 \qquad\quad + 5x_3
\end{aligned}
$$

a) What is the proper notation for the transformation? (i.e. $T: R^? \rightarrow R^?$)
b) What is the standard matrix for the transformation?

c) Using two different methods what is T(-1,5,3)?

d) Given the vector $\begin{bmatrix}-3 & 2\end{bmatrix}$ show the resulting vector after performing the following operations. in turn. Include the proper standard matrices that correlate with the operations

1. Reflection about the line x=y
2. Reflection about the y-axis
3. Rotation through an angle $\pi/2$ C.C.W.

SOLUTION

a) $T: R^3 \rightarrow R^4$

b) $\begin{bmatrix} 3 & 5 & -1 \\ 0 & -7 & 3 \\ -1 & -7 & 4 \\ 2 & 0 & 5 \end{bmatrix}$

c) Through substitution

1.) $w_1 = 3x_1 + 5x_2 - x_3 = 3(-1) + 5(5) - 1(3) = 19$

$w_2 = -7x_2 + 3x_3 = -7(5) + 3(3) = -26$

$w_3 = -x_1 - 7x_2 + 4x_3 = -1(-1) - 7(5) + 4(3) = -22$

$w_4 = 2x_1 + 5x_3 = 2(-1) + 5(3) = 13$

$$\begin{bmatrix} 19 \\ -26 \\ -22 \\ 13 \end{bmatrix}$$

2.) Through matrix multiplication

$$\begin{bmatrix} 3 & 5 & -1 \\ 0 & -7 & 3 \\ -1 & -7 & 4 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 19 \\ -26 \\ -22 \\ 13 \end{bmatrix}$$

d)

1) $\begin{bmatrix}-3 & 2\end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix}2 & -3\end{bmatrix}$

2) $\begin{bmatrix}2 & -3\end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix}-2 & -3\end{bmatrix}$

3) $\begin{bmatrix}-2 & -3\end{bmatrix} \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} =$

$\begin{bmatrix}-2 & -3\end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix}-3 & 2\end{bmatrix}$

Question 3 (5 points total)
Indicate whether the following statements are true or false. Justify your answer by giving a logical argument or a counter example. (5 points total)

a. The rotation operator $T_A:R^2 \rightarrow R^2$ is one to one.
b. The orthogonal projection onto the xy-plane $T_A:R^3 \rightarrow R^3$ is "sometimes" one to one.

?

SOLUTION

a. True. We know by theorem that for $T_A:R^2 \rightarrow R^2$ to be one to one, then the standard matrix A, where A is an 2 x 2 matrix and $T:R^2 \rightarrow R^2$ is multiplication by A, then for T_A to be one to one, then A must be invertible. The standard matrix for a rotation in R^2 is:

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

This matrix is invertible because $\det(A) \neq 0$:

$$\det\left(\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\right) = \cos^2(\theta) + \sin^2(\theta) = 1 \neq 0$$

Therefore, $T_A:R^2 \rightarrow R^2$ is one to one.

b. False. For $T_A:R^3 \rightarrow R^3$ to be one to one, then the standard matrix A, where A is an n x n matrix and $T_A:R^3 \rightarrow R^3$ is multiplication by A, then for T_A to be one to one, then A must be invertible. The standard matrix for an orthogonal projection onto the xy-plane in R^3 is:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is not invertible because $\det(A) = 0$:

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = (1)(1)(0) = 0$$

Therefore, $T_A:R^3 \rightarrow R^3$ is never one to one.

Question 4 (5 points each, 10 points total)
Determine whether the linear operator $T: R^3 \rightarrow R^3$ defined by the equations is one to one; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2, w_3)$. (5 points each, 10 points total)

a.
$$\begin{aligned} w_1 &= 2x_1 + 7x_2 + 4x_3 \\ w_2 &= x_1 - 4x_2 + 5x_3 \\ w_3 &= 3x_1 - 2x_2 + 11x_3 \end{aligned}$$

b. $$\begin{aligned} w_1 &= 3x_1 - 10x_2 + 25x_3 \\ w_2 &= x_1 - 4x_2 + 10x_3 \\ w_3 &= 2x_1 + 0x_2 + 5x_3 \end{aligned}$$

SOLUTION

a.
$$\begin{aligned} w_1 &= 2x_1 + 7x_2 + 4x_3 \\ w_2 &= x_1 - 4x_2 + 5x_3 \\ w_3 &= 3x_1 - 2x_2 + 11x_3 \end{aligned} \sim \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 4 \\ 1 & -4 & 5 \\ 3 & -2 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The standard matrix (A) for this transformation is: $\begin{bmatrix} 2 & 7 & 4 \\ 1 & -4 & 5 \\ 3 & -2 & 11 \end{bmatrix}$

If det(A) $\neq 0$, then A is invertible and T is one to one.

$$\det\left(\begin{bmatrix} 2 & 7 & 4 \\ 1 & -4 & 5 \\ 3 & -2 & 11 \end{bmatrix}\right) = (-1)\det\left(\begin{bmatrix} 1 & -4 & 5 \\ 2 & 7 & 4 \\ 3 & -2 & 11 \end{bmatrix}\right) = (-1)\det\left(\begin{bmatrix} 1 & -4 & 5 \\ 0 & 10 & -4 \\ 0 & 15 & -6 \end{bmatrix}\right)$$

$$= (-1)\det\left(\begin{bmatrix} 1 & -4 & 5 \\ 0 & 5 & -2 \\ 0 & 15 & -6 \end{bmatrix}\right) = (-1)\det\left(\begin{bmatrix} 1 & -4 & 5 \\ 0 & 5 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right) = (-1)(1)(5)(0) = 0$$

Because det(A) = 0, A is not invertible and T is not one to one.

b.
$$\begin{aligned} w_1 &= 3x_1 - 10x_2 + 25x_3 \\ w_2 &= x_1 - 4x_2 + 10x_3 \\ w_3 &= 2x_1 + 0x_2 + 5x_3 \end{aligned} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 & -10 & 25 \\ 1 & -4 & 10 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The standard matrix (A) for this transformation is: $\begin{bmatrix} 3 & -10 & 25 \\ 1 & -4 & 10 \\ 2 & 0 & 5 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 3 & -10 & 25 \\ 1 & -4 & 10 \\ 2 & 0 & 5 \end{bmatrix}\right) = (-1)\det\left(\begin{bmatrix} 1 & -4 & 10 \\ 3 & -10 & 25 \\ 2 & 0 & 5 \end{bmatrix}\right) = (-1)\det\left(\begin{bmatrix} 1 & -4 & 10 \\ 0 & 2 & -5 \\ 0 & 8 & -15 \end{bmatrix}\right)$$

$$= (-1)\det\left(\begin{bmatrix} 1 & -4 & 10 \\ 0 & 2 & -5 \\ 0 & 0 & 5 \end{bmatrix}\right) = (-1)(1)(2)(5) = -10 \neq 0$$

Because det(A) = -6 $\neq 0$, A is invertible and T is one to one.

$$\left[\begin{array}{ccc|ccc} 3 & -10 & 25 & 1 & 0 & 0 \\ 1 & -4 & 10 & 0 & 1 & 0 \\ 2 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -4 & 10 & 0 & 1 & 0 \\ 3 & -10 & 25 & 1 & 0 & 0 \\ 2 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -4 & 10 & 0 & 1 & 0 \\ 0 & 2 & -5 & 1 & -3 & 0 \\ 0 & 8 & -15 & 0 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -4 & 10 & 0 & 1 & 0 \\ 0 & 2 & -5 & 1 & -3 & 0 \\ 0 & 0 & 5 & -4 & 10 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -4 & 10 & 0 & 1 & 0 \\ 0 & 2 & -5 & 1 & -3 & 0 \\ 0 & 0 & 1 & -4/5 & 2 & 1/5 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & -4 & 0 & 8 & -19 & -2 \\ 0 & 2 & 0 & -3 & 7 & 1 \\ 0 & 0 & 1 & -4/5 & 2 & 1/5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14 & -33 & -4 \\ 0 & 2 & 0 & -3 & 7 & 1 \\ 0 & 0 & 1 & -4/5 & 2 & 1/5 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14 & -33 & -4 \\ 0 & 1 & 0 & -3/2 & 7/2 & 1/2 \\ 0 & 0 & 1 & -4/5 & 2 & 1/5 \end{array}\right] \sim \quad [I \mid A^{-1}] \Rightarrow [A^{-1}] = \begin{bmatrix} 14 & -33 & -4 \\ -3/2 & 7/2 & 1/2 \\ -4/5 & 2 & 1/5 \end{bmatrix}$$

$$T^{-1}(w_1,w_2,w_3) = [A^{-1}]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 14 & -33 & -4 \\ -3/2 & 7/2 & 1/2 \\ -4/5 & 2 & 1/5 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$= (14w_1 - 33w_1 - 4w_3, -3/2w_1 + 7/2w_2 + 1/2w_3, -4/5w_1 + 2w_2\ 1/5w_3)$$

Question 5 (10 points)

Show your reasoning behind whether or not the following may be considered to be a vector space.

The set of all pairs of real numbers V=(x,y)" with the operation"

x+y=0

I don't understand – x & y are components, not vectors.

If this cannot be considered to be a vector space, list the axioms which do not hold.

SOLUTION

The set of all pairs of real numbers V=(x,y) with the operation
x+y=0, is a vector space because all of the axioms hold.

I define the vectors **u**=(x,y) **v**=(a,b) **w**=(c,d) as three arbitrary pairs of real numbers, elements of V.

Axiom 1 holds, since u+v is a pair of real numbers.
u+**v**=(x+a,y+b)

Axiom 2 holds, since **u**+**v**=**v**+**u**.
v+**u**=(a+x,b+y)=[(a+x),(b+y)]=[(x+a),(y+b)]=**u**+**v**.

Axiom 3 holds, since **u**+(**v**+**w**)=(**u**+**v**)+**w**.
(**u**+**v**)+**w**=[(x+a),(y+b)]+(c,d)=[(x+a)+c,(y+b)+d]=[x+(a+c),y+(b+d)]=(x,y)+[(a+c),(b+d)]
=**u**+(**v**+**w**).

Axiom 4 holds because 0+**u** is equal to **u**.
0+**u**=(0,0)+(x,y)=(0+x,0+y)=(x,y) — weird defined addition

Axiom 5 holds, since **u**+(-**u**)=(x-x,y-y)=(0,0)

Axiom 6 holds since k**u**=k(x,y)=(kx,ky)=[(kx),(ky)], which is an element of V.

Axiom 7 holds, since k(**u**+**v**)= [(kx+ka),(ky+kb)]=(kx+ka),(ky+kb)=(kx,ky)+(ka,kb)=k(x,y)+k(a,b)=k**u**+k**v**.

Axiom 8 holds, since (k+m)**u**= [(kx+mx),(ky+my)]=(kx,ky)+(mx,my)
=k(x,y)+m(x,y)=k**u**+m**u**.

Axiom 9 holds, since
k(m**u**)=k[m(x,y)]=k(mx,my)=[k(mx),k(my)]=[(km)x,(km)y]=km(x,y)=(km)**u**.

Axiom 10 holds, since 1**u**=1(x,y)=(1x,1y)=(x,y)=**u**.

Question 6 (10 points)

How can (2,3,1) be expressed as a linear combination of (1,2,1) and (1,1,0)?

SOLUTION

The matrix (2,3,1) can be expressed as a linear combination of (1,2,1) and (1,1,0) if there exists scalars {a,b}such that (2,3,1)=a(1,2,1)+b(1,1,0). We find that this gives the following system of linear equations:

$$\begin{array}{rcrcl} a & + & b & = & 2 \\ 2a & + & b & = & 3 \\ a & & & = & 1 \end{array}$$

The augmented matrix for this system of equations is represented as follows:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, by Gauss-Jordan elimination, we have found that when the scalars $\{a,b\}$ are $a=1$ and $b=1$, (2,3,1) can be express as a linear combination as follows:

$(2,3,1) = 1(1,2,1) + 1(1,1,0)$.

Question 7 (10 points)

Are the vectors $\mathbf{v}=(2,6,3)$, $\mathbf{w}=(4,2,1)$, and $\mathbf{z}=(3,3,1)$ linearly independent?

SOLUTION

The vectors $\mathbf{v}=(2,6,3)$, $\mathbf{w}=(4,2,1)$, and $\mathbf{z}=(3,3,1)$ are linearly independent if ~~there exists the scalars , $\{k_1,k_2,k_3\}$ such that,~~ $k_1\mathbf{v} + k_2\mathbf{w} + k_3\mathbf{z}=0$ implies that $k_1=k_2=k_3=0$.

This equation yields the following system of equations:

$2k_1 + 4k_2 + 3k_3 = 0$
$6k_1 + 2k_2 + 3k_3 = 0$
$3k_1 + 1k_2 + 1k_3 = 0$

The augmented matrix for this system of equations is represented as follows:

$$\begin{bmatrix} 2 & 4 & 3 & 0 \\ 6 & 2 & 3 & 0 \\ 3 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 3 & 0 \\ 0 & -10 & -6 & 0 \\ 0 & -10 & -7 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 3 & 0 \\ 0 & -10 & -6 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Solving by Gauss-Jordanian elimination we find that there exist only the scalars $\{k_1,k_2,k_3\}$

such that $k_1\mathbf{v} + k_2\mathbf{w} + k_3\mathbf{z} = 0$ implies that $k_1=k_2=k_3=0$.

Question 8 (10 points)

Let $\mathbf{v}_1 = (1,3,2)$, $\mathbf{v}_2 = (5,18,13)$, and $\mathbf{v}_3 = (4,2,4)$. (10 points)

a. Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for R^3.
b. Find coordinate vector of $\mathbf{v} = (6,-5,1)$ with respect to S.

SOLUTION

a. To show that the set S spans R^3 and is linearly dependent, we will consider the equation:

$$\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

or

$$(0,0,0) = c_1(1,3,2) + c_2(5,18,13) + c_3(4,2,4)$$

or

$$(0,0,0) = (1c_1 + 5c_2 + 4c_3,\ 3c_1 + 18c_2 + 2c_3,\ 2c_1 + 13c_2 + 4c_3)$$

or equating corresponding components,

$$\begin{array}{rcrcrcl} c_1 & + & 5c_2 & + & 4c_3 & = & 0 \\ 3c_1 & + & 18c_2 & + & 2c_3 & = & 0 \\ 2c_1 & + & 13c_2 & + & 4c_3 & = & 0 \end{array}$$

To show that S spans R^3 and is linearly independent, (by theorem) we can look at the determinate of the matrix of coefficients:

$$\det\left(\begin{bmatrix} 1 & 5 & 4 \\ 3 & 18 & 2 \\ 2 & 13 & 4 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & -10 \\ 0 & 3 & -4 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & -10 \\ 0 & 0 & 6 \end{bmatrix}\right) = (1)(3)(6) = 18$$

Since the determinant is not equal to zero, the vectors are linearly independent and they span R^3 because an inverse can be determined.

b. We need to find scalars c_1, c_2, c_3 such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

or considering the components,

$$(6,-5,1) = c_1(1,3,2) + c_2(5,18,13) + c_3(4,2,4)$$

Which can be considered by equating corresponding components and creating a matrix from the coefficients as:

$$\begin{bmatrix} 1 & 5 & 4 & 6 \\ 3 & 18 & 2 & -5 \\ 2 & 13 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 4 & 6 \\ 0 & 3 & -10 & -23 \\ 0 & 3 & -4 & -11 \end{bmatrix} \quad \begin{bmatrix} 1 & 5 & 4 & 6 \\ 0 & 3 & -10 & -23 \\ 0 & 0 & 6 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 4 & 6 \\ 0 & 3 & -10 & -23 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 & -2 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \quad \begin{array}{l} c_1 = 3 \\ c_2 = -1 \\ c_3 = 2 \end{array}$$

Therefore $(v)_s = (3, -1, 2)$.

Question 9 (10 points)

Given the matrix A = $\begin{bmatrix} 1 & 5 & 4 & -2 \\ -2 & -9 & -11 & 5 \\ 3 & 15 & 13 & 1 \\ -1 & -5 & -2 & 17 \end{bmatrix}$,which of the following are in the column space~~s~~ of A ?

a) $\begin{bmatrix} 5 \\ -9 \\ 12 \\ 27 \end{bmatrix}$ b) $\begin{bmatrix} -4 \\ 8 \\ -5 \\ 19 \end{bmatrix}$ c) $\begin{bmatrix} 11 \\ -3 \\ 5 \\ -22 \end{bmatrix}$ d) $\begin{bmatrix} 4 \\ -12 \\ 6 \\ -17 \end{bmatrix}$

<u>SOLUTION</u>

a) Not ~~a~~ in the column space, it cannot be written as a linear combination of any of the columns in A because the first two terms are identical to two of the terms in a column of A, but the last two terms are different. – not a good explanation.

b) Column space

This is not a sentence.

$$\begin{bmatrix} 1 & 5 & 4 & -2 \\ -2 & -9 & -11 & 5 \\ 3 & 15 & 13 & 1 \\ -1 & -5 & -2 & 17 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ -5 \\ 19 \end{bmatrix}$$

By Gaussian elimination

$$\left[\begin{array}{cccc|c} 1 & 5 & 4 & -2 & -4 \\ -2 & -9 & -11 & 5 & 8 \\ 3 & 15 & 13 & 1 & -5 \\ -1 & -5 & -2 & 17 & 19 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 5 & 4 & -2 & -4 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & 7 & 7 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

z=1	z=1
y+7z = 7	y=0
x-3y+z=0	x=-1
w +5x +4y -2z = -4	w=3

c) Not ~~a~~ in the column space of A because the column vector cannot be written as a linear combination of any of the columns in A. – how do we know?

d) Column space

$$\begin{bmatrix} 1 & 5 & 4 & -2 \\ -2 & -9 & -11 & 5 \\ 3 & 15 & 13 & 1 \\ -1 & -5 & -2 & 17 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \\ 6 \\ -17 \end{bmatrix}$$

By Gaussian elimination

$$\left[\begin{array}{cccc|c} 1 & 5 & 4 & -2 & 4 \\ -2 & -9 & -11 & 5 & -12 \\ 3 & 15 & 13 & 1 & 6 \\ -1 & -5 & -2 & 17 & -17 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 5 & 4 & -2 & 4 \\ 0 & 1 & -3 & 1 & -8 \\ 0 & 0 & 1 & 7 & -6 \\ 0 & 0 & 2 & 15 & -13 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 5 & 4 & -2 & 4 \\ 0 & 1 & -3 & 1 & -8 \\ 0 & 0 & 1 & 7 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$z=-1$	$z=-1$
$y+7z = -8$	$y=1$
$x-3y+z=-6$	$x=0$
$w +5x +4y -2z = -1$	$w=-2$

Question 10 (10 points)

a. What are the <u>fundamental matrix spaces</u> associated with a given matrix A? (4 points)
b. Define rank and nullity for ~~the column and row spaces of~~ a matrix. (3 points)
c. If rank(A) + nullity (A) = n what can be said about the dimensions of A? (3 points)

<u>SOLUTION</u>

a. The fundamental matrix spaces of A are the 1) the row space of A, 2) the column space of A, 3) the nullspace of A, and 4) the nullspace of A^T. The row space and column space of A^T are not fundamental because they are equivalent to the column space and row space of A respectively.
b. Rank is the common dimension of the row space and column space in a given matrix A. Nullity is the dimension of the nullspace in a given matrix A.
c. Matrix A will have n number of columns. Nothing can be said about the number of rows that A has.

5 pts

1. Show that if **v** is a nonzero vector in R^3, then $(1/\|\mathbf{v}\|)\mathbf{v}$ has Euclidean norm 1.

Solution:

$\|\mathbf{v}\| = \sqrt{\mathbf{v} * \mathbf{v}} \neq \mathbf{v}$ therefore

$$\left(\frac{1}{\|\mathbf{v}\|}\right) * \mathbf{v} \neq \frac{\mathbf{v}}{\mathbf{v}} = 1$$

doesn't make sense to divide by a vector.

10 pts

2. Find the standard matrix for the linear operator T: $R^3 \rightarrow R^3$ given by

$$w_1 = 3x_1 + 5x_2 - x_3$$
$$w_2 = 4x_1 - x_2 + x_3$$
$$w_3 = 3x_1 + 2x_2 - x_3$$

and then calculate T(-1,2,4) by directly substituting in the equations and also by matrix multiplication.

Solution:

The standard matrix is: $\begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}$.

T(-1,2,4) solving directly we insert the values into the equations:

$$w_1 = 3(-1) + 5(2) - 4 \qquad w_1 = 3$$
$$w_2 = 4(-1) - 2 + 4 \qquad w_2 = -2$$
$$w_3 = 3(-1) + 2(2) - 4 \qquad w_3 = -3$$

Solving using matrix multiplication we get:

$$\begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}.$$

10 pts

3.Show that the range of the linear operator defined by the equations

$$\begin{aligned} w_1 &= x_1 - 2x_2 + x_3 \\ w_2 &= 5x_1 - x_2 + 3x_3 \\ w_3 &= 4x_1 + x_2 + 2x_3 \end{aligned}$$

is not all of R^3.

<u>Solution:</u>
Writing the equations in matrix form we get:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 5 & -1 & 3 \\ 4 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

This matrix does not have a determinant - it's not square.

where $\det = \begin{vmatrix} 1 & -2 & 1 & 1 & -2 \\ 5 & -1 & 3 & 5 & -1 \\ 4 & 1 & 2 & 4 & 1 \end{vmatrix} = -2 - 24 + 5 + 20 - 3 + 4 = 0$

Since the determinant is equal to zero the range is not all of R^3.

5 pts

4. Let $\mathbf{v} = (2, -1, 2)$. Find all k such that $||k\mathbf{v}|| = 6$.

Solution:

$k\mathbf{v} = (2k, -k, 2k)$.
$||k\mathbf{v}|| = ((2k)^2 + (-k)^2 + (2k)^2)^{\wedge}½$
$6 = (4k^2 + k^2 + 4k^2)^{\wedge}½$
$6 = (9k^2)^{\wedge}½$
$6 = \pm 3k$
$\pm 2 = k$

20 pts

5. What is the standard matrix used to perform the stated compositions on $\mathbf{R}^2$?
   a. rotation of 90° followed by reflection about the line y = x
   b. orthogonal projection onto y axis, followed by contraction k = ½
   c. reflection about the x axis, followed by dilation k = 3

Solution:

a. The standard matrix for rotation of 90° is $\begin{bmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{bmatrix}$ which is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

The standard matrix for reflection about the line y = x is: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

To compose functions, multiply the standard matrices in reverse order:

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ So the standard matrix is $\boldsymbol{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$

b. The standard matrix for orthogonal projection onto y axis is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

The standard matrix for contraction k = ½ is $\begin{bmatrix} ½ & 0 \\ 0 & ½ \end{bmatrix}$

To compose functions, multiply the standard matrices in reverse order:

$\begin{bmatrix} ½ & 0 \\ 0 & ½ \end{bmatrix} * \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & ½ \end{bmatrix}$ So the standard matrix is $\boldsymbol{\begin{bmatrix} 0 & 0 \\ 0 & ½ \end{bmatrix}}$

c. The standard matrix for reflection about the x axis is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

The standard matrix for dilation k = 3 is $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

To compose functions, multiply the standard matrices in reverse order:

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$ So the standard matrix is $\mathbf{\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}}$

15 pts

6. Show whether or not the set of all positive integers with the operations x + y = xy and kx = x^k is a vector space.

Solution:

To check whether or not a set is a vector space, we need to check closure on addition and on scalar multiplication.

Check addition: x + y = xy. x and y are both positive integers. The multiplication of two positive integers will always yield a positive integer. So addition holds.

Check multiplication: kx = x^k for all real k. If k is not an integer, then x^k not necessarily an integer. For example, if k = ½ and x = 3, x^k is not an integer. So closure does not hold on scalar multiplication, so it is not a vector space.

25 pts

7. Which of the following can be expressed as linear combinations of the vectors **u** = (1,-1,2) and **v** = (2,0,1) If so, what is the linear combination?

a. (-1,3,-5)
b. (3,2,4)
c. 0,-1,3/2)

Solution:

a. $(-1,3,-5) = \alpha(1,-1,2) + \beta(2,0,1)$

This can be written with respect to each slot:

$\alpha + 2\beta = -1$

$-\alpha + 0\beta = 3$

$2\alpha + \beta = -5$

Placing this information into an augmented matrix, we solve for α and β:

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 3 \\ 2 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus from this last matrix we have $\alpha = -3$ and $\beta = 1$. **Thus the vector can be expressed in the linear combination (-1,3,5) = -3α + β**

b. $(3,2,4) = \alpha(1,-1,2) + \beta(2,0,1)$

This can be written with respect to each slot:

$\alpha + 2\beta = 3$

$-\alpha + 0\beta = 2$

$2\alpha + \beta = 4$

Placing this information into an augmented matrix, we solve for α and β:

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5/2 \\ 0 & 1 & 2/3 \end{bmatrix}$$ From here we see that $\beta = 5/2$ and $\beta = 2/3$.

Because there are two conflicting values for β, the matrix is inconsistent. **Hence this matrix can not be expressed as a linear combination of u and v.**

not what you wanted to do,

c. $(0,-1,3/2) = \alpha(1,-1,2) + \beta(2,0,1)$

This can be written with respect to each slot:

$\alpha + 2\beta = 0$

$-\alpha + 0\beta = -1$

$2\alpha + \beta = 3/2$

Placing this information into an augmented matrix, we solve for α and β:

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 2 & 1 & 3/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & -1 \\ 0 & -3 & 3/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus from the last matrix we have $\alpha = 1$ and $\beta = -1/2$. **Thus the vector can be expressed in the linear combination $(0,-1,3/2) = u - v/2$.**

30 pts

8. Determine if the following set of objects, together with the given operations of addition and scalar multiplication, is a vector space:

The set of all pairs of real numbers of the form $(1\,,x)$ with the operations

$$(1\,,x) + (1\,,x') = (1\,,x+x') \text{ and } k(1\,,x) = (1\,,kx)$$

Solution:

By the definition of a vector space, we need to determine if the given set and operations comply with the "ten axioms."

Let $\mathbf{u} = (1\,,x)$, $\mathbf{v} = (1\,,x')$, and $\mathbf{w} = (1\,,x'')$:

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V.

$\mathbf{u} + \mathbf{v} = (1\,, x + x')$ or, rewritten: $\mathbf{u} + \mathbf{v} = (1\,, (x + x'))$.

Which is of the form $(1\,, x)$, and is a pair of real numbers, therefore $\mathbf{u} + \mathbf{v}$ is in V.

2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

$\mathbf{u} + \mathbf{v} = (1\,, x + x') = (1\,, x' + x) = \mathbf{v} + \mathbf{u}$

3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (1\,, x) + (1\,, x' + x'') = (1\,, x + (x' + x''))$

$= (1\,, (x + x') + x'') = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

4. There is an object $\mathbf{0}$ in V, called a zero vector for V, such that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ in V.

Let $\mathbf{0} = (1\,, 0)$. First off, this is of the form of $(1\,, x)$, and is a pair of real numbers, so it is in V.

$\mathbf{u} + \mathbf{0} = (1\,, x + 0) = (1\,, 0 + x) = \mathbf{0} + \mathbf{u}$

$= (1\,, x) = \mathbf{u}$

5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a negative of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.

Let $-\mathbf{u} = (1\,, -x)$. Once again, this is of the form $(1\,, x)$, and is a pair of real numbers, so it is in V.

$\mathbf{u} + (-\mathbf{u}) = (1\,, x + {}^{-}x) = (1\,, {}^{-}x + x) = (-\mathbf{u}) + \mathbf{u}$

$= (1\,, 0) = \mathbf{0}$

6. If k is any scalar $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.

$k\mathbf{u} = (1\,, kx)$ or, rewritten: $k\mathbf{u} = (1\,, (kx))$.

Which is of the form $(1\,, x)$, and is a pair of real numbers, therefore $\mathbf{u} + \mathbf{v}$ is in V.

7. $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

$$k(\mathbf{u}+\mathbf{v}) = k(1, x+x') = (1, k(x+x'))$$
$$= (1, (kx)+(kx')) = k\mathbf{u} + k\mathbf{v}$$

8. $(k+m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

$$(k+m)\mathbf{u} = (1, (k+m)x) = (1, kx+mx)$$
$$= k\mathbf{u} + m\mathbf{u}$$

9. $k(m\mathbf{u}) = (km)(\mathbf{u})$

$$k(m\mathbf{u}) = k(1, mx) = (1, k(mx)) = (1, (km)x)$$
$$= (km)(\mathbf{u})$$

10. $1\mathbf{u} = \mathbf{u}$

$$1\mathbf{u} = (1, 1\cdot\mathbf{u}) = (1, \mathbf{u}) = \mathbf{u}$$

All of the axioms hold, so this is a vector space.

15 pts

9. Is the following a subspace of M_{nn}?
All $n \times n$ matrices A such that $\text{tr}(A) = 0$

Solution:

By Theorem 5.2.1 (and what we learned in class), all we need to check is:
(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u}+\mathbf{v}$ is in W.
(b) If k is any scalar and $\mathbf{u}$ is any vector in W, then $k\mathbf{u}$ is in W.
Where W is a set of one or more vectors from a vector space V. Iff these conditions are met, then W is a subspace of V.

(a) $A + A' = (A + A')$. $tr(A+A') = tr(A) + tr(A')$ (proof omitted)
By premise, $tr(A) = 0$ and $tr(A') = 0 \Rightarrow tr(A) + tr(A') = 0 + 0 = 0$.
Therefore, $tr(A+A') = tr(A) + tr(A') = 0 \Rightarrow A + A'$ is in W

(b) $tr(kA) = k \cdot tr(A)$ (proof omitted)
Once again by premise, $tr(A) = 0$.
Therefore, $tr(kA) = k \cdot tr(A) = k \cdot 0 = 0 \Rightarrow kA$ is in W

All of the conditions are met, so this is a subspace of M_{nn}.

15 pts

10. Express the following as a linear combination of $\mathbf{p}_1 = 2 + x + 4x^2$, $\mathbf{p}_2 = 1 - x + 3x^2$, and $\mathbf{p}_3 = 3 + 2x + 5x^2$:

$$6 + 11x + 6x^2$$

Solution:

By the definition of a linear combination, we are trying to solve the equation:
$\mathbf{r} = k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3$ for k_1, k_2, and k_3.

$(6 + 11x + 6x^2) = k_1(2 + x + 4x^2) + k_2(1 - x + 3x^2) + k_3(3 + 2x + 5x^2)$

speak of standard basis of P_2.

By grouping and dividing out like terms, this can be expressed as a system of equations:

$$\begin{aligned} k_1(2) + k_2(1) + k_3(3) &= 6 \\ k_1(1) + k_2(-1) + k_3(2) &= 11 \\ k_1(4) + k_2(3) + k_3(5) &= 6 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right]$$

Now we solve the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 2 & 1 & 3 & 6 \\ 4 & 3 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 3 & -1 & -16 \\ 4 & 3 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 3 & -1 & -16 \\ 0 & 7 & -3 & -38 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & -\frac{1}{3} & -\frac{16}{3} \\ 0 & 7 & -3 & -38 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & -\frac{1}{3} & -\frac{16}{3} \\ 0 & 0 & -\frac{2}{3} & -\frac{2}{3} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & -\frac{1}{3} & -\frac{16}{3} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Using back substitution we get:
$k_1 = 4$, $k_2 = -5$, and $k_3 = 1$.

So, $6 + 11x + 6x^2 = 4\mathbf{p}_1 - 5\mathbf{p}_2 + \mathbf{p}_3$.

343 Exam #2

1. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ are (n x 1) vectors in R^n and A is an (n x n) matrix, Show that the following statements are always true or give a counter example.

a) $\mathbf{u} \cdot \mathbf{v} = \frac{1}{4} \| \mathbf{u} + \mathbf{v} \|^2 - \frac{1}{4} \| \mathbf{u} - \mathbf{v} \|^2$

b) $\| \mathbf{u} + \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2$

c) $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$

Solution: Assuming result,

a) $\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}(\mathbf{u} + \mathbf{v})(\mathbf{u} + \mathbf{v}) - \frac{1}{4} (\mathbf{u} - \mathbf{v})(\mathbf{u} - \mathbf{v})$ By the definition of Exponents

$= \frac{1}{4} [\| \mathbf{u} \|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \| \mathbf{u} \|^2] - \frac{1}{4} [\| \mathbf{u} \|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + \| \mathbf{u} \|^2]$ By multiplication

$= \frac{1}{4} [2(\mathbf{u} \cdot \mathbf{v})] - \frac{1}{4} [- 2(\mathbf{u} \cdot \mathbf{v})]$ Grouping like terms

$= \mathbf{u} \cdot \mathbf{v}$ By distribution and addition

b) FALSE

$\| \mathbf{u} + \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \| \mathbf{u} \|^2 \neq \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2$

If the two vectors are not orthogonal then $2(\mathbf{u} \cdot \mathbf{v})$ does not equal zero and the statement is false.

c) Let $\mathbf{w} = A\mathbf{u}$

$\mathbf{w} \cdot \mathbf{v} = \mathbf{w}_1 v_1 + w_2 v_2 + \ldots + w_n v_n$ By the definition of the Euclidean inner product

$\mathbf{w} \cdot \mathbf{v} = \mathbf{v}^T\mathbf{w}$ By the definition of the transpose and matrix multiplication

$= \mathbf{v}^T(A\mathbf{u})$ Substitution

$= (\mathbf{v}^T A)\mathbf{u}$ Associative Law for Matrix Multiplication

$= (A^T\mathbf{v})^T\mathbf{u}$ Theorem 1.4.9

$= \mathbf{u} \cdot A^T\mathbf{v}$ From the second step

2. For the system of equations given,

$$3x_1 - 4x_2 + x_3 = w_1$$
$$2x_1 - 7x_2 - 4x_3 = w_2$$
$$x_1 + 5x_2 - 8x_3 = w_3$$

a) Write the system in Dot Product Form (a column matrix with each entry being the dot product of two vectors, equal to another column matrix of constants)

b) Write the standard matrix, A, for this system that encodes this linear transformation $T: \mathrm{R}^3 \to \mathrm{R}^3$

c) By matrix multiplication, calculate the image of the point $(x_1, x_2, x_3) = (1,2,3)$

__Solution:__

a) $$\begin{bmatrix} (3,-4,1)\cdot(x_1,x_2,x_3) \\ (2,-7,-4)\cdot(x_1,x_2,x_3) \\ (1,5,-8)\cdot(x_1,x_2,x_3) \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

b) $$A = \begin{bmatrix} 3 & -4 & 1 \\ 2 & -7 & -4. \\ 1 & 5 & 8 \end{bmatrix}$$

c) $$\begin{bmatrix} 3 & -4 & 1 \\ 2 & -7 & -4. \\ 1 & 5 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3-8+3 \\ 2-14-21 \\ 1+10+24 \end{bmatrix} = \begin{bmatrix} -2 \\ -33 \\ 35 \end{bmatrix}$$

The image of the point is (-2,-33,35)

3. For the following **u** = (0, 3, -2, 1), **v** = (4, -3, -2, 3), **w** = (2, 1, -1, -4)

a) Find the Euclidean Distance between **u** and **v**.

b) Find the Euclidean inner product while completing the following equation:
$(\mathbf{u} \cdot \mathbf{v})\,\mathbf{w}$ – not an equation.

c) The linear transformation T:$R^4 \to R^3$ is defined by:

$\mathbf{x} = 2\mathbf{q} + \mathbf{r} + 3\mathbf{s} + 4\mathbf{t}$
$\mathbf{y} = 4\mathbf{q} + 2\mathbf{r} - 2\mathbf{t}$
$\mathbf{z} = 2\mathbf{q} + 4\mathbf{r} - 3\mathbf{s} - \mathbf{t}$

Find the image of the vector found in part b using a standard matrix ~~and~~ of this transformation.

Solution:

(a) The Euclidean distance is easily be found by taking the following steps:
$$d(\mathbf{u},\mathbf{v}) = \| \mathbf{u} - \mathbf{v} \| = \sqrt{(0-4)^2 + (3+3)^2 + (-2-(-2))^2 + (1-3)^2} = \sqrt{16+36+0+4} = \sqrt{56}$$

why do this here?

(b) The Euclidean inner product is calculated in the following manner:
$$(\mathbf{u}\cdot\mathbf{v})\mathbf{w} = \left(\tfrac{1}{4}\| \mathbf{u}+\mathbf{v} \|^2 - \tfrac{1}{4}\| \mathbf{u}-\mathbf{v} \|^2\right)\mathbf{w}$$
$$= \left(\tfrac{1}{4}\sqrt{(0+4)^2 + (3-3)^2 + (-2-2)^2 + (1+3)^2} - \tfrac{1}{4}\sqrt{(0-4)^2 + (3+3)^2 + (-2-(-2))^2 + (1-3)^2}\right)\mathbf{w}$$
$$= \left[\tfrac{1}{4}(56) - \tfrac{1}{4}(48)\right]\mathbf{w} = (14-12)\mathbf{w} = 2\mathbf{w} = (4, 2, -2, 8)$$

(c) Provided with the answer from part b, the requested linear transformation is accomplished by creating a standard matrix and multiplying, as such:
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 2 & 0 & -2 \\ 2 & 4 & -3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -2 \\ 8 \end{bmatrix} = \begin{bmatrix} 8+2-6+32 \\ 16+4+0-16 \\ 8+8+6-8 \end{bmatrix} = \begin{bmatrix} 36 \\ 4 \\ 14 \end{bmatrix}$$

4. Calculate for the vector, V, in R^3 where
$$V = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

a) its rotation about the x-axis through an angle of $\pi/2$
b) then the orthogonal projection of the resulting vector on the xz-plane

Solution:

$$\text{a)}\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\dfrac{\pi}{2}\right) & -\sin\left(\dfrac{\pi}{2}\right) \\ 0 & \sin\left(\dfrac{\pi}{2}\right) & \cos\left(\dfrac{\pi}{2}\right) \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$\text{b)}\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

5. Is the set of vectors S = {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} linearly independent if $\mathbf{v}_1 = (8, 6, -2)$, $\mathbf{v}_2 = (5, 3, 2)$, and $\mathbf{v}_3 = (4, 3, 2)$?

Solution:

Upon observation it can be seen that $\mathbf{v}_3$ is a scalar multiple of $\mathbf{v}_1$. This means that $\mathbf{v}_3$ can be expressed as a linear combination of the other vectors and by theorem 5.3.1 set S is linearly dependent.

6. a) Show that vectors $\mathbf{v}_1 = (6, 2, -6, 3)$, $\mathbf{v}_2 = (5, 4, -10, 1)$, and $\mathbf{v}_3 = (7, 0, -2, 5)$ form a linearly dependent set in R^4.
b) Express each vector as a linear combination of the other two.

Solution:

(a) In order for the set of vectors to be linearly dependent there must be a nontrivial solution for the equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$.

$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0} \Leftrightarrow$

$k_1 (6, 2, -6, 3) + k_2 (5, 4, -10, 1) + k_3 (7, 0, -2, 5) = (0, 0, 0, 0) \Leftrightarrow$

$(6k_1 + 5k_2 + 7k_3, 2k_1 + 4k_2 + 0k_3, -6k_1 - 10k_2 - 2k_3, 3k_1 + 1k_2 + 5k_3) = (0, 0, 0, 0) \Leftrightarrow$

$6k_1 + 5k_2 + 7k_3 = 0$
$2k_1 + 4k_2 + 0k_3 = 0$
$-6k_1 - 10k_2 - 2k_3 = 0$
$3k_1 + 1k_2 + 5k_3 = 0$

Performing elementary row operations on the matrix of coefficients we see

$$\begin{bmatrix} 6 & 5 & 7 \\ 2 & 4 & 0 \\ -6 & -10 & -2 \\ 3 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 0 \\ 3 & 1 & 5 \\ 6 & 5 & 7 \\ -6 & -10 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 5 \\ 6 & 5 & 7 \\ -6 & -10 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 5 \\ 0 & -7 & 7 \\ 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -7 & 7 \\ 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is code for $k_1 + 2k_2 = 0$ and $k_2 - k_3 = 0$, choosing $k_3 = 1$ we set $k_2 = 1$ and $k_1 = -2$.

$\Leftrightarrow$

$-2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$ is nontrivial solution. $\Rightarrow$ The vectors are linearly dependent.

(b) Performing manipulations on $-2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$ we obtain

$\mathbf{v}_1 = 1/2\ \mathbf{v}_2 + 1/2\ \mathbf{v}_3$
$\mathbf{v}_2 = 2\ \mathbf{v}_1 - \mathbf{v}_3$
$\mathbf{v}_3 = 2\ \mathbf{v}_1 - \mathbf{v}_2$

7. What does the Wronskian tell us about the following sets of vectors.

(a) sinx, cosx
(b) $1,\ x - 3,\ 2x^2 + 4$
(c) $x,\ 3x + 2,\ 9x - 5$

<u>Solution:</u>

(a)

$$W(x) = \det\begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} = -\sin 2x - \cos 2x = -1 \neq 0, \text{ for some value of x.}$$

$\Rightarrow$ The set is linearly independent.

(b)

$$W(x) = \det\begin{bmatrix} 1 & x-3 & 2x^2+4 \\ 0 & 1 & 4x \\ 0 & 0 & 4 \end{bmatrix} = (1)(1)(4) = 4 \neq 0, \text{ for some value of x.}$$

$\Rightarrow$ The set is linearly independent.

(c)

$$W(x) = \det\begin{bmatrix} x & 3x+2 & 9x-5 \\ 1 & 3 & 9 \\ 0 & 0 & 0 \end{bmatrix} = 0, \text{ for all values of x.}$$

$\Rightarrow$ The set could be linearly independent or linearly dependent.

8. Determine which sets are vector spaces under the given operations. Specify which axioms do not hold if it is not a vector space.

a) The set of all pairs of real numbers (x, y) with the operations
$(x, y) + (x', y') = (x * x', y * y')$ and $k(x, y) = (x + k, y + k)$

b) The set of all 2 x 2 matrices of the form

$$\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$$

Can't pick specific vectors

Solution:

a) To show that this pair of real number is a vector space, we must "prove" the ten axioms.

1. Let **u** = (2, 3) and **v** = (4, 5) then
   (2, 3) + (4, 5) = (2*4, 3*5) = (8, 15)
   which is a pair of real numbers. Passes.
2. Let **u** and **v** have the same values as stated in axiom 1. So
   (2, 3) + (4, 5) ?= (4, 5) + (2, 3)
   (2*4, 3*5) ?= (4*2, 5*3)
   (8, 15) = (8, 15)
   and passes.
3. Let **u** and **v** have the same values as stated in axiom 1 and let **w** = (6, 7). So
   (2, 3) + ((4, 5) + (6, 7)) ?= ((4, 5) + (2, 3)) + (6, 7)
   (2, 3) + (4*6, 5*7) ?= (4*2, 5*3) + (6, 7)
   (2, 3) + (24, 35) ?= (8, 15) + (6, 7)
   (2*24, 3*35) ?= (8*6, 15*7)
   (48, 105) = (48, 105)
   and passes.
4. Let **0** (the zero vector) = (1, 1), and let **u** = (2, 3) ~~then~~
   (1, 1) + (2, 3) ?= (2, 3) + (1, 1) ?= (2, 3)
   (1*2, 1*3) ?= (2*1, 3*1) ?= (2, 3)
   (2, 3) = (2, 3) = (2, 3)
   and passes.
5. Let **u** have the same value as in axiom 4.
   (2, 3) + (-(2,3)) ?= (1,1)
   (2, 3) + (2-1,3-1) ?= (1,1)
   (2, 3) + (1,2) ?= (1,1)

this is not axiom 5, because $-u \neq (-1)u$.

$$(2, 6) \neq (1,1)$$

so axiom 5 fails.

6. Let **u** have the same value as in axiom 4 and let $k = 2$ then
$$2(2, 3) = (2+2, 3+2) = (4, 5)$$
which is a pair of real numbers. Passes.

can't be specific,

7. Let **u** and **v** have the values from axiom 1 and $k = 2$. Then
$$2((2, 3) + (4, 5)) ?= 2(2, 3) + 2(4, 5)$$
$$2(8, 15) ?= (4, 5) + (6, 7)$$
$$(10, 17) \neq (24, 35)$$
the conditions do not hold, so axiom 7 fails.

can be specific for counter-example,

8. Let **u** and k have the same values from axiom 7 and let $m = 3$. Then
$$(2 + 3)(2, 3) ?= 2(2, 3) + 3(2, 3)$$
$$5(2, 3) ?= (4, 5) + (5, 6)$$
$$(7, 8) \neq (20, 30)$$
so axiom 8 fails.
9. Let **b** and k and m have the same values from axiom 8. Then
$$2(3(2, 3)) ?= (2*3)(2,3)$$
$$2(5, 6) ?= 6(2,3)$$
$$(7, 12) \neq (8,9)$$
so axiom 9 fails.
10. Let **u** have the same value from axiom 1. Then
$$1(2, 3) ?= (2, 3)$$
$$(3, 4) \neq (2, 3)$$
so axiom 10 fails.

Therefore (a) is not a vector space, as axioms 5, 7-10 failed.

b) To show that $\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$ is a vector space, we must prove the 10 vector space axioms.

1. Let $\mathbf{u} = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}$ then
$$\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & b+c \\ b+c & 0 \end{bmatrix}.$$
$b + c = b + c$ which matches our 2x2 matrix description, so it passes.

2. Let **u** and **v** have the same values as stated in axiom 1. So
$$\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} ?= \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & b+c \\ b+c & 0 \end{bmatrix} ?= \begin{bmatrix} 0 & c+b \\ c+b & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & b+c \\ b+c & 0 \end{bmatrix} = \begin{bmatrix} 0 & b+c \\ b+c & 0 \end{bmatrix}$$

and passes.

3. Let **u** and **v** have the same values as stated in axiom 1 and let $\mathbf{w} = \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix}$. So

$$\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \left(\begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} \right) ?= \left(\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \right) + \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & c+d \\ c+d & 0 \end{bmatrix} ?= \begin{bmatrix} 0 & b+c \\ b+c & 0 \end{bmatrix} + \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & b+c+d \\ b+c+d & 0 \end{bmatrix} = \begin{bmatrix} 0 & b+c+d \\ b+c+d & 0 \end{bmatrix}$$

and passes.

4. Let **0** (the zero vector) $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and let **u** have the same value from axiom 1

then

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} ?= \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} ?= \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$$

and passes.

5. Let **u** have the same value as in axiom 1, so

$$\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \left(-\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} \right) ?= \left(-\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} \right) + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} ?= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and passes.

6. Let **u** have the same value as in axiom 1 and let $k = n$ then

$$k\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & kb \\ kb & 0 \end{bmatrix}$$

$kb = kb$ which matches our 2x2 matrix description, so it passes.

7. Let **u** and **v** have the values from axiom 1 and $k = n$. Then

$$k\left(\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \right) ?= k\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + k\begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}$$

$$k\begin{bmatrix} 0 & b+c \\ b+c & 0 \end{bmatrix} ?= \begin{bmatrix} 0 & kb \\ kb & 0 \end{bmatrix} + \begin{bmatrix} 0 & kc \\ kc & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & k(b+c) \\ k(b+c) & 0 \end{bmatrix} ?= \begin{bmatrix} 0 & kb+kc \\ kb+kc & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & k(b+c) \\ k(b+c) & 0 \end{bmatrix} = \begin{bmatrix} 0 & k(b+c) \\ k(b+c) & 0 \end{bmatrix}$$

and passes.

8. Let **u** and k have the same values from axiom 7 and let $m = o$. Then

$$(no)\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} ?= n\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + o\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & (n+o)b \\ (n+o)b & 0 \end{bmatrix} ?= \begin{bmatrix} 0 & nb \\ nb & 0 \end{bmatrix} + \begin{bmatrix} 0 & ob \\ ob & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & nb+ob \\ nb+ob & 0 \end{bmatrix} = \begin{bmatrix} 0 & nb+ob \\ nb+ob & 0 \end{bmatrix}$$

and passes.

9. Let **b** and k and m have the same values from axiom 8. Then

$$n\left(o\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}\right) ?= (no)\left(\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}\right)$$

$$n\begin{bmatrix} 0 & ob \\ ob & 0 \end{bmatrix} ?= \left(\begin{bmatrix} 0 & nob \\ nob & 0 \end{bmatrix}\right)$$

$$\begin{bmatrix} 0 & nob \\ nob & 0 \end{bmatrix} = \begin{bmatrix} 0 & nob \\ nob & 0 \end{bmatrix}$$

and passes.

10. Let **u** have the same value from axiom 1. Then

$$1\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1*b \\ 1*b & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$$

and passes.

Therefore (b) is a vector space, as all the axioms pass.

9. Determine which of the following are subspaces of R^3.

a) All vectors of the form $(a, 0, a)$.

b) All vectors of the form (a, b, c), where c = a / b.

Solution:

a) To show that (a) is a subspace of R^3, we must prove the 2 conditions of Theorem 5.2.1.

1. Let $\mathbf{u} = (b, 0, b)$ and $\mathbf{v} = (d, 0, d)$, then
$$(b, 0, b) + (d, 0, d) = (b + d, 0\ b + d)$$
which matches the description $(a, 0, a)$, as $b + d = b + d$. Passes.
2. Let $\mathbf{u} = (b, 0, b)$ and $k = n$, then
$$n(b, 0, b) = (nb, 0, nb)$$
which matches the description $(a, 0, a)$, as $nb = nb$. Passes.

Therefore (a) is a subspace of R^3 as both conditions hold.

b) To show that (b) is a subspace of R^3, we must prove the 2 conditions of Theorem 5.2.1.

1. Let $\mathbf{u} = (a, b, a/b)$ and $\mathbf{v} = (c, d, c/d)$, then
$$(a, b, a/b) + (c, d, c/d) = (a+c, b+d, a/b + c/d)$$
which doesn't match the description (a, b, c) where c = a / b, as $a/b + c/d \neq (a+c)/(b+d)$. Fails.

2. Let $\mathbf{u} = (a, b, a/b)$ and $k = n$, then

$$n(a, b, a/b) = (na, nb, na/b)$$

which doesn't match the description (a, b, c) where $c = a / b$, as $na/b \neq (na/nb = a/b)$. Fails.

Therefore (b) is not a subspace of R^3 as both conditions don't hold.

10. Determine whether the given vectors span R^3:

a) $\mathbf{v_1} = (1, 2, 3)$, $\mathbf{v_2} = (0, 2, 3)$, $\mathbf{v_3} = (0, 0, 3)$.

b) $\mathbf{v_1} = (2, 1, 1)$, $\mathbf{v_2} = (4, 1, 3)$, $\mathbf{v_3} = (7, -1, 8)$.

__Solution:__

a) The vectors here can be expressed as an augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The determinant of this final matrix is 1, and the system has only the trivial solutions and is linearly independent. Thus for any arbitrary vector (a, b, c), there are values of k_1, k_2 and k_3 that can be multiplied to the vectors in (a) to span R^3.

b) The vectors here can be expressed as an augmented matrix:

$$\begin{bmatrix} 2 & 4 & 7 \\ 1 & 1 & -1 \\ 1 & 3 & 8 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 9 \\ 1 & 1 & -1 \\ 0 & 2 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

The determinant of this final matrix is 0 and the solution has infinitely many solutions and is linearly dependent. Thus for any arbitrary vector (a, b, c), there are no values for k_1, k_2 and k_3 that can be multiplied to the vectors in (a) to span R^3. So it does not span R^3.

POINT VALUE OF PROBLEMS

1. (**24 Points**)
a) 8
b) 8
c) 8

2. (**12 Points**)
a) 3
b) 4
c) 5

3. (**14 Points**)
a) 4
b) 4
c) 6

4. (**14 Points**)
a) 7
b) 7

5. (**8 Points**)

6. (**12 Points**)
a) 7
b) 5

7.(**12 Points**)
a) 4
b) 4
c) 4

8. (**24 Points**)
a) 12
b) 12

9. (**14 Points**)
a) 7
b) 7

10. (**12 Points**)
a) 6
b) 6

Total = 146 Points

1) Find two vectors in $\mathbf{R}^3$ such that the norm of their sum is greater than the sum of their norms.

Solution:

By the triangle inequality, this is impossible. In other words, no side of a triangle (in two dimensions by ~~definition~~ theorem and in n dimensions by analogy) is longer than the sum of the other sides, and is only equal in the limiting case in which one of the angles of the triangle is 180 degrees.

2) Find the standard matrix for the following linear operators in R^3:

a) A rotation of 90° about the z-axis, followed by a rotation of 90° about the y-axis, followed by a rotation of 90° about the x-axis.

b) A reflection about the yz-plane, followed by a dilation of ½.

Solution:

a) The linear transformation T can be expressed as the composition $T = T_3 \circ T_2 \circ T_1$ where

$$[T_1]=\begin{bmatrix}\cos(90) & -\sin(90) & 0\\ \sin(90) & \cos(90) & 0\\ 0 & 0 & 1\end{bmatrix}\quad [T_2]=\begin{bmatrix}\cos(90) & 0 & \sin(90)\\ 0 & 1 & 0\\ -\sin(90) & 0 & \cos(90)\end{bmatrix}\quad [T_3]=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(90) & -\sin(90)\\ 0 & \sin(90) & \cos(90)\end{bmatrix}$$

From Eqn 22, pg. 193: $[T_3 \circ T_2 \circ T_1]=[T_3][T_2][T_1]$ so:

one more step – not obvious

$$[T]=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(90) & -\sin(90)\\ 0 & \sin(90) & \cos(90)\end{bmatrix}\begin{bmatrix}\cos(90) & 0 & \sin(90)\\ 0 & 1 & 0\\ -\sin(90) & 0 & \cos(90)\end{bmatrix}\begin{bmatrix}\cos(90) & -\sin(90) & 0\\ \sin(90) & \cos(90) & 0\\ 0 & 0 & 1\end{bmatrix}\overset{\checkmark}{=}\begin{bmatrix}0 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$$

b) From above:

$$[T_1]=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\quad [T_2]=\begin{bmatrix}\frac{1}{2} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\text{ and}$$

$$[T_2 \circ T_1]=[T_2][T_1]=\begin{bmatrix}\frac{1}{2} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-\frac{1}{2} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}.$$

3) If the linear operator $T : R^3 \rightarrow R^3$ defined by the following equations is one-to-one, find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2, w_3)$.

a) $\begin{aligned} w_1 &= x_1 + 4x_2 - 3x_3 \\ w_2 &= -x_1 + 3x_2 + 3x_3 \\ w_3 &= -x_1 - 4x_2 - 3x_3 \end{aligned}$ b) $\begin{aligned} w_1 &= x_1 - x_2 + x_3 \\ w_2 &= -2x_1 + 2x_2 + 2x_3 \\ w_3 &= 3x_1 - 3x_2 + 3x_3 \end{aligned}$

Solution:

lot's of work

a) Standard matrix for T: $[T] = \begin{bmatrix} 1 & 4 & -3 \\ -1 & 3 & 3 \\ -1 & -4 & -3 \end{bmatrix}$ and $\det[T] \neq 0$ so the matrix is invertible and by Theorem 4.3.1 the standard matrix for

lot's of work

$$[T^{-1}] = [T]^{-1} = \begin{bmatrix} \frac{-1}{14} & \frac{-4}{7} & \frac{-1}{2} \\ \frac{1}{7} & \frac{1}{7} & 0 \\ \frac{-1}{6} & 0 & \frac{-1}{6} \end{bmatrix}, \text{ thus:}$$

$$[T^{-1}]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{14} & \frac{-4}{7} & \frac{-1}{2} \\ \frac{1}{7} & \frac{1}{7} & 0 \\ \frac{-1}{6} & 0 & \frac{-1}{6} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{14}w_1 + \frac{-4}{7}w_2 + \frac{-1}{2}w_3 \\ \frac{1}{7}w_1 + \frac{1}{7}w_2 + 0w_3 \\ \frac{-1}{6}w_1 + 0w_2 + \frac{-1}{6}w_3 \end{bmatrix} \text{ ~~and~~ or}$$

$$T^{-1}(w_1, w_2, w_3) = \left(-\frac{1}{14}w_1 - \frac{4}{7}w_2 - \frac{1}{2}w_3, \frac{1}{7}w_1 + \frac{1}{7}w_2, -\frac{1}{6}w_1 - \frac{1}{6}w_3\right)$$

lot's of work

b) Following the same logic as above: $T : [T] = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 2 & 2 \\ 3 & -3 & 3 \end{bmatrix}$ and $\det[T] = 0$ so this matrix is not invertible or one-to-one and no standard matrix for T^{-1} exists.

4) Show that the range of the linear operator defined by the following equations is not all of R^3, and find a vector that is not in the range.

$$w_1 = x_1 - 2x_2 + 8x_3$$
$$w_2 = 2x_1 - x_2 + 2x_3$$
$$w_3 = 6x_1 - 3x_2 + 1x_3$$

Solution:

By theorem 4.3.1, if A is an n x n matrix and $T_A : R^n \rightarrow R^n$ is a multiplication by A, then the following statements are equivalent:
a) A is invertible
b) The range of T_A is R^n
c) T_A is one-to-one

lots of work

So $A = \begin{bmatrix} 1 & -2 & 8 \\ 2 & -1 & 2 \\ 6 & -3 & 1 \end{bmatrix}$ and $\det[A] = 0$ thus A is not invertible and the range of A is not

is it? Reduce (A|b), with y arbitrary.

all of R^n. A vector outside the range could be: (-6, 3, -1).

Assuming W is a subset

5) What two axioms of a vector space must be shown to prove that W is a subspace of V, and show using these axioms whether the sets of vectors of the form (a, 0, b) and (a, b, c), (where c = a + 1) are subspaces of $\mathbf{R}^3$ with normal vector addition and scalar multiplication.

Solution: For the first part of this question axioms in question are 1 and 6, which are the axioms implying closure under addition (**u** and **v** are vectors in W, then **u** + **v** are in W for "+" the addition of the space V) and closure under scalar multiplication (k is any scalar, then k***u** is in W as well for "*" the multiplication of the space V).

a.) According to axiom 1, if (a,0,b) + (x,0,y) is an element of W then W is closed under addition and that part of the theorem defining subspaces is satisfied. Since (a + x, 0 + 0, b + y) is the sum of these two vectors under $\mathbf{R}^3$'s vector addition and it is still of the form (a,0,b) (trivially because of **R**'s closure under addition) this follows naturally. To demonstrate axiom 6, given any scalar k, when multiplied to the system with normal scalar multiplication, yielding (ka, 0, kb), we find that the product vector is still of the form (a,0,b) and so this part of the theorem is likewise satisfied. Since both axioms 1 and 6 are satisfied for this space, by theorem 5.2.1 the vectors of the form (a,0,b) together with real scalars, real vector addition, and real scalar multiplication form a subspace of $\mathbf{R}^3$.

Once again we choose an arbitrary vector identical in form to add to our posited one,

(x, y, x+1), giving us after vector addition a vector that looks like (a + x, b + y,

a+x+2), which is not of the form (a, b, a+1) and therefore addition is not closed, therefore by 5.2.1 vectors of this form (with real scalars, multiplication, addition) do not form a subspace of $\mathbf{R}^3$.

6) Determine whether the vectors $\mathbf{v}_1 = (1, 2, 4)$, $\mathbf{v}_2 = (5, 6, -1)$, $\mathbf{v}_3 = (3, 2, 1)$ form a linearly dependent or independent set.

Solution

Write the vectors in a vector equation:

$k_1\mathrm{v}_1 + k_2\mathrm{v}_2 + k_3\mathrm{v}_3 = 0 \Rightarrow k_1(1, 2, 4) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$

Equating corresponding components gives:

$$\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ 2k_1 + 6k_2 + 2k_3 &= 0 \\ 4k_1 - k_2 + k_3 &= 0 \end{aligned}$$

This can be expressed in matrix form to solve for the coefficients k_1, k_2, and k_3

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ 2 & 6 & 2 & 0 \\ 4 & -1 & 1 & 0 \end{bmatrix} \text{ which reduces to the augmented matrix } \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & \frac{3}{13} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\rightarrow k_2 = k_3 = 0$, k_1 arbitrary.

So $k_3 = 0$ and by substitution $k_2 = 0$ and so $k_1 = 0$ after considering the original equations, which necessarily implies that the vectors form a linearly independent set. Something's wrong

7) Determine the dimension of and a basis for the solution space of the system

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ 2x_1 - 6x_2 + 2x_3 &= 0 \\ 3x_1 - 9x_2 + 3x_3 &= 0 \end{aligned}$$

Solution

The general solution which can be determined from a reduced row echelon form matrix (or by inspection since the second two equations are multiples of the first) is

$$x_1 - 3x_2 + x_3 = 0$$

which can also be written in the form

$$x_1 = 3x_2 - x_3$$
$$3x_2 = x_1 + x_3$$
$$x_3 = 3x_2 - x_1$$

which can also be written

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t - s \\ t \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

vectors

The bas~~e~~s are then (-1, 0, 1) and (3, 1, 0) and the dimension is 2 since there are two basis vectors **s** and **t**.

8) Find a basis for the space spanned by the vectors

$\mathbf{a}_1 = (1, 3, 2, 4, 2)$, $\mathbf{a}_2 = (0, 1, 3, 2, 1)$,
$\mathbf{a}_3 = (0, -4, 2, -4, 0)$, $\mathbf{a}_4 = (0, 0, 2, 1, 2)$.

Solution: We begin by putting these vectors into an augmented matrix and reduce to row-echelon form, like so,

$$\begin{bmatrix} 1 & 3 & 2 & 4 & 2 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & -4 & 2 & -4 & 0 \\ 0 & 0 & 2 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 2 & 4 & 2 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 14 & 4 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 2 & 4 & 2 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1/2 & 1 \\ 0 & 0 & 7 & 2 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 2 & 4 & 2 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1/2 & 1 \\ 0 & 0 & 7 & 2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -7 & -2 & -1 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 3/2 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3/2 & 6 \\ 0 & 1 & 0 & 1/2 & -2 \\ 0 & 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 1 & 10/3 \end{bmatrix}.$$

Therefore ~~the~~ some vectors that form a basis for the row space are

$\mathbf{b}_1 = (1, 0, 0, 3/2, 6)$, $\mathbf{b}_2 = (0, 1, 0, 1/2, -2)$, $\mathbf{b}_3 = (0, 0, 1, 1/2, 1)$, $\mathbf{b}_4 = (0, 0, 0,1, 10/3)$.

9) Given the augmented matrix with m rows and n columns corresponding to an overdetermined linear system with no redundant equations, what must be the relationship between m and n?

Solution: augmented?. If its augmented, then there is one more column than unaugmented.

The number of rows m of this matrix must be greater than or equal to the number of columns n of the matrix, since only if this is true are there more independent equations than unknowns, or in other words more constraints than possible degrees of freedom.

10) Does the set of all elements of $\mathbf{R}^2$ always define a vector space? What can be added, changed or taken away to change its status as a vector space?

Solution:

The operations "+" and "*" must also be defined along with a scalar set, probably the reals, in order to satisfy the definition of a scalar, so these elements alone are not a vector space. Some possible changes are eliminating the zero vector, making "+"or "*" pathologically not-closed, or using rationals, irrationals, integers, naturals, complex numbers, surreal numbers, or flying iguanas as the scalars, all of which changes would destroy $\mathbf{R}^2$'s chances of being a vector space.

unless such form a "field"

Problem 1

Here is a vector in R^4 (-3,2,6,8), find the Euclidean Norm

Answer

$$\sqrt{(-3)^2 + 2^2 + 6^2 + 8^2} \qquad = \sqrt{113}$$

Problem 2

Find the standard matrix for the stated composition of linear operators on R^3

A projection onto the yz-plane, followed by a rotation of 45 degrees counter-clockwise about the positive z-axis, followed by a reflection about the xy-plane.

Answer

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{-\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Problem 3

Determine whether the linear operator T: $R^3 \to R^3$ defined by the equations is one-to-one; if so, find the standard matrix for the inverse operator.

$$\begin{aligned} w_1 &= 4x_1 + 2x_2 + x_3 \\ w_2 &= 1x_1 + 1x_2 + 6x_3 \\ w_3 &= 3x_1 + 2x_2 + 6x_3 \end{aligned}$$

Answer

let's check

$$A := \begin{pmatrix} 4 & 2 & 1 \\ 1 & 1 & 6 \\ 3 & 2 & 6 \end{pmatrix} \qquad |A| = -1$$ That implies that the transformation is one-to-one

let's check

$$A^{-1} = \begin{pmatrix} 6 & 10 & -11 \\ -12 & -21 & 23 \\ 1 & 2 & -2 \end{pmatrix}$$

Problem 4

Determine whether the following statement is true or false. Justify your answer by giving a logical argument or a counter example.

All vector spaces have an infinite number of vectors

Answer

False!

The zero vector space has only one vector .

Problem 5

Use known theorems to determine which of the following are subspaces of R^3. If not a subspace of R^3 then specify which of the axioms it violates.

(A) all vectors of the form (a,b,a+b)

(B) all vectors of the form (a,b,0)

(C) all vectors of the form (a,b,1)

Answer

(A) and (B) are subspaces of R^3 (C) violates axiom 1 because the set is not closed with respect to addition.

and many other problems.

Problem 6

Which of the following sets of vectors in P_2 are linearly dependent?

(A) $3 + 2x + 6x^2 = v_1$

$1 + 2x^2 = v_2$

$1 + 2x + 2x^2 = v_3$

(B) $1 + x^2 = w_1$

$2 + 2x + x^2 = w_2$

$3 + 2x + 4x^2 = w_3$

(C) $4 - 3x + 5x^2 = u_1$

$1 + 3x + 2x^2 = u_2$

$2 + x + 3x^2 = u_3$

Answer

The set of vectors $\{v_1,v_2,v_3\}$ is linearly dependent

The set of vectors $\{w_1,w_2,w_3\}$ is linearly independent

The set of vectors $\{u_1,u_2,u_3\}$ is linearly dependent

lots of work (cubed).

Problem 7

For the linearly dependent sets of vectors in problem 6 express one vector as a linear combination of the other two.

Answer

let's do work,

$3u_3 - 2u_2 = u_1$ $\qquad$ $v_1 - 2v_2 = v_3$

Problem 8

Find the coordinate vector of v relative to the basis $S = \{v_1, v_2, v_3\}$

$v := (4,8,2)$

$v_1 = (4,0,0)$

$v_2 = (2,1,0)$

$v_3 = (6,4,2)$

Answer

$$\begin{pmatrix} 4 & 2 & 6 & 4 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & 2 & 2 \end{pmatrix} \rightarrow$$

$\alpha - 4 + 3 = 2$ $\qquad$ $\gamma := 1$

$2b + 3 = -1$ $\qquad$ $\alpha := 3$

$3\gamma = 13$ $\qquad$ $\beta := -2$

write out as a coordinate vector, answer still unclear.

Problem 9

A matrix in row-echelon form is given, By inspection, find bases for the row and column spaces of A

$$\begin{pmatrix} 1 & -5 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Answer

$r_1 := (1\ \ -5\ \ 0\ \ 0)$

$r_2 := (0\ \ 1\ \ 0\ \ 0)$

$c_1 := (1\ \ 0\ \ 0)$

$c_2 := (-5\ \ 1\ \ 0)$

Problem 10

In each part, use the information in the table to find the dimension of the row space, column space, and null space of A.

what are you referring to?

(A) A is 4x4

(B) A is 3x5

Answer

(A) The largest possible value of the rank of A is 4, nullity is 0 $0 \le \operatorname{rank} A \le 4$, $4 \ge$ Nullity ≥ 0

(B) The largest possible value of the rank of A is 3, nullity is 2

$0 \le \operatorname{rank} A \le 3$,

$5 \ge$ Nullity ≥ 2

Not a well-~~thought out~~ stated problem, Ambiguous on a couple of levels.

Question 1:
Show that the linear operator T: R^2-> R^2 defined by the equations

$$w_1 = 2x_1 + 3x_2$$
$$w_2 = -x_1 + 5x_2$$

is one-to-one, and find T^{-1} (w_1, w_2).

Solution:

$$\begin{bmatrix} w1 \\ w2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix}$$

Standard matrix for T is $[T] = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$. $[T^{-1}] = 1/13 \begin{bmatrix} 5 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5/13 & -3/13 \\ 1/13 & 2/13 \end{bmatrix}$.

Because [T] is invertible it is one-to-one.

$$[T^{-1}] \begin{bmatrix} w1 \\ w2 \end{bmatrix} = \begin{bmatrix} 5/13w1 - 3/13w2 \\ 1/13w1 + 2/13w2 \end{bmatrix}$$

T^{-1} (w_1, w_2) = ($5/13w_1$-$3/13w_2$, $1/13\ w_1$ +$2/13\ w_2$)

Question 2:
a. Find the standard matrix for the linear operator $T : R3 \to R3$ given by

$$w_1 = 2x_1 + 4x_2 + 6x_3$$
$$w_2 = 3x_1 + 5x_2 + 11x_3$$
$$w_3 = 2x_1 + 2x_2 + 11x_3$$

then calculate T(3,-2,4) by (b.) directly substituting in the equations and (c.) by matrix multiplication.

Solution:

a. The standard matrix for the linear operator is $\begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 11 \\ 2 & 2 & 11 \end{bmatrix}$.

b. By substituting 3 for x_1, -2 for x_2, and 4 for x_3 we get $w_1 = 2(3) + 4(-2) + 6(4) = 22$

$$w_2 = 3(3) + 5(-2) + 11(4) = 43$$
$$w_3 = 2(3) + 2(-2) + 11(4) = 46$$

c. Using matrix multiplication we have $\begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 11 \\ 2 & 2 & 11 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2(3)+4(-2)+6(4) \\ 3(3)+5(-2)+11(4) \\ 2(3)+2(-2)+11(4) \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \\ 46 \end{bmatrix}$

Question 3:

a) Use matrix multiplication to find the image of (2, -1) when it is rotated through an angle of 60°.

b)Use matrix multiplication to find the image of (3, 1, 0) when it is rotated 45° about the x-axis.

Solution:

a) We take the given vector and multiply it with the vector $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ using the appropriate angle, as follows:

$$\begin{bmatrix} \cos(60) & -\sin(60) \\ \sin(60) & \cos(60) \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{2+\sqrt{3}}{2} \\ \frac{2\sqrt{3}-1}{2} \end{bmatrix}.$$

So the rotated vector is now $\left(\frac{2+\sqrt{3}}{2}, \frac{2\sqrt{3}-1}{2} \right)$.

b) We multiply the given vector with $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ using 45° as the angle:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(45) & -\sin(45) \\ 0 & \sin(45) & \cos(45) \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \\ \cos(45) \\ \sin(45) \end{bmatrix}$$
$$= \begin{bmatrix} 3 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}.$$

So the rotated vector is $(3, \sqrt{2}/2, \sqrt{2}/2)$.

Question 4:

$$A=\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$$

?

a) Find the Inverse of A and if the reduced row-echelon form of A.
b) What does this tell you about whether T_A is one-to-one?
c) What does this tell you about what solutions Ax=0 has?

Solution:

a) We can solve both parts of a using an augmented matrix. The row-reduction using elementary row operations is as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 4 & -3 & 0 & 1 & 0 \\ 3 & 6 & -5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -7 & -2 & 1 & 0 \\ 0 & 3 & -11 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -7/2 & -1 & 1/2 & 0 \\ 0 & 0 & -1/2 & 0 & -3/2 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -7/2 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 3 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -6 & 12 \\ 0 & 1 & 0 & -1 & 11 & -7 \\ 0 & 0 & 1 & 0 & 3 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -17 & 19 \\ 0 & 1 & 0 & -1 & 11 & -7 \\ 0 & 0 & 1 & 0 & 3 & -2 \end{array}\right]$$

b) By Theorem 4.3 we know that if the matrix has an inverse or its row-echelon form is I_n, then T_A is one-to-one. Therefore since we showed that it has an inverse and that its row-echelon form is I_n, T_A is one-to-one.

c) By the same theorem we know that Ax=0 has only the trivial solution.

Question 5:

Express the following as linear combinations of u = (2,1,-3), v = (4,3,-1), and w = (8,5,-11).

a. (0,0,0) b. (2,2,6) c. (-6,-4,12)

Solution:

Expressing the vectors as a linear combination of u, v, and w, with constants a, b, and c we have

$$\begin{aligned} (0,0,0) &= a(2,1,-3) + b(4,3,-1) + c(8,5,-11) \\ (2,2,6) &= a(2,1,-3) + b(4,3,-1) + c(8,5,-11) \\ (-6,-4,12) &= a(2,1,-3) + b(4,3,-1) + c(8,5,-11) \end{aligned}$$

Equating corresponding components give

2a + 4b + 8c = 0	2a + 4b + 8c = 2	2a + 4b + 8c = -6
a + 3b + 5c = 0	a + 3b + 5c = 2	a + 3b + 5c = -4
-3a + -b -11c = 0	-3a + -b -11c = 6	-3a + -b -11c = -12

Because all of these systems of equations have the same coefficient matrix they can solved simultaneously by augmenting the matrix with the solutions of all three and row reducing

$$\left[\begin{array}{ccc|c|c|c} 2 & 4 & 8 & 0 & 2 & -6 \\ 1 & 3 & 5 & 0 & 2 & -4 \\ -3 & -1 & -11 & 0 & 6 & 12 \end{array}\right] \quad \begin{array}{l} R1 \leftrightarrow R2 \\ \rightarrow \end{array} \quad \left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 0 & 2 & -4 \\ 2 & 4 & 8 & 0 & 2 & -6 \\ -3 & -1 & -11 & 0 & 6 & 12 \end{array}\right]$$

$$\begin{array}{l} R2 = R2 - 2R1 \\ R3 = R3 + 3R1 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 0 & 2 & -4 \\ 0 & -2 & -2 & 0 & -2 & 2 \\ 0 & 8 & 4 & 0 & 12 & 0 \end{array}\right]$$

$$\begin{array}{l} R2 = R2/-2 \\ R3 = R3 + 4R2 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 0 & 2 & -4 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & -4 & 0 & 4 & 8 \end{array}\right]$$

$$\begin{array}{l} R4 = R4/-4 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 0 & 2 & -4 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -2 \end{array}\right]$$

$$\begin{array}{l} R1 = R1 - 5R3 \\ R2 = R2 - R3 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c|c|c} 1 & 3 & 0 & 0 & 7 & 6 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & -1 & -2 \end{array}\right]$$

$$\begin{array}{l} R1 = R1 - 3R2 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & -1 & -2 \end{array}\right]$$

So a. a = 0, b = 0, c = 0 and (0,0,0) = 0u + 0v + 0w

b. a = 1, b = 2, c = -1 and (2,2,6) = u + 2v - w

c. a = 3, b = 1, c = -2 and (-6,-4,12) = 3u + v – 2w

Question 6:

Determine whether the following vectors span the vector space R^3:
$\underline{v}$ = (1, 3, 2), $\underline{u}$ = (1, 1, 4), and $\underline{w}$ = (2, 0, -1).

Solution:

To find out whether or not these vectors span R^3 we need to find out if we can make a linear combination of them to form any arbitrary vector in the vector space. By theorem 4.3.4 this is consistent for R^3 if and only if the coefficient matrix has a non-zero determinant, so we will use the vector $\underline{a}$ = (a1, a2, a3). So we get the following equation:

$\underline{a} = j\underline{v} + k\underline{u} + m\underline{w}$, or:

(a1, a2, a3) = j(1, 3, 2) + k(1, 1, 4) + m(2, 0, -1).

So, using this equation we can set up a system of equations as follows:

$$\begin{aligned} j + k + 2m &= a1 \\ 3j + k \quad &= a2 \\ 2j + 4k - m &= a3 \end{aligned}$$

This system will be consistent if the coefficient matrix is consistent. We will write the matrix and test to see if the determinant $\neq 0$.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ 2 & 4 & -1 \end{bmatrix}$$

det(A) = (1)(1)(-1) + 0 + (2)(3)(4) – 0 – (1)(3)(-1) – (2)(1)(2)
= -1 + 24 +3 – 4
= 22 $\neq$ 0, so the given vectors do span R^3.

Question 7:

Determine if the vectors in R^3 are linearly dependent.

a. (1,2,0), (-2,1,-4)
b. (-3,0,2), (1,1,-2), (-2,5,1)
c. (1,5,1), (1,2,3), (5,1,9), (0,1,9)

Solution:

a. $\begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & -3 \\ 0 & -4 \end{bmatrix}$ Since there is only the trivial solution, to what? these vectors are not linearly dependent

this is not in echelon form - hard to conclude anything just yet.

b. $\left[\begin{array}{cc|c} -3 & 1 & -2 \\ 0 & 1 & 5 \\ 2 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 1/2 \\ 0 & 1 & 5 \\ 0 & -2 & 1 \end{array}\right]$ Since there is only the trivial solution, these vectors are not linearly dependent

c. is linearly dependent by inspection, by theorem 5.3.3 r>n
~~(and none of the vectors are scalar multiples of each other)~~

Question 8:

Determine the dimension of and basis for the solution space of the system

$$\begin{aligned} 3x_1 - x_2 + 2x_3 - 4x_4 + x_5 &= 0 \\ 9x_1 - 3x_2 + 6x_3 - 12x_4 + 3x_5 &= 0 \end{aligned}$$

Solution:

Because the bottom equation is just a scalar multiple of the first we just need to solve for one of them

Let $x_2 = r$, $x_3 = s$, $x_4 = t$, $x_5 = u$, then $x_1 = r - 2s + 4t - u$

Putting this in matrix form we have

$$\begin{bmatrix} \dfrac{r-2s+4t-u}{3} \\ r \\ s \\ t \\ u \end{bmatrix} = \begin{bmatrix} \frac{1}{3}r \\ r \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{2}{3}s \\ 0 \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}t \\ 0 \\ 0 \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3}u \\ 0 \\ 0 \\ 0 \\ u \end{bmatrix} =$$

$$r\begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -\frac{2}{3} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{4}{3} \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So a basis for the solution space is $\left\{ \begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{4}{3} \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

and the solution space is four-dimensional.

Question 9:

$$A = \begin{bmatrix} 1 & 4 & 6 & 3 & 2 & 1 \\ 0 & 6 & 3 & 9 & 6 & 3 \\ 2 & 2 & 3 & 9 & -2 & 3 \\ -1 & 1 & -3 & 11/2 & 3 & 2 \end{bmatrix}$$

a) Find ~~the~~ a row basis for the row space of A.
b) Find ~~the~~ a basis for the column space of A.

Solution:
First we need to put A in row reduced form. The row-reduction using elementary row operations is as follows:

$$\begin{bmatrix} 1 & 4 & 6 & 3 & 2 & 1 \\ 0 & 6 & 3 & 9 & 6 & 3 \\ 2 & 2 & 3 & 9 & -2 & 3 \\ -1 & 1 & -3 & 11/2 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 6 & 3 & 2 & 1 \\ 0 & 6 & 3 & 9 & 6 & 3 \\ 0 & -6 & -9 & 3 & -6 & 1 \\ 0 & 5 & 2 & 17/2 & 5 & 3 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & 4 & 6 & 3 & 2 & 1 \\ 0 & 1 & 1/2 & 3/2 & 1 & 1/2 \\ 0 & 0 & -6 & 12 & 0 & 6 \\ 0 & 0 & -1/2 & 1 & 0 & 1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 6 & 3 & 2 & 1 \\ 0 & 1 & 1/2 & 3/2 & 1 & 1/2 \\ 0 & 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & -1/2 & 1 & 0 & 1/2 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & 4 & 6 & 3 & 2 & 1 \\ 0 & 1 & 1/2 & 3/2 & 1 & 1/2 \\ 0 & 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we can find a. From Theorem 5.5.6 we find that the vectors that ~~form~~ span the row space of A are:

$[1 \quad 4 \quad 6 \quad 3 \quad 2 \quad 1]$, $[0 \quad 1 \quad 1/2 \quad 3/2 \quad 1 \quad 1/2]$, and $[0 \quad 0 \quad 1 \quad -2 \quad 0 \quad -1]$.

For b, usine the same theorem we find that ~~the~~ same vectors that ~~form~~ span the column space of A are:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 6 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}.$$

- not true!

False!

Question 10:

Find the rank and nullity of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 7 \\ -2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}.$$

Solution:

To find first the rank, we need to get A in reduced-row echelon form.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 7 \\ -2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 7 \\ 2 & 1 & 1 \\ -2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -7 \\ 0 & 5 & 15 \\ 0 & -3 & -9 \\ 0 & 9 & 27 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So rank(A) = 2 because there are two rows with leading ones.

The nullity is obtained by finding out how many dimensions there are as follows:

$x - z = 0$
$y + 3z = 0$

The general solution is:

$x = t$
$y = -3t$
$z = t$

So nullity(A) = 1 because
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}.$$

1. (**11 points**) In each part, calculate the Euclidean norm of the vector
(a) (4,7)
(b) (3, -1,9)
(c) (0,1,-2,3)
(d) (5,0,1,1,-3)

Solution

(a) $\sqrt{}((4)^{\wedge}2 + (7)^{\wedge}2\ ) = \sqrt{}(16+49) = \sqrt{}(65)$

(b) $\sqrt{}((3)^{\wedge}2 + (-1)^{\wedge}2 + (9)^{\wedge}2) = \sqrt{}(9+1+81) = \sqrt{}(91)$

(c) $\sqrt{}((0)^{\wedge}2 + (1)^{\wedge}2 + (-2)^{\wedge}2 + (3)^{\wedge}2) = \sqrt{}(0+1+4+9) = \sqrt{}(14)$

(d)$\sqrt{}((5)^{\wedge}2 + (0)^{\wedge}2 + (1)^{\wedge}2 + (1)^{\wedge}2 +(-3)^{\wedge}2) = \sqrt{}(25+0+1+1+9) = \sqrt{}(36) = 6$

2. (**11 points**) Determine whether the following vectors are orthogonal
(a) u = (1,2,3), v = (1,2,3)
(b) u = (-2,4,1), v = (4,2,0)
(c) u = (x,y,z), v = (0,0,0)
(d)u = (9,0,1,5) v = (0,1,0,1)

Solution

(a) u·v = (1)(1) + (2)(2) + (3)(3) = 14 ,not orthogonal
(b) u·v = (-2)(4) + (4)(2) + (1)(0) = 0, orthogonal
(c)u·v = (x)(0) + (y)(0) +(z)(0) = 0, orthogonal
(d)u·v = (9)(0) + (0)(1) + (1)(0) + (5)(1) = 5, not orthogonal

3. (**17 points**) Determine a basis and dimension for the following solution space of the system.

2(x1) + 4(x2) - 2(x3) = 0
1(x1) + 2(x2) - 1(x3) = 0
3(x1) + 6(x2) - 3(x3) = 0

Solution

First, represent the system of equations through an augmented matrix.

[2	4	-2	0]
[1	2	-1	0]
[3	6	-3	0]

by inspection you can determine that row 3 = 3*(row 2), and that row 1 = 2*(row 2) so the matrix ends up looking like [1 2 -1 0]

this means we have to use parameters. The solution vectors are: $(x1) = -2r + s$, $(x2) = r$, $(x3) = s$

The solution vectors can be written as

$$\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} -2r+s \\ r \\ s \end{bmatrix} = \begin{bmatrix} -2r \\ r \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \\ s \end{bmatrix} = r\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

this shows that the vectors $(v1) = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $(v2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

span the solution space. Because they are also linearly independent, they form the basis and the solution space is two-dimensional.

4. (**13 points**) Find the standard Matrix for the linear operator defined by the equations, and determine whether the operator is one-to-one (hint: use Theorem 4.3.4)

$$\begin{aligned} 4x_1 + 2x_2 + 3x_3 + x_4 &= w_1 \\ -x_2 + 13x_3 + 7x_4 &= w_2 \\ 7x_3 &= w_3 \\ 7x_4 &= w_4 \end{aligned}$$

Solution

The standard matrix can be displayed as the following:

$$\begin{bmatrix} 4 & 2 & 3 & 1 \\ 0 & -1 & 13 & 7 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

Theorem 4.3.4

If A is an n x n matrix, and if $T_A: R^n \rightarrow R^n$ is a multiplication by A, then the following are equivalent:

a) A is invertible
b) A**x** = **0** has the only trivial solution
c) The reduced row echelon form of A is I_n.
d) A is expressible as a product of elementary matrices.
e) A**x** = **b** is consistent for every n x 1 matrix **b**.
f) A**x** = **b** has exactly one solution for every n x 1 matrix **b**.
g) $\det(A) \neq 0$
h) The range of T_A is R^n.
i) T_A is one-to-one.

Since det(A) = -196 $\neq 0$, therefore the operator is one-to-one

5. (**15 points**) Determine if the following vectors are linearly dependent or linearly independent.

$$v_1 = (1, -2, -2),\ v_2 = (2, 0, 4),\ v_3 = (0, 1, -1)$$

Solution

$$k_1v_1 + k_2v_2 + k_3v_3 = 0$$

becomes

$$k_1(1, -2, -2) + k_2(2, 0, 4) + k_3(0, 1, -1) = (0, 0, 0)$$

or, equivalently:

$$(k_1 + 2k_1,\ 2k_1 + 2k_3,\ -2k_1 + 4k_2 - k_3) = (0, 0, 0)$$

Equations corresponding components gives:

$$\begin{aligned} k_1 + k_2 \quad &= 0 \\ -2k_1 + \quad + k_3 &= 0 \\ -2 k_1 + 4k_2 - k_3 &= 0 \end{aligned}$$

Which can be shown as the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ -2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which reduces by Gaussian elimination to

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & ½ \\ 0 & 0 & 1 \end{bmatrix}$$

Which is code for:

$$\begin{aligned} k_1 + k_2 \quad &= 0 \\ k_2 + ½\, k_3 &= 0 \\ k_3 &= 0 \end{aligned}$$

Which reduces further to:

$$k_2 + ½(0) = 0 = k_2$$

Which reduces further to:

$$k_1 + (0) = 0 = k_1$$

Therefore:

$$k_1 = k_2 = k_3 = 0$$

Therefore the set of vectors $\{v_1, v_2, v_3\}$ is linearly independent.

6. **(18 points)** Find the rank and nullity of the matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 & 7 \\ 2 & 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 3 & -1 \end{bmatrix}$$

Solution

Calculating the reduced row echelon form of A yields:

$$\begin{bmatrix} 1 & 0 & 2 & 4 & 7 \\ 0 & 1 & -2 & -7 & -11 \\ 0 & 1 & 0 & -1 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 4 & 7 \\ 0 & 1 & -2 & -7 & -11 \\ 0 & 0 & 2 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & 4 \\ 0 & 1 & 0 & -1 & -8 \\ 0 & 0 & 2 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & 4 \\ 0 & 1 & 0 & -1 & -8 \\ 0 & 0 & 1 & 3 & 3/2 \end{bmatrix}$$

Since there are three nonzero rows, the row and column space are both three dimensional, so rank(A) = 3

A in reduced row echelon form is code for:

$$\begin{aligned} x_1 \qquad\qquad - 2x_4 + 4x_5 &= 0 \\ x_2 \qquad - x_4 - 8x_5 &= 0 \\ x_3 + 3x_4 + 3/2x_5 &= 0 \end{aligned}$$

Solve for leading variables:

$$\begin{aligned} x_1 &= 2x_4 - 4x_5 \\ x_2 &= x_4 + 8x_5 \\ x_3 &= -3x_4 - 3/2x_5 \end{aligned}$$

It follows that the several solution of the system is

$$\begin{aligned} x_1 &= 2r - 4s \\ x_2 &= r + 8s \\ x_3 &= -3r - 3/2s \\ x_4 &= r \\ x_5 &= s \end{aligned}$$

more work than needed to answer question.

or equivalently:

$$** \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r\begin{bmatrix} 2 \\ 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 8 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$$

Because the two vectors on the right side of ** form a basis for the solution space, nullity(A) = 2

7. **(13 points)** "Find matrix $A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ rotated 90 degrees about the z axis and then reflected about the xz plane."

interpretated as per your solution, I suppose,

Solution

$$T_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(T_2 \bullet T_1)A = T_2(T_1 A) =$$

$$T_2 \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 & 3 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$T_2 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 & 3 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & -1 & -4 \\ 1 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\boldsymbol{\begin{bmatrix} -2 & -1 & -4 \\ -1 & -5 & -3 \\ 0 & 0 & 1 \end{bmatrix}}$$

8. **(17 points)** Do the polynomials $1 + 2x + 3x^2$, $4 + 7x + 10x^2$, $3 + 5x + 7x^2$, $2 + 3x + 4x^2$ span P_2?

Solution

how?. Talk about basis for P_2, etc.

Rewrite the polynomials in matrix form.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 10 \\ 3 & 5 & 7 \\ 2 & 3 & 4 \end{bmatrix}$$ which can be reduced as follows

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix} \sim$$

[1 2 3]
[0 -1 -2]
[0 0 0]
[0 0 0]

So these polynomials do not span P_2 because there are only 2 independent polynomials.

9. **(18 points)** Find bases for the row and column spaces of
A = [4 2 8 11]
[2 1 6 -2]
[1 9 5 8]

<u>Solution</u>

A = [4 2 8 11]
[2 1 6 -2]
[1 9 5 8] which can be reduced as follows

[1 9 5 8]
[0 -17 -4 -18] ~
[0 -34 -12 -21]

[1 9 5 8]
[0 17 4 18] ~
[0 0 -4 15]

[1 9 5 8]
[0 1 4/17 18/17] = R
[0 0 1 -15/4]

The basis for the row space of R or A is
r_1 = [1 9 5 8]
r_2 = [0 1 4/17 18/17]
r_3 = [0 0 1 -15/4]

The basis for the column space of R is
C^1_1 = [1] C^1_2 = [9] C^1_3 = [5]
[0] [1] [4/17]
[0] [0] [1]
The corresponding basis for the column space of A is
C_1 = [4] C_2 = [2] C_3 = [8]
[2] [1] [6]
[1] [9] [5]

good problem.

10.(**17 points**)If $T: R2 \to R2$ is linear, what is $T\begin{bmatrix}2\\2\end{bmatrix}$ given that $T\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$ and $T\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}6\\5\end{bmatrix}$?

<u>Solution</u>

$$\begin{bmatrix}2\\2\end{bmatrix} = a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}1 & -2\\2 & 1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} \Rightarrow \begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}1 & -2\\2 & 1\end{bmatrix}^{-1}\begin{bmatrix}2\\2\end{bmatrix}$$

Also,

$$T\begin{bmatrix}2\\2\end{bmatrix} = a\bullet T\begin{bmatrix}1\\2\end{bmatrix} + b\bullet T\begin{bmatrix}-2\\1\end{bmatrix} = a\begin{bmatrix}3\\2\end{bmatrix} + b\begin{bmatrix}6\\5\end{bmatrix} = \begin{bmatrix}3 & 6\\2 & 5\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$$

$$= \begin{bmatrix}3 & 6\\2 & 5\end{bmatrix}\begin{bmatrix}1 & -2\\2 & 1\end{bmatrix}^{-1}\begin{bmatrix}2\\2\end{bmatrix} = (1/5)\begin{bmatrix}3 & 6\\2 & 5\end{bmatrix}\begin{bmatrix}1 & 2\\-2 & 1\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} = (1/5)\begin{bmatrix}-9 & 12\\-8 & 9\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} = (1/5)\begin{bmatrix}6\\2\end{bmatrix}$$

$$= \mathbf{\begin{bmatrix}6/5\\2/5\end{bmatrix}}$$

Exam 2

1. (5 points) Prove the Pythagorean Theorem in R^n; If u and v are orthogonal vectors in R^n with the Euclidean inner product then, $\|u\|^2 + \|v\|^2 = \|u+v\|^2$.

Solution:

$$\|u+v\|^2 := (u+v)\cdot(u+v)$$

$$\|u+v\|^2 = u\cdot u + 2u\cdot v + v\cdot v$$

by definition the Euclidean inner product of orthogonal vectors is zero, therefore:

$$\|u+v\|^2 = u\cdot u + v\cdot v =:$$

$$\|u\|^2 + \|v\|^2,$$

2. (20 points) Given that the set S={(2,3,0), (3,2,1), (1,0,1)} is a basis for R^3 — good problem

a) Find the coordinate vector of **v** = (2,2,2) with respect to S.

b) Find a vector **v** in R^3 whose coordinate vector with respect to the basis S is (2,2,2).

Solution:

a) **v** = (2,2,2) = c_1(2,3,0)+ c_2(3,2,1) + c_3(1,0,1), where $(\mathbf{v})_s = (c_1, c_2, c_3)$.

We can express this as a system of linear equations as:

$$\begin{aligned} 2 &= 2c_1 + 3c_2 + 1c_3 \\ 2 &= 3c_1 + 2c_2 \\ 2 &= \phantom{3c_1 +{}} 1c_2 + 1c_3 \end{aligned}$$

Expressed as a matrix:

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 & 2 \\ 3 & 2 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -5/2 & -3/2 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to$$

$$\begin{bmatrix} 2 & 3 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

We conclude $(\mathbf{v})_s$ = (2, -2, 4).

b) Since $(\mathbf{v})_s = (c_1, c_2, c_3)$ in $\mathbf{v} = c_1(2,3,0) + c_2(3,2,1) + c_3(1,0,1)$, and since it is given that $(\mathbf{v})_s = (2,2,2)$, we can say that:

$$\mathbf{v} = (2)(2,3,0) + (2)(3,2,1) + (2)(1,0,1)$$
$$\mathbf{v} = (4,6,0) + (6,4,2) + (2,0,2)$$
$$\mathbf{v} = (12,10,4)$$

3. (15 points) Find the bases for the row and column spaces of A.

$$A = \begin{bmatrix} 4 & 12 & 8 \\ 0 & 3 & 6 \\ 4 & 9 & 2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 4 & 12 & 8 \\ 0 & 3 & 6 \\ 4 & 9 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 12 & 8 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$r_1 = [1 \quad 3 \quad 2] \quad r_2 = [0 \quad 1 \quad 2]$$

the first and second columns of the row-reduced form of A contain leading ones so the first and second columns of A ~~*form*~~ span *the column* ~~*basis*~~ space *of A*

$$c_1 = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} \quad c_2 = \begin{bmatrix} 12 \\ 3 \\ 9 \end{bmatrix}$$

Therefore, (1,3,2) and (0,1,2) form ~~the~~ a basis for the row space of A and $\begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$ *and* $\begin{bmatrix} 12 \\ 3 \\ 9 \end{bmatrix}$ form ~~the~~ a basis for the column space of A.

4. (15 points) Determine whether $\mathbf{v}_1 = (2,5,4), \mathbf{v}_2 = (1,-2,-1)$, and $\mathbf{v}_3 = (3,3,3)$ span the vector space $\mathbf{R}^3$.

Solution:

?

We must find $\mathbf{b} \in \mathbf{R}^3$, such that $\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$, (where $\mathbf{b} \neq 0$ and $k_1, k_2, k_3 \neq 0$.)?. Close to irrelevant.

*Continued on the following page.

Expressing this as a linear equation we write $\begin{pmatrix} 2k_1 + k_2 + 3k_3 = b_1 \\ 5k_1 - 2k_2 + 3k_3 = b_2 \\ 4k_1 - k_2 + 3k_3 = b_3 \end{pmatrix}$.

This system of equations is consistent for all entries of **b** if and only if the coefficient matrix A,

$$A=\begin{bmatrix} 2 & 1 & 3 \\ 5 & -2 & 3 \\ 4 & -1 & 3 \end{bmatrix}$$

has a nonzero determinant.

$$\det(A) = \begin{vmatrix} 2 & 1 & 3 \\ 5 & -2 & 3 \\ 4 & -1 & 3 \end{vmatrix} \sim \begin{vmatrix} 2 & 1 & 3 \\ 0 & 9 & 9 \\ 0 & -3 & -3 \end{vmatrix} \sim \begin{vmatrix} 2 & 1 & 3 \\ 0 & 9 & 9 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

However, det (A) = 0, therefore $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ do not span $\mathbf{R}^3$.

5. (15 points) Show that the polynomials $\mathbf{p}_1 = 2x^2 - 3, \mathbf{p}_2 = x + 1$, and $\mathbf{p}_3 = 2x^2 + 3x + 6$ form a linearly independent set in P_2.

Solution:

If $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$ form a linearly independent set in P_2, then $k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 = \mathbf{0}$, only when ~~with~~ $k_1 = k_2 = k_3 = 0$. ~~as the only solutions.~~

The system of equations can be put in the matrix form and reduced:

talk about standard basis for P_2.

$$k_1(2x^2 - 3) + k_2(x+1) + k_3(2x^2 + 3x + 6) = \mathbf{0} \sim$$

$$\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ -3 & 1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Solving the system yields $k_1 = k_2 = k_3 = 0$, the trivial solution, as the only solution. Therefore $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$ form a linearly independent set in P_2.

6. (15 points) Determine whether $\mathbf{v}_1 = (2,5,4), \mathbf{v}_2 = (1,-2,-1)$, and $\mathbf{v}_3 = (3,3,3)$ form a linearly dependent or linearly independent set in $\mathbf{R}^3$. Verify your answer.

Solution:

By definition of linear dependency, if $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ form a linearly dependent set, then

$\exists \mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3 \in \mathbf{R}$ s.t. $\mathrm{k}_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ for $(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3) \neq (0, 0, 0)$.

To see if there exists such $(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3) \neq \underline{0}$, the system of equations is set up as the matrix:

$$\mathrm{A} = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 5 & -2 & 3 & 0 \\ 4 & -1 & 3 & 0 \end{bmatrix}$$

Evaluating we find:

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 5 & -2 & 3 & 0 \\ 4 & -1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & -9 & -9 & 0 \\ 0 & -3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is code for:

$$\begin{array}{lcl} k_1 + k_3 = 0 & & k_1 = -t \\ k_2 + k_3 = 0, & \text{or} & k_2 = -t \\ k_3 = t & & k_3 = t \end{array}$$

Substituting this into the equation $\mathrm{k}_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ yields

$$-\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$$

so what is the answer to your question?

To verify our answer, $-\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0} \sim -(2,5,4) - (1,-2,1) + (3,3,3) = \mathbf{0} \sim$

$(-2-1+3, -5+2+3, -4+1+3) = \mathbf{0}$

7. (20 points) Find the standard matrix for the stated composition of linear operators on R^3:

a) A reflection about the xy-plane, followed by dilation with factor k = 3/2, followed by a rotation of θ about the x-axis.

b) A rotation about the y-axis of θ, followed by a reflection on the yz-plane, followed by a contraction with factor k = ¼.

Solution:

a) First we find the standard matrices for these linear transformations.

$$[\mathrm{T1}] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \quad [\mathrm{T2}] = \begin{vmatrix} 3/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 3/2 \end{vmatrix} \quad [\mathrm{T3}] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{vmatrix}$$

*Continued on following page.

The standard matrix T is [T] = [T3][T2][T1]; that is

$$[T] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{vmatrix} \begin{vmatrix} 3/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 3/2 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} =$$

$$\begin{vmatrix} 3/2 & 0 & 0 \\ 0 & (3/2)\cos\theta & -(3/2)\sin\theta \\ 0 & (3/2)\sin\theta & (3/2)\cos\theta \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 3/2 & 0 & 0 \\ 0 & (3/2)\cos\theta & (3/2)\sin\theta \\ 0 & (3/2)\sin\theta & -(3/2)\cos\theta \end{vmatrix}$$

b) First we find the standard matrices for these linear transformations.

$$[T1] = \begin{vmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{vmatrix} \qquad [T2] = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad [T3] = \begin{vmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{vmatrix}$$

The standard matrix T is [T] = [T3][T2][T1]; that is

$$[T] = \begin{vmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{vmatrix} \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{vmatrix} =$$

$$\begin{vmatrix} -1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{vmatrix} \begin{vmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{vmatrix} = \begin{vmatrix} -(1/4)\cos\theta & 0 & -(1/4)\sin\theta \\ 0 & 1/4 & 0 \\ -(1/4)\sin\theta & 0 & (1/4)\cos\theta \end{vmatrix}$$

8. (15 points) Show that the linear operator T: **R2** → **R2** defined by the equations:

$$w_1 = 4x_1 + 2x_2$$
$$w_2 = 5x_1 + 3x_2$$

is one-to-one , and find T^{-1}(w1,w2).

Solution:

The matrix form of these equations is $\begin{vmatrix} w_1 \\ w_2 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$

So the standard matrix T is $[T] = \begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix}$

*Continued on following page.

If we are able to find that [T] is invertible, we will at the same time show that T is one to one.

$$\begin{bmatrix} 4 & 2 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 & 1 & 0 \\ 0 & -2 & 5 & -4 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 6 & -4 \\ 0 & -2 & 5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3/2 & -1 \\ 0 & 1 & -5/2 & 2 \end{bmatrix}$$

We see that [T] is invertible, proving it is one to one, and that:

$$[T]^{-1} = \begin{bmatrix} 3/2 & -1 \\ -5/2 & 2 \end{bmatrix}$$

Thus $[T]^{-1} \begin{vmatrix} w_1 \\ w_2 \end{vmatrix} = \begin{vmatrix} 3/2 & -1 \\ -5/2 & 4/2 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \end{vmatrix} = \begin{bmatrix} (3/2)w_1 & - w_2 \\ -(5/2)w_1 & +2w_2 \end{bmatrix}$ same info.

From that we conclude that $T^{-1}(w_1, w_2) = (3/2\ w_1 - w_2, -5/2\ w_1 + 2w_2)$.

9. (10 points) Let $A = \begin{vmatrix} c & d & e \\ 1 & 1 & 1 \end{vmatrix}$. Find conditions on c, d, e such that:

(a) rank A = 1
(b) rank A = 2

Solution:

a) To reach the desired rank the condition c = d = e is needed.
b) To reach the desired rank at least two of c, d, and e must be distinct.

10. (20 points) Determine whether the following "vector space" is valid. If it is not valid, list which axiom(s) cause it to fail.

The set of all pairs of real numbers where vector addition is defined as x + y = x*y, and vector multiplication is defined as kx = 2x.

Solution:

By evaluating the axioms, it is found that 7,8,9 and 10 are violated. Hence the vector space is invalid.

show!

Also x = 0 does not have an "addative inverse" since $\frac{1}{0}$ is not a real number - so axiom 5 fails too.

1)

A. Determine whether $\underline{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\underline{y} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$, $\underline{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$ are linearly dependent in $\mathbb{R}^4$.

B. For which real numbers λ are the following vectors linearly independent in $\mathbb{R}^3$?

$\underline{x} = \begin{bmatrix} \lambda \\ -1 \\ -1 \end{bmatrix}$, $\underline{y} = \begin{bmatrix} -1 \\ \lambda \\ -1 \end{bmatrix}$, $\underline{z} = \begin{bmatrix} -1 \\ -1 \\ \lambda \end{bmatrix}$

SOLUTION:

A. $\underline{x}$, $\underline{y}$, and $\underline{z}$, are linearly dependent if $k_1\underline{x} + k_2\underline{y} + k_3\underline{z} = 0$ does not contain only the trivial solution. So,

$\underline{x} + 0\underline{y} + \underline{z} = 0$
$0\underline{x} + \underline{y} + \underline{z} = 0$
$\underline{x} + \underline{y} + \underline{z} = 0$
$2\underline{x} + 2\underline{y} + 3\underline{z} = 0$

What is this? change vectors to K's?

Where the coefficient matrix is: $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$

$k_1 = 0$, $k_2 = 0$, $k_3 = 0$ $\rightarrow$ NOT linearly dependent

B. $\underline{x}$, $\underline{y}$, and $\underline{z}$, are linearly independent if $k_1\underline{x} + k_2\underline{y} + k_3\underline{z} = 0$ contains only the trivial solution. So,

$\lambda\underline{x} - \underline{y} - \underline{z} = 0$
$-\underline{x} - \lambda\underline{y} - \underline{z} = 0$
$-\underline{x} - \underline{y} - \lambda\underline{z} = 0$

Where the coefficient determinant is: $\det\begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix}$

$= \lambda(\lambda^2 - 1) + (-\lambda - 1) - (1 + \lambda)$
$= \lambda^3 - 3\lambda - 2 = (\lambda+1)^2(\lambda-2)$
$= \lambda(\lambda^2 - 3) - 2$

$(\lambda+1)(\lambda-2) = \lambda^2 - \lambda - 2$

$$\begin{array}{r} \lambda + 1 \\ \lambda^2-\lambda-2 \,\overline{)\, \lambda^3 - 3\lambda - 2} \\ \underline{\lambda^3 - \lambda^2 - 2\lambda} \\ \lambda^2 - \lambda - 2 \end{array}$$

So, the values of λ that make the vectors linearly independent are those λ satisfying $\det(A) \neq 0 \rightarrow \lambda \neq -1, \lambda \neq 2$

? some work

2) Show that $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = (\|\vec{u}\| + \|\vec{v}\|)(\|\vec{u}\| - \|\vec{v}\|)$ or give a counter-example if the ~~in~~equality does not hold.

SOLUTION:

$$(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) \neq$$
$$\vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{v} \neq$$
$$\vec{u}\cdot\vec{u}+\vec{v}\cdot\vec{v} \neq$$
$$\left(\sqrt{\vec{u}\cdot\vec{u}}\right)^2-\left(\sqrt{\vec{v}\cdot\vec{v}}\right)^2 =$$
$$\|\vec{u}\|^2-\|\vec{v}\|^2 =$$
$$\|\vec{u}\|\|\vec{u}\|-\|\vec{u}\|\|\vec{v}\|+\|\vec{u}\|\|\vec{v}\|-\|\vec{v}\|\|\vec{v}\| =$$
$$\left(\|\vec{u}\|+\|\vec{v}\|\right)\left(\|\vec{u}\|-\|\vec{v}\|\right)$$

3) [True/False and explain]:
If A is a 7×12 matrix, then the linear transformation $x \to Ax$ cannot be one-to-one.

SOLUTION:

True – If there are more columns than rows, there must be some free variables in solution of $Ax=0$.

4) Let $\mathbb{L}: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, where

$$\mathbb{L}\left(\begin{bmatrix}2\\3\end{bmatrix}\right)=\begin{bmatrix}2\\3\end{bmatrix} \quad \text{and} \quad \mathbb{L}\left(\begin{bmatrix}3\\4\end{bmatrix}\right)=\begin{bmatrix}0\\4\end{bmatrix}$$

Good problem.

Find $\mathbb{L}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix} \\ \end{bmatrix}$

SOLUTION:

Solve the system $x\begin{bmatrix}2\\3\end{bmatrix}+y\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$ -getting x = -4, y = 3

So, $\mathbb{L}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=-4\mathbb{L}\left(\begin{bmatrix}2\\3\end{bmatrix}\right)+3\mathbb{L}\left(\begin{bmatrix}3\\4\end{bmatrix}\right)=-4\begin{bmatrix}2\\3\end{bmatrix}+3\begin{bmatrix}0\\4\end{bmatrix}=\begin{bmatrix}-8\\0\end{bmatrix}$

5) Perform the following operations, where:

A = {6 , 1 , -3 , 9} **B** = {3 , -2 , -5 , 0} **C** = {0 , 0 , 1 , -3}

a) A – B b) 2 (A + 3B) c) (3A – C) – (5B + A)
d) Find the Euclidean inner product of A and B.
e) Find the Euclidean length of A.

Answers:

a) A – B = {3 , 3 , 2 , 9}.

b) 3B = {9 , -6 , -15 , 0}.
A + 3B = {15 , -5 , -18 , 9}.
2 (A + 3B) = {30 , -10 , -36 , 18}.

c) 3A = {18 , 3 , -9 , 27}.
3A – C = {18 , 3 , -10 , 24}.
5B = {15 , -10 , -25 , 0}.
5B + A = {21 , -9 , -28 , 9}.
(3A – C) – (5B + A) = {-3 , 12 , 18 , 15}.

d) A • B = 6(3) + 1(-2) + (-3)(-5) + 9(0)
= 18 -2 + 15 + 0
= 31

e) ||A||

$$= \sqrt{6^2 + 1^2 + (-3)^2 + 9^2}$$
$$= \sqrt{36 + 1 + 9 + 81}$$
$$= \sqrt{36 + 1 + 9 + 81}$$
$$= \sqrt{127}$$

6) Consider the vectors **a** = (1 , 2 , -1) and **b** = (2 , 5 , -3). Is **c** = (-4 , -11 , 6) a linear combination of **a** and **b**?

Answer:

Assume **c** is a linear combination of **a** and **b**.
There must be scalars x and y where:

(-4 , -11 , 6) = x (1 , 2 , -1) + y (2 , 5 , -3).

Or

(-4 , -11 , 6) = (x + 2y , 2x + 5y , -x – 3y)

Can be put into the augmented matrix:

$$\begin{bmatrix} 1 & 2 & -4 \\ 2 & 5 & -11 \\ -1 & -3 & 7 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -3 \\ 0 & -1 & 3 \end{bmatrix}$$ Add (-2 x row 1) to row 2 and add row 1 to row 3.

$$\sim\begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$ Add row 2 to row 3.

$$\sim\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$ Add (-2 x row 2) to row 1.

x = 2 and y = -3.
Therefore, **c** = 2**a** – 3**b**, can be made, which qualifies **c** as a linear combination.

7) <u>Find the image of the vector (2, 3) when it is rotated through the angle.</u>

A. $\theta = 30^\circ$ B. $\theta = 60^\circ$ C. $\theta = 90^\circ$

<u>Solution</u>

A. $$\begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} * \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} ½\sqrt{3} & -½ \\ ½ & ½\sqrt{3} \end{bmatrix} * \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \sqrt{3}-(3/2) \\ 1+(3/2)\sqrt{3} \end{bmatrix}$$

B. $$\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} * \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} ½ & -½\sqrt{3} \\ ½\sqrt{3} & ½ \end{bmatrix} * \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1-(3/2)\sqrt{3} \\ \sqrt{3}+(3/2) \end{bmatrix}$$

C. $$\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} * \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

8) Express $7 + 17x + 13x^2$ as a linear combination of the following:

P1: $1 + 2x^2$
P2: $5 + 2x$
P3: $2 + 3x + x^2$

Solution

Talk about standard basis of P_2, to get matrix.

$$\begin{bmatrix} 1 & 5 & 2 & 7 \\ 0 & 2 & 3 & 17 \\ 2 & 0 & 1 & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & 2 & 7 \\ 0 & 2 & 3 & 17 \\ 0 & -10 & -3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & 2 & 7 \\ 0 & 2 & 3 & 17 \\ 0 & 0 & 12 & 84 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & 2 & 7 \\ 0 & 2 & 3 & 17 \\ 0 & 0 & 1 & 7 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 5 & 0 & -7 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & 1 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{bmatrix} \rightarrow$$

$3\mathbf{P1} - 2\mathbf{P2} + 7\mathbf{P3} = 7 + 17x + 13x^2$

9) Use the Wronskian to show that the following sets of vectors are linearly independent.

A. $2, 2x, e^x$ B. $\sin x, \cos x$

Solution

A. $\text{Det}\begin{vmatrix} 2 & 2x & e^x \\ 0 & 2 & e^x \\ 0 & 0 & e^x \end{vmatrix} = 4e^x$

Since the determinant is not 0 at some point, it is independent.

B. $\text{Det}\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1$

Since the determinant is not 0 at some point, it is independent.

10) Show that the following set of matrices is a basis of M22

$$\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix},$$

Solution

Since matrices work just like vectors we can put them into vector form and multiply with different scalars

a (3,6,3,-6) + b (0,-1,-1,0)+ c (0,-8,-12,-4)+ d (1,0,-1,2)

We get the augmented matrix as shown below

3a+0b+0c+d=0

6a+-1b+-8c+0d=0

3a+-1b+-12c+-1d=0

-6a+0b-4c+2d=0

We have to show that the vectors are linear independent and that they span r4

We set equations equal to 0 and proof that the determinant is not zero.

$$\begin{bmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{bmatrix}$$ Perform row reduction and you get the following matrix

$$\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$ The determinant for the following matrix is the product of the diagonal entries.

Det =6

Thus it's a basis

Questions

1. *(15 points)* Consider Matrix A

$$A = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

/ Put answers next to questions

Determine if the vectors that compose $(A^{-1})^T$ form a linearly dependant set.

2. *(15 points)* Use the Wronskian to verify whether $\mathbf{f_1} = 3x^2$, $\mathbf{f_2} = \sin(x)$ and $\mathbf{f_3} = -\cos(x)$ form a linearly independent set of vectors in $C^2(-\infty,\infty)$.

3. *(15 points)* Find the coordinate vector of A relative to the basis of $S = \{A_1, A_2, A_3, A_4\}$

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}; A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

4: *(15 points)* Find the Euclidean inner product $u \bullet v$

a) u= (2,5), v=(-4,3)
b) u= (4,8,2), v= (0,1,3)
c) u= (3,1,4,-5), v= (2,2,-4,-3)
d) u= (-1,1,0,4,3), v=(-2,-2,0,2,-1)

5: *(15 points)* Find the image of the vector (-2,1,3), if it is rotated

a) 30° about x-axis
b) 45° about y-axis
c) 90° about z-axis

6. *(15 points)* Show that the range of these equations:

$W_1 = 8x_1+5x_2+6x_3$
$W_2 = 2x_1+3x_2+x_3$
$W_3 = 4x_1+6x_2+2x_3$

Is not all of R^3, and find a vector that is not in it's range.

7. *(18 points)* Given the following ten axioms

1. If **u** and **v** are objects in *V*, then **u** + **v** is in *V*.
2. **u + v = v + u**
3. **u + (v + w)=(u + v) + w**
4. There is an object **0** in V, called a ***zero vector*** for V, such that **0** + **u** = **u**
5. For each **u** in V, there is an object –**u** in V, called a ***negative*** of **u**, such that **u + (-u) = (-u) + u = 0**
6. If k is a scalar and **u** is any object in V, then k**u** is in V.

7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)(\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u}$

prove whether the following set of vectors is or is not a vector space. **Show your work!!**

The set of all 3 x 3 matrices of the form

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

With the standard matrix addition and scalar multiplication.

8. *(12 points)* Express the vector **v** =(8, 8, 9) as a linear combination of the vectors (2, 3, 0), (-1, 0, 4),
and (4, -1, 1).

9. *(15 points)* Find a basis for the nullspace of

$$A = \begin{bmatrix} 3 & 0 & 0 & -2 & -2 \\ -2 & 1 & 1 & -2 & -1 \\ 1 & 0 & 0 & -1 & -1 \\ 0 & -2 & -2 & 1 & -1 \end{bmatrix}$$

10. *(15 points)* What conditions must be satisfied by b_1, b_2, b_3, b_4, and b_5 for the overdetermined linear system

$$\begin{aligned} x_1 - x_2 &= b_1 \\ x_1 + 2x_2 &= b_2 \\ x_1 - 3x_2 &= b_3 \\ x_1 + 4x_2 &= b_4 \\ x_1 - 5x_2 &= b_5 \end{aligned}$$

to be consistant?

Solutions

1.

$$A = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$$(A^{-1})^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 3 & 3 & 8 \end{bmatrix}$$

$v1 = (1,2,1), v2 = (2,5,0), v3 = (3,3,8)$

much easier way to do this - just take det A.

Expressing the coefficients as components of equations

$$k1 + 2k2 + k3 = 0$$
$$2k1 + 5k2 = 0$$
$$3k1 + 3k2 + 8k3 = 0$$

Take the determinant of the augmented matrix, which equals -1 meanings that the system is independent, and that the only solution set for the above system of equations is the trivial solution.

2. $\mathbf{f_1} = 3x^2$
$\mathbf{f_2} = \sin(x)$
$\mathbf{f_3} = -\cos(x)$

$$f_1(x) = 3x^2 \quad f_2(x) = \sin x \qquad f_3(x) = -\cos x$$
$$f'_1(x) = 6x \quad f'_2(x) = \cos x \quad f'_3(x) = \sin x$$
$$f''_1(x) = 6 \quad f''_2(x) = -\sin x \quad f''_3(x) = \cos x$$

Setting these up in the Wronskian is as follows

$$W(x) = \begin{vmatrix} 3x^2 & \sin x & -\cos x \\ 6x & \cos x & \sin x \\ 6 & -\sin x & \cos x \end{vmatrix}$$

$= 3x^2(\cos x \cos x - -\sin x \sin x) - 6x(\sin x \cos x - \sin x \cos x) + 6(\sin x \sin x - -\cos x \cos x)$ Since

$\cos x \cos x + \sin x \sin x = (\cos x)^2 + (\sin x)^2$ is the identity equal to 1, the above equation simplifies to

$3x^2+6=3(x2+2)$ which never equals 0 at any point, therefore $\mathbf{f}_1$, $\mathbf{f}_2$ and $\mathbf{f}_3$ form a linearly dependant set. No!.

3. $A=\begin{bmatrix}2 & 0\\ -1 & 3\end{bmatrix}; A_1=\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix}, A_2=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, A_3=\begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}, A_4=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$

$A = x_1A_1+x_2A_2+x_3A_3+x_4A_4$

$$\begin{bmatrix}2 & 0\\ -1 & 3\end{bmatrix}=x_1\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix}+x_2\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}+x_3\begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}+x_4\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

$$v_1=-x_1+x_2$$
$$v_2=x_1+x_2$$
$$v_3=x_3$$
$$v_4=-x_3+x_4$$

Then converting to augmented matrix form and row reducing

$$\left[\begin{array}{cccc|c}-1 & 1 & 0 & 0 & 2\\ 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & -1\\ 0 & 0 & -1 & 1 & 3\end{array}\right]\sim\left[\begin{array}{cccc|c}0 & 2 & 0 & 0 & 2\\ 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & 3\end{array}\right]\sim\left[\begin{array}{cccc|c}1 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & 3\end{array}\right]\sim\ \cdots$$

$x_1=-1$
$x_2=1$
$x_3=-1$
$x_4=3$

$(A)_S=(-1,1,-1,3)$

4.

a) u_v = 2_4+5_3 = 7
b) u_v = 4_0+8_1+2_3 = 14
c) u_v = 3_2+1_2+-4_4+-5_-3 = 7
d) u_v = -1_-2+1_-2+0_0+4_2+-1_-3 = 11

5.

a) $$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos 30 & -\sin 30\\ 0 & \sin 30 & \cos 30\end{bmatrix}\_\begin{bmatrix}-2\\ 1\\ 3\end{bmatrix}=\begin{bmatrix}-2\\ \dfrac{(\sqrt{3}-3)}{2}\\ \dfrac{3\cdot\sqrt{3}+1}{2}\end{bmatrix}$$

b) $\begin{bmatrix} \cos 45 & 0 & \sin 45 \\ 0 & 1 & 0 \\ -\sin 45 & 0 & \cos 45 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ 1 \\ 5\sqrt{2}/2 \end{bmatrix}$

c) $\begin{bmatrix} \cos 90 & -\sin 90 & 0 \\ \sin 90 & \cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$

6.

The range is not all of R^3 because A, the coefficient matrix, is not invertible.

$\begin{bmatrix} 8 & 5 & 6 \\ 2 & 3 & 1 \\ 4 & 6 & 2 \end{bmatrix}$ - is not invertible.

show.

A vector that is not in the range is the (0,0,1) vector because it makes the system inconsistent.

7. We'll prove each axiom.

1. $\begin{bmatrix} a1 & 0 & 0 \\ 0 & b1 & 0 \\ 0 & 0 & c1 \end{bmatrix} + \begin{bmatrix} a2 & 0 & 0 \\ 0 & b2 & 0 \\ 0 & 0 & c2 \end{bmatrix} = \begin{bmatrix} (a1+a2) & 0 & 0 \\ 0 & (b1+b2) & 0 \\ 0 & 0 & (c1+c2) \end{bmatrix}$

2. $\begin{bmatrix} a1 & 0 & 0 \\ 0 & b1 & 0 \\ 0 & 0 & c1 \end{bmatrix} + \begin{bmatrix} a2 & 0 & 0 \\ 0 & b2 & 0 \\ 0 & 0 & c2 \end{bmatrix} = \begin{bmatrix} (a1+a2) & 0 & 0 \\ 0 & (b1+b2) & 0 \\ 0 & 0 & (c1+c2) \end{bmatrix} =$

$\begin{bmatrix} (a2+a1) & 0 & 0 \\ 0 & (b2+b1) & 0 \\ 0 & 0 & (c2+c1) \end{bmatrix} = \begin{bmatrix} a2 & 0 & 0 \\ 0 & b2 & 0 \\ 0 & 0 & c2 \end{bmatrix} + \begin{bmatrix} a1 & 0 & 0 \\ 0 & b1 & 0 \\ 0 & 0 & c1 \end{bmatrix}$

3. $\begin{bmatrix} a1 & 0 & 0 \\ 0 & b1 & 0 \\ 0 & 0 & c1 \end{bmatrix} + \left(\begin{bmatrix} a2 & 0 & 0 \\ 0 & b2 & 0 \\ 0 & 0 & c2 \end{bmatrix} + \begin{bmatrix} a3 & 0 & 0 \\ 0 & b3 & 0 \\ 0 & 0 & c3 \end{bmatrix} \right) = \begin{bmatrix} a1 & 0 & 0 \\ 0 & b1 & 0 \\ 0 & 0 & c1 \end{bmatrix} +$

$\begin{bmatrix} (a2+a3) & 0 & 0 \\ 0 & (b2+b3) & 0 \\ 0 & 0 & (c2+c3) \end{bmatrix} = \begin{bmatrix} (a1+a2+a3) & 0 & 0 \\ 0 & (b1+b2+b3) & 0 \\ 0 & 0 & (c1+c2+c3) \end{bmatrix} =$

$\begin{bmatrix} (a1+a2)+a3 & 0 & 0 \\ 0 & (b1+b2)+b3 & 0 \\ 0 & 0 & (c1+c2)+c3 \end{bmatrix} =$

$$\begin{bmatrix} (a1+a2) & 0 & 0 \\ 0 & (b1+b2) & 0 \\ 0 & 0 & (c1+c2) \end{bmatrix} + \begin{bmatrix} a3 & 0 & 0 \\ 0 & b3 & 0 \\ 0 & 0 & c3 \end{bmatrix}$$

4. $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

5. $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} + (-\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} + \begin{bmatrix} -a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{bmatrix} = \begin{bmatrix} a-a & 0 & 0 \\ 0 & b-b & 0 \\ 0 & 0 & c-c \end{bmatrix} =$
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

6. k $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} ka & 0 & 0 \\ 0 & kb & 0 \\ 0 & 0 & kc \end{bmatrix} = \begin{bmatrix} ka & 0 & 0 \\ 0 & kb & 0 \\ 0 & 0 & kc \end{bmatrix} = \begin{bmatrix} (ka) & 0 & 0 \\ 0 & (kb) & 0 \\ 0 & 0 & (kc) \end{bmatrix}$

7. k($\begin{bmatrix} a1 & 0 & 0 \\ 0 & b1 & 0 \\ 0 & 0 & c1 \end{bmatrix} + \begin{bmatrix} a2 & 0 & 0 \\ 0 & b2 & 0 \\ 0 & 0 & c2 \end{bmatrix}$) = k $\begin{bmatrix} (a1+a2) & 0 & 0 \\ 0 & (b1+b2) & 0 \\ 0 & 0 & (c1+c2) \end{bmatrix} =$
$\begin{bmatrix} k(a1+a2) & 0 & 0 \\ 0 & k(b1+b2) & 0 \\ 0 & 0 & k(c1+c2) \end{bmatrix} = \begin{bmatrix} (ka1+ka2) & 0 & 0 \\ 0 & (kb1+kb2) & 0 \\ 0 & 0 & (kc1+kc2) \end{bmatrix} =$
$\begin{bmatrix} ka1 & 0 & 0 \\ 0 & kb1 & 0 \\ 0 & 0 & kc1 \end{bmatrix} + \begin{bmatrix} ka2 & 0 & 0 \\ 0 & kb2 & 0 \\ 0 & 0 & kc2 \end{bmatrix} =$ k $\begin{bmatrix} a1 & 0 & 0 \\ 0 & b1 & 0 \\ 0 & 0 & c1 \end{bmatrix}$ + k $\begin{bmatrix} a2 & 0 & 0 \\ 0 & b2 & 0 \\ 0 & 0 & c2 \end{bmatrix}$

8. (k+m) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} (k+m)a & 0 & 0 \\ 0 & (k+m)b & 0 \\ 0 & 0 & (k+m)c \end{bmatrix} = \begin{bmatrix} (ka+ma) & 0 & 0 \\ 0 & (kb+mb) & 0 \\ 0 & 0 & (kc+mc) \end{bmatrix} =$
$\begin{bmatrix} ka & 0 & 0 \\ 0 & kb & 0 \\ 0 & 0 & kc \end{bmatrix} + \begin{bmatrix} ma & 0 & 0 \\ 0 & mb & 0 \\ 0 & 0 & mc \end{bmatrix} =$ k $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ + m $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

9. $k(m\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}) = k\begin{bmatrix} ma & 0 & 0 \\ 0 & mb & 0 \\ 0 & 0 & mc \end{bmatrix} = \begin{bmatrix} kma & 0 & 0 \\ 0 & kmb & 0 \\ 0 & 0 & kmc \end{bmatrix} = \begin{bmatrix} (km)a & 0 & 0 \\ 0 & (km)b & 0 \\ 0 & 0 & (km)c \end{bmatrix} = (km)$
$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

10. $1\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} 1a & 0 & 0 \\ 0 & 1b & 0 \\ 0 & 0 & 1c \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

8. We can represent this linear combination as a system of equations.

-1k+4m+2n = 8
0k-1m+3n = 8
4k+1m+0n=9

We can then solve this by row-reducing the augmented matrix.

$$\begin{bmatrix} -1 & 4 & 2 & 8 \\ 0 & -1 & 3 & 8 \\ 4 & 1 & 0 & 9 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & 2 & 8 \\ 0 & -1 & 3 & 8 \\ 0 & 17 & 8 & 41 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & 2 & 8 \\ 0 & -1 & 3 & 8 \\ 0 & 0 & 59 & 177 \end{bmatrix}$$

This represents the equations
-1k+4m+2n=8
-1m+3n=8
59n=177
Solving for n and the back substituting gives us

n=3, m=1, and k=2

So we can represent **v** by the linear combination of

(8, 8, 9) = 2(-1, 0, 4) + (4, -1, 1) + 3(2, 3, 0)

9.
The nullspace of A is the solution space of the system (all set equal to 0)

$3x_1 - 2x_4 - 2x_5 = 0$
$-2x_1 + x_2 + x_3 - 2x_4 - x_5 = 0$
$x_1 - x_4 - x_5 = 0$
$-2x_2 - 2x_3 + x_4 - x_5 = 0$

The matrix of this system (in augmented matrix form) reduces as follows

$$\begin{bmatrix} 3 & 0 & 0 & -2 & -2 \\ -2 & 1 & 1 & -2 & -1 \\ 1 & 0 & 0 & -1 & -1 \\ 0 & -2 & -2 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & -4 & -3 \\ 0 & -2 & -2 & 1 & -1 \\ 2 & 1 & 1 & -2 & -1 \\ 1 & 0 & 0 & -1 & -1 \end{bmatrix} \begin{matrix} R2 + R1 // and \\ R2 \leftrightarrow R3 \\ R2 \leftrightarrow R4 \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & -4 & -3 \\ 0 & -2 & -2 & 1 & -1 \\ 0 & -1 & -1 & 6 & 5 \\ 0 & -1 & -1 & 3 & 2 \end{bmatrix} \begin{matrix} -2R1 + R3 \\ -R1 + R4 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & -4 & -3 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 5\frac{1}{2} & 5\frac{1}{2} \\ 0 & 0 & 0 & 2\frac{1}{2} & 2\frac{1}{2} \end{bmatrix} \begin{matrix} -2(R2) // and \\ R2 + R3 \\ R2 + R4 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & -4 & -3 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 5\frac{1}{2}(R3) // and \\ -2\frac{1}{2}R3 + R4 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & -4 & -3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{1}{2}R3 + R2$$

$$\begin{bmatrix} 1 & 0 & 0 & -4 & -4 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} -R2 + R1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} 4R3 + R1$$

$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which has the solutions: $\begin{matrix} x_1 = 0 \\ x_2 + x_3 + x_5 = 0 \\ x_4 + x_5 = 0 \end{matrix}$

so, $\begin{matrix} x_1 = 0 \\ x_2 = -x_3 - x_5 \\ x_4 = -x_5 \end{matrix}$ and substituting $\begin{matrix} x_3 = s \\ x_5 = t \end{matrix}$ we write solutions for all x as $\begin{matrix} x_1 = 0 \\ x_2 = -s - t \\ x_3 = s \\ x_4 = -t \\ x_5 = t \end{matrix}$.

And to find the basis for matrix A we write the vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -s - t \\ s \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -t \\ 0 \\ -t \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

so, the basis for the matrix A are the vectors

$$v_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

10.

Written as a augmented matrix and reduced by Guass-Jordan elimination gives

$$\begin{bmatrix} 1 & -1 & b_1 \\ 1 & 2 & b_2 \\ 1 & -3 & b_3 \\ 1 & 4 & b_4 \\ 1 & -5 & b_5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & b_1 \\ 0 & 3 & b_1+b_2 \\ 0 & -2 & b_1+b_3 \\ 0 & 5 & b_1+b_4 \\ 0 & -4 & b_1+b_5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 4b_1+3b_2 \\ 0 & 1 & 3b_1+3b_2 \\ 0 & 0 & 7b_1+6b_2+b_3 \\ 0 & 0 & -14b_1-15b_2+b_4 \\ 0 & 0 & 13b_1+12b_2+b_5 \end{bmatrix}$$

So to be consistant the following must apply:

$$7b_1+6b_2+b_3=0$$
$$-14b_1-15b_2+b_4=0$$
$$13b_1+12b_2+b_5=0$$

or, on solving these equations:

$$b_3=-7b_1-6b_2$$
$$b_4=14b_1+15b_2$$
$$b_5=-13b_1-12b_2$$

$$b_1=s$$
$$b_2=t$$
$$b_3=-7s-6t$$
$$b_4=14s+15t$$
$$b_5=-13s-12t$$

also

$$b = x_1\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} + x_2\begin{bmatrix}-1\\2\\-3\\4\\-5\end{bmatrix}$$

which ~~[illegible]~~

is easy to find!

1. Given **u**=(4,6,2), **v**=(5,3,4), and **w**=(1,7,3), evaluate the following operations:

a) **u+v+w**

b) **u • v**

c) **(u+v) • w**

d) **(u+v) • (v-w)**

Solution

a) (4,6,2)+(5,3,4)+(1,7,3)=(10,16,9)

b) (4,6,2) • (5,3,4)=(20,18,8)

c) [(4,6,2)+(5,3,4)] • (1,7,3)=(9,9,6) • (1,7,3)=(9,63,18)

d) [(4,6,2)+(5,3,4)] • [(5,3,4)-(1,7,3)]=(9,9,6) • (4,-4,1)=(36,-36,6)

2. The linear transformation $T:R^4 \to R^3$ is defined by the equations

$w_1=3x_1+x_2-4x_3+x_4$

$w_2=x_1-2x_2+x_3-3x_4$

$w_3=2x_1+5x_2+2x_3-x_4$

a) Find the standard matrix for T

Solutions

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -4 & 1 \\ 1 & -2 & 1 & -3 \\ 2 & 5 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$, so the standard matrix for T is $\begin{bmatrix} 3 & 1 & -4 & 1 \\ 1 & -2 & 1 & -3 \\ 2 & 5 & 2 & -1 \end{bmatrix}$.

b) Calculate T(3,1,4,2) by matrix multiplication and check your answers by direct substitution into the equations.

Solution

-By multiplication:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -4 & 1 \\ 1 & -2 & 1 & -3 \\ 2 & 5 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ 17 \end{bmatrix}$$

By direct substitution:

$w_1=3(3)+1-4(4)+2= -4$

$w_2=3-2(1)+4-3(2)= -1$

$w_3=2(3)+5(1)+2(4)-2= 17$

4.3

Q. What are the eigenvalues (λ) of the transformation T: $R_3 \rightarrow R_3$ involving first a dilation by 4 and then an orthogonal projection on the xz-plane.

Solution:

$$T/M_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\det(\lambda \text{I-A}) = \det \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda - 4 \end{bmatrix} = 0$$

$$(\lambda\text{-}4)(\ \lambda)(\lambda\text{-}4) = 0$$

$$\boldsymbol{\lambda = 4,0}$$

5.2

Q. Let S= $\{v_1, v_2, v_3\}$ such that $v_1 = (3,2,1)$ $v_2 = (1,1,0)$ and $v_3 = (2,1,3)$.
Let P= $\{w_1, w_2\}$ such that $w_1 = (7,2,5)$ and $w_2 = (3,3,4)$
Does span(S) = span (P)? Hint: Use linear combinations.

Solution:
By theorem 5.2.4, the spans are equal iff each vector in S is a linear combination of those in P and vise versa. Each linear combination needs to be tried.

1. Check to see if the vectors in S combine to create a linear combination of w_1.

$$\begin{array}{ccc|cc} 3 & 1 & 2 & 7 & 3 \\ 2 & 1 & 1 & 2 & 3 \\ 1 & 0 & 3 & 5 & 4 \end{array} \quad \begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 2 & 1 & 1 & 2 \\ 3 & 1 & 2 & 7 \end{array} \sim \begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & -5 & -8 \\ 0 & 1 & -7 & -8 \end{array} \sim \begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & -5 & -8 \\ 0 & 0 & -2 & 0 \end{array}$$

At this point it can be determined that

a+c = 5
b-5c = -8
-2c = 0

Therefore a=5 b=-8 and c= 0. Thus $w_1 = 5\,v_1 \text{ -}8\,v_2 + 0\,v_3$

do at same time.

Similarly, w_2 is a linear combination of the vectors in S.

2. Check to see if the vectors in P combine to create a linear combination of v_1.

$$\begin{array}{cc|ccc} 3 & 7 & 3 & 1 & 2 \\ 3 & 2 & 2 & 1 & 1 \\ 4 & 5 & 1 & 0 & 3 \end{array} \quad \begin{array}{cc|c} 3 & 7 & 3 \\ 0 & -5 & -1 \\ 0 & \frac{-13}{3} & -3 \end{array} \sim \begin{array}{cc|c} 3 & 7 & 3 \\ 0 & -5 & -1 \\ 0 & -13 & -9 \end{array} \sim \begin{array}{cc|c} 3 & 7 & 3 \\ 0 & 5 & 1 \\ 0 & 0 & \frac{-32}{5} \end{array}$$

Since it is impossible for 0 to equal -32/5, there is no linear combination of P that will create v_1.
Therefore the span(S) $\neq$ span(P).

5.1

1) A set of objects is given, together with operations of addition and scalar multiplication. Determine which sets are vector spaces under the given operations. For those that are not vector spaces, list all the axioms that fail to hold.

The set of all triples of integers with the operations

$(x, y, z) + (x', y', z') = (x+x', y+y', z+z')$ and $k(x, y, z) = (x, ky, z)$

1) If **u** and **v** are arbitrary vectors with integer components with **u** = (a, b, c) and **v** =(d, e, f), then **u+v** = (a+d, b+e, c+f). All of the resulting components of **u+v** are **integers,**
therefore axiom 1 holds.
2) **u** = (a, b, c), **v** = (d, e, f)
u+v = (a+d, b+e, c+f)
v+u = (d+a, e+b, f+c)
therefore axiom 2 holds.
3) **u** = (a, b, c), **v**=(d, e, f), and **w**=(g, h, i).
u+(v+w) = (a, b, c) + (d+g, e+h, f+j) = (a+d+g, b+e+j, c+f+j)
(u+v)+w = (a+d, b+e, c+f) + (g, h, i) = (a+d+g, b+e+j, c+f+j)
therefore axiom 3 holds.
4) **0+u** = **u+0** by axiom 2. **u+0** = (a+0, b+0, c+0) = (a, b, c) = **u**
therefore axiom 4 holds.
5) **u+ (-u)** = 0 => (a, b, c)+(a', b', c') = 0 => (a+a', b+b', c+c') = (0, 0, 0) therefore the negative of **u** = (-a, -b, -c) and **(-u)+u = 0 = u+(-u)** by axiom 2
therefore axiom 5 holds.
6) k**u** is not in V for any scalar k because V contains the set of triples of integers. if k were not an integer k**u** = (a, kb, c) is not necessarily a vector with integer components and is not in the set V
therefore axiom 6 fails.
7) **u** = (a, b, c), **v**=(d, e, f)
k(**u+v**) = k(a+d, b+e, c+f) = (a+d, k(b+e), c+f)
k**u**+k**v** = k(a, b, c) + k(d, e, f) = (a, kb, c) + (d, ke, f) = (a+d, k(b+e), c+f)
therefore axiom 7 holds.
8) **u** = (a, b, c)
(k+m)**u** = (a, (k+m)b, c)
k**u**+m**u**= (a, kb, c) + (a, mb, c) = (2a, 2(k+m)b, 2c)
therefore axiom 8 fails.
9) **u** = (a, b, c)
k(m**u**) = k(a, mb, c) = (a, kmb, c)
(km)**u** = (a, kmb, c)
therefore axiom 9 holds.
10) **u** = (a, b, c)
1**u** = (a, 1b, c) = (a, b, c)
therefore axiom 10 holds.

The set is not a vector space because axioms 6 and 8 fail.

2) The following proves theorem 5.1.1. Justify each step by filling in the blank line by specifying the number of one of the vector space axioms, or by stating a basic mathematic property.

$0\mathbf{u} + 0\mathbf{u} = (0+0)\mathbf{u}$ ____________**axiom 8**________________

$= 0\mathbf{u}$ ____________**property of 0**, scalar 0____________

By axiom 5 the vector $0\mathbf{u}$ has a negative, $-0\mathbf{u}$. By adding this negative to both sides above yields

$[0\mathbf{u} + 0\mathbf{u}] + (-0\mathbf{u}) = 0\mathbf{u} + (-0\mathbf{u})$

or

$0\mathbf{u} + [0\mathbf{u} + (-0\mathbf{u})] = 0\mathbf{u} + (-0\mathbf{u})$ ______**axiom 3**____________

$0\mathbf{u} + \mathbf{0} = \mathbf{0}$ ____________**axiom 5**____________

$0\mathbf{u} = \mathbf{0}$ ____________**axiom 4**____________

Section 5.3 – Linear Independence

Determine if S = {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} is Linearly Independent.

$\mathbf{v}_1 = (1, 2, 3)$
$\mathbf{v}_2 = (4, 5, 6)$
$\mathbf{v}_3 = (7, 8, 9)$

Solution: If $(k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3 = 0 \Rightarrow k_1, k_2, k_3 = 0)$ then L.I.

Let $A = [\mathbf{v}_1^T, \mathbf{v}_2^T, \mathbf{v}_3^T]$, so $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$, and let $k = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$.

So if $Ak = 0 \Rightarrow k_1, k_2, k_3 = 0$ then L.I.

This is true iff A^{-1} exists, ~~=> $k_1, k_2, k_3 = 0$~~, which is true iff A^{-1} exists, which it does iff det(A) $\neq 0$. Thus we will solve for det(A).

$$\det(A) = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 7 \\ 0 & 3 & 6 \\ 0 & 6 & 12 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 7 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Thus A^{-1} does not exist and S is Linearly Dependent.

Section 5.4 – Basis

In 1 Nephi 20:13, Jehovah says to Nephi, "my right hand hath spanned the heavens."
Let "the heavens" be a vector space V.
Let his "right hand" be a vector set S = {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{v}_5$}, one vector for each finger.
Answer each part independently of the previous one (e.g., a. doesn't imply anything about b.)

a. What condition can be placed on S (not V) to make it a basis for V?
   Solution: Make sure S is Linearly Independent.

b. If S is a basis for V, then what is dim(V)?
   Solution: dim(V) = 5 because S has 5 vectors and is a basis.

c. If dim(V) = 4, and S is L.I., then what can we do to make S a basis for V?
   Solution: Remove one vector from S. – the right one, though, and only if the right hand is in the heavens –

d. If S and a new vector set S' are both bases for V, then how many vectors are in S'?
   Solution: S' has the same number of vectors as S: 5.

see "preamble" to theorem

this statement seems a bit ambiguous, not!

Find the bases for the column space of the following matricies ~~expressed in reduced row, echelon form~~ and express any remaining column vectors in terms of the basis.

A) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ B) $\begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 9 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ C) $\begin{bmatrix} 1 & 3 & 0 & 8 & 6 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 9 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Answer:

A) Basis: $C1= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $C3= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ Others: C2= 3(C1)+2(C2)

B) Basis: $C1= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $C3= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ C2=.. Others: ~~C2=3(C1)~~, C4=2(~~C1~~)(C3) ?
C5= (C1)+6(C3)

C) Basis: $C1= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $C2= \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $C3= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $C4= \begin{bmatrix} 8 \\ -3 \\ 9 \\ 1 \end{bmatrix}$ Others: C5=6(C1)-2(C3)

For each matrix A find the *largest* possible value of rank(A) and the *smallest* possible value of nullity(A):

A) A is 2×3 B) A is 4×3 C) A is 7×7 D) A is 2×6

Answer:

A) rank(A)$\leq$2 nullity(A)$\geq$1

B) rank(A)$\leq$3 nullity(A)$\geq$0

C) rank(A)$\leq$7 nullity(A)$\geq$0

D) rank(A)$\leq$2 nullity(A)$\geq$4

Given that **u**= (3,2,-1), **v**=(4,6,2), **w**=(10,-10,0), **z**=(3,3,3)
1. Compute the following: (5pts each)
a) **v** · **2w**

v·2**w**=-32+20+12=0

b)(3**u**+**w**) · (**v**-**u**)

(3**u**+**w**) · (**v**-**u**) = [3(3,2,-1)+ (10,-10,0)] · [(4,6,2)- (3,2,-1)] =(5,4,2) · (1,7,3)=5-28+6=-17

c) From the vectors **u,v,w** which are orthogonal to each other?

u·**v**=12-10-2=0
u·**w**=-12-4-3=-19
v·**w**=-16+10+6=0

Hence **u**$\perp$**v** and **v**$\perp$**w** are orthogonal

2. a) Compute: 4||**z**+**v**|| (5pts)

$4\|\mathbf{z}+\mathbf{v}\| = 4\|(3+4,\ 3\text{-}5{,}3+2)\| = 4\|(7,-2,5)\| = 4\sqrt{(49+4+25)} = 4\sqrt{78}$

b) Find the vector x that satisfies **4x-z**=2(**x+w**) (5pts)

$4x - z = 2x + 2w$
$x = w + z/2$
$x = (-5/2, -1/2, 9/2)$

c) Solve the following system of linear equations: (5pts)
2u · (x,y,z)=4
2z · (x,y,z)=8
w · (x,y,z) =1

$6x + 4y - 2z = 4$
$6x + 6y + 6z = 12$
$-4x - 2y + 3z = 1$

$$\left[\begin{array}{ccc|c} 6 & 4 & -2 & 4 \\ 6 & 6 & 6 & 12 \\ -4 & -2 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 6 & 4 & -2 & 4 \\ -4 & -2 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 8 & 8 \\ 0 & 2 & 7 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad x = -5 \qquad y = 8 \qquad z = -1$$

3. Find the standard matrix for the linear transformation T defined by the formula
a) $T(x_1,x_2) = (x_2,\ 2x_1+x_2,\ -x_1-x_2)$ (7pts)
$0x_1 + 2x_2$
$2x_1 + x_2$

$-x_1 - x_2$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \\ -1 & -1 \end{bmatrix}$$

b) $T(x_1,x_2,x_3) = (2x_2, x_1+x_2, -x_1-x_3, x_1+x_2+x_3)$ (8 pts)

$0x_1 + 2x_2 + 0x_3$
$x_1 + x_2 + 0x_3$
$-x_1 + 0x_2 + -x_3$
$x_1 + x_2 + x_3$

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

4. Determine whether the linear operator $T:R^3 \longrightarrow R^3$ defined by the equations is one-to-one; if so find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2, w_3)$. (8pts)

a) $w_1 = 3x - 2y + z$
$w_2 = 3y - z$
$w_3 = x + 2z$

$$\left[\begin{array}{ccc|ccc} 3 & -2 & 1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 3 & -2 & 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & -2 & -5 & 1 & 0 & -3 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & -17/3 & 1 & 2/3 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & -3/17 & -2/17 & 9/17 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6/17 & 4/17 & -18/17 \\ 0 & 1 & 0 & -1/17 & 15/51 & 9/51 \\ 0 & 0 & 1 & -3/17 & -2/17 & 9/17 \end{array}\right]$$

$T^{-1}(w_1, w_2, w_3) = (6/17w_1 + 4/17w_2 - 18/17w_3, -1/17w_1 + 15/51w_2 + 9/51w_3, -3/17w_1 - 2/17w_2 + 9/17w_3)$

(7pts)

b) $w_1 = x + 2y$
$w_2 = -x + 4y + z$
$w_3 = 2x + 4y$

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 4 & 1 \\ 2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ → Row of zeroes, therefore not one to one.

5. Prove the 2x2 **0** matrix is actually a vector space in M_{22}.
(15 pts)

Define addition to be matrix addition and multiplication to be scalar matrix multiplication.

$$\underline{\mathbf{u}} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

1). $\underline{\mathbf{u}}+\underline{\mathbf{v}} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \underline{\mathbf{u}}+\underline{\mathbf{v}}$ is an element of the vector space M_{22}.

2). $\underline{\mathbf{u}}+\underline{\mathbf{v}} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_2 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_2 + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 = \underline{\mathbf{v}}+\underline{\mathbf{u}}$

3). $\underline{\mathbf{u}}+(\underline{\mathbf{v}}+\underline{\mathbf{w}}) = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 + (\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_2 + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_3) = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_2 + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_3 =$
$(\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_2) + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_3 = (\underline{\mathbf{u}}+\underline{\mathbf{v}})+\underline{\mathbf{w}}$

4). $\underline{\mathbf{u}}+\underline{\mathbf{0}} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_0 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \underline{\mathbf{u}}$; $\underline{\mathbf{0}}+\underline{\mathbf{u}} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_0 + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \underline{\mathbf{u}}$

5). $\underline{\mathbf{u}} + (-\underline{\mathbf{u}}) = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 + -\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \underline{\mathbf{0}}$; $(-\underline{\mathbf{u}}) + \underline{\mathbf{u}} = -\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 +$
$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \underline{\mathbf{0}}$

6). $k\underline{\mathbf{u}} = k\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 = k\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = k\underline{\mathbf{u}}$ is part of the vector space in M_{22}.

7). $k(\underline{\mathbf{u}}+\underline{\mathbf{v}}) = k(\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_2) = k\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 + k\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_2 = k\underline{\mathbf{u}} + k\underline{\mathbf{v}}$

8). $(k+m)\underline{\mathbf{u}} = (k+m)\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 = k\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1 + m\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_2 = k\underline{\mathbf{u}} + m\underline{\mathbf{u}}$

9). $k(m(\underline{\mathbf{u}})) = k(m(\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1)) = k(m\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}) = km\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = km(\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}_1) = (km)\underline{\mathbf{u}}$

10). $1\underline{\mathbf{u}} = 1(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_1) = (1\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \underline{\mathbf{u}}$

Because the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ satisfies all ten axioms it is determined to be a vector space and is denoted by $\underline{\mathbf{0}}$.

6. Does the span of set $S = \{v_1, v_2, v_3\}$ where $v_1 = (1,2,1)$, $v_2 = (2,1,1)$, $v_3 = (0,1,1)$ equal the span of set $T = \{w_1, w_2, w_3\}$ where $w_1 = (14,16,5)$, $w_2 = (7,9,4)$, $w_3 = (7,6,4)$?
(15 pts)

$v_1 = k_1w_1 + k_2w_2 + k_3w_3 = (1,2,1) = k_1(14,16,5) + k_2(7,9,4) + k_3(7,6,3)$
(to save space and be more efficient solve for v_1 v_2 and v_3 simultaneously)

Equating the eqn.s yields :

Since both sets are of 3 elements, and in $\mathbb{R}^3$, just check 2 determinants.

$$14k_1+7k_2+7k_3= (1,2,0)$$
$$16k_1+9k_2+6k_3= (2,1,1)$$
$$5k_1+4k_2+3k_3= (1,1,-1)$$

$\begin{bmatrix} 14 & 7 & 7 \\ 16 & 9 & 6 \\ 5 & 4 & 3 \end{bmatrix}$ ~~divide R1 by 7~~

$\left[\begin{array}{ccc|c|c|c} 14 & 7 & 7 & 1 & 2 & 0 \\ 16 & 9 & 6 & 2 & 1 & 1 \\ 5 & 4 & 3 & 1 & 1 & -1 \end{array}\right]$

$\left[\begin{array}{ccc|c|c|c} 2 & 1 & 1 & 1/2 & 2/7 & 0 \\ 16 & 9 & 6 & 2 & 1 & 1 \\ 5 & 4 & 3 & 1 & 1 & -1 \end{array}\right]$ {R1· -8+R2, {R1· -5/2 +R3

$\left[\begin{array}{ccc|c|c|c} 2 & 1 & 1 & 1/7 & 2/7 & 0 \\ 0 & 1 & -2 & 6/7 & 9/7 & 1 \\ 0 & 3/2 & 1/2 & 9/14 & -5/7 & -1 \end{array}\right]$ {R2· -3/2+ R3

$\left[\begin{array}{ccc|c|c|c} 2 & 1 & 1 & 1/7 & 2/7 & 0 \\ 0 & 1 & -2 & 6/7 & 9/7 & 1 \\ 0 & 0 & 7/2 & -9/14 & -37/14 & -5/2 \end{array}\right]$ {R3· 2/7

$\left[\begin{array}{ccc|c|c|c} 2 & 1 & 1 & 1/7 & 2/7 & 0 \\ 0 & 1 & -2 & 6/7 & 9/7 & 1 \\ 0 & 0 & 1 & -9/49 & -37/49 & -5/7 \end{array}\right]$ {R3·-1+R1, {R3·2+R2

$$\left[\begin{array}{ccc|c|c|c} 2 & 1 & 0 & 16/49 & 51/49 & 5/7 \\ 0 & 1 & 0 & 24/49 & -11/49 & -3/7 \\ 0 & 0 & 1 & -9/49 & -37/49 & -5/7 \end{array}\right] \{R2\cdot -1+R1,\ \{R1\cdot 1/2$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -4/49 & 31/49 & 4/7 \\ 0 & 1 & 0 & 24/49 & -11/49 & -3/7 \\ 0 & 0 & 1 & -9/49 & -37/49 & -5/7 \end{array}\right]$$

Since all three k's for the linear combinations of the vectors w that will create all three vectors v are nonzero we know that each vector in S is a linear combination of the vectors in T.
We must now show that each vector in T is a linear combination of the vectors in S.

$w_1 = k_1v_1 + k_2v_2 + k_3v_3$
(to save space and be more efficient solve for w_1 w_2 and w_3 simultaneously)

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & 0 & 14 & 7 & 7 \\ 2 & 1 & 1 & 16 & 9 & 6 \\ 1 & 1 & -1 & 5 & 4 & 3 \end{array}\right] \{R1\cdot -1 +R3,\ \{R1\cdot -2 +R2$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & 0 & 14 & 7 & 7 \\ 0 & -3 & 1 & -12 & -5 & -8 \\ 0 & -1 & -1 & -9 & -3 & -4 \end{array}\right] \{R2\cdot -1,\ \{R3\cdot 3$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & 0 & 14 & 7 & 7 \\ 0 & 3 & -1 & 12 & 5 & 8 \\ 0 & 0 & -4 & -15 & -4 & -4 \end{array}\right] \{R2+R3$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & 0 & 14 & 7 & 7 \\ 0 & 3 & 0 & 63/4 & 6 & 9 \\ 0 & 0 & 1 & 15/4 & 1 & 1 \end{array}\right] \{R3\div -4,\ \{R3+R2$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 7/2 & 3 & 1 \\ 0 & 1 & 0 & 21/4 & 2 & 3 \\ 0 & 0 & 1 & 15/4 & 1 & 1 \end{array}\right] \{R2\div 3,\ \{R2\cdot -2 +R1$$

Since all three k's for the linear combinations of the vectors v that will create all three vectors w are nonzero we know that each vector in T is a linear combination of the vectors in S.
Because both sets of vectors are linear combinations of each other the two ~~sets~~ spans are equal.

7. Determine if the set S= $\{v_1, v_2, v_3\}$ where $v_1 = (1,4,3)$, $v_2 = (3,6,2)$, $v_3 = (5,1,1)$ is linearly independent or linearly dependent.
(15 pts)

You can rewrite the eqn.s to be: $k_1v_1+k_2v_2+k_3v_3=0$ which implies

$K_1+3k_2+5k_3=0$
$4k_1+6k_2+k_3=0$
$3k_1+2k_2+k_3=0$

{By row reducing this matrix we will be able to see if one of the variables becomes solvable in terms of one of the other variables. If this is so then the set S is linearly dependent and if not the set S is linearly independent.

$$\begin{bmatrix} 1 & 3 & 5 \\ 4 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix} \{R1\cdot-4+R2,\ \{R1\cdot-3+R3$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -6 & -19 \\ 0 & -7 & -19 \end{bmatrix} \{R2\cdot-1,\ \{R3\cdot-1$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 6 & 19 \\ 0 & 7 & 19 \end{bmatrix} \{R2\cdot-7,\ \{R3\cdot 6$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -42 & -133 \\ 0 & 42 & 114 \end{bmatrix} \{R2+R3,\ \{R2\div-7$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 6 & 19 \\ 0 & 0 & -19 \end{bmatrix} \{R3+R2,\ \{R3\div-19$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \{R3\cdot-5+R1,\ \{R2\div 6,\ \{R2\cdot-3+R1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because the matrix reduced to reduced row echelon form and did not yield any vectors as scalar multiples of any other two vectors in the set we can conclude that the set $S = \{v_1, v_2, v_3\}$ is linearly independent.

8. Determine the dimension of and a basis for the solution space of the system (15pts)

$$\begin{aligned} x + y + z &= 0 \\ 3x + 2y - 2z &= 0 \\ 4x + 3y - z &= 0 \\ 6x + 5y + z &= 0 \end{aligned}$$

Encode the equations in matrix form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{array}\right]$$

Row-reduce to

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right]$$

Which can be re-written as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right]$$ Making the equations 3-dimentional

Which can be reduced to

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right]$$

1 vector

Therefore the solution vectors can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} +4t \\ -5t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ 2t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2t \\ t \end{bmatrix} + \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

but not independent

Which shows that the vectors **v**, **u**, and **w** ~~**span** the~~ solution space.
v = (-2,2,0) **u** = (0,2,1) w = (-2,1,0) in 3-dimension space

1 dimensional !

9. Find the bases for the row and column space of A (15 pts)

$$A = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By theorem 5.5.6 the row vectors are

$$r_1 = \begin{bmatrix} 1 & 2 & -1 & 5 \end{bmatrix}$$
$$r_2 = \begin{bmatrix} 0 & 1 & 4 & 3 \end{bmatrix}$$
$$r_3 = \begin{bmatrix} 0 & 0 & 1 & -7 \end{bmatrix}$$
$$r_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

And the column vectors

$c_1 = [1 \quad 0 \quad 0 \quad 0]$

$c_2 = [2 \quad 1 \quad 0 \quad 0]$

$c_3 = [-1 \quad 4 \quad 1 \quad 0]$

$c_4 = [5 \quad 3 \quad -7 \quad 1]$

Form ~~a~~ the column space of A

10. Find the rank and nullity of the matrix (15 pts)

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

Row-reduces to:

$$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 2 & 1 \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}$$

There are 2 leading ones, so row and column space are 2-dimensional, rank (A) = 2.

And the matrix is code for:

$x_1 + 4x_2 + 5x_3 + 6x_4 + 9x_5 = 0$

$x_2 + x_3 + x_4 + 2x_5 = 0$

And solving for the leading variables

$x_1 = -r - 2s - 1t$

$x_2 = -r - s - 2t$

$x_3 = r$

$x_4 = s$

$x_5 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Three vectors form the basis for the solution space, nullity(A) = 3

1) Let $\mathbf{u}$ = (5, -7, 3, 0), $\mathbf{v}$ = (4, 8, -3, 5) and $\mathbf{w}$ = (-9, 0, 2, 1). Find:

a) $5\mathbf{w} - 3\mathbf{u} + 2\mathbf{v}$

b) $(\mathbf{w} + 2\mathbf{v}) - 3\mathbf{u}$

Solution

a) 5(-9, 0, 2, 1) - 3(5, -7, 3, 0) + 2(4, 8, -3, 5) =

(-45, 0, 10, 5) – (15, -21, 9, 0) + (8, 16, -6, 10) = (-52, 37, -7, 15)

b) [(-9, 0, 2, 1) + 2(4, 8, -3, 5)] -3(5, -7, 3, 0) =

[(-3, 16, -4, 11)] – (12, -21, 9, 0) = (-15, 37, -13, 11)

2) Find the standard matrix for the linear transformation T defined by the formula:

a) $T(x_1, x_2, x_3) = (x_1 + 7x_3, x_2, x_1 + 3x_2 + 9x_3, x_1)$

b) $T(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$

Solution

a)
$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & 3 & 9 \\ 1 & 0 & 0 \end{bmatrix}$$

b)

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3)

a) The linear operation $T: R^2 \rightarrow R^2$ is defined by the following equations.

$x_1 = 5y + 2z$

$x_2 = 4y + 3z$

Find $T^{-1}(x_1, x_2)$

c) Is this one-to-one?

Solutions

a)

$$[T] = \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix} \qquad [T^{-1}] = \begin{bmatrix} 3/7 & -2/7 \\ -4/7 & 5/7 \end{bmatrix}$$

$T^{-1}(x_1, x_2) = (3/7y - 2/7z, -4/7y + 5/7z)$

b) since the matrix is invertible T is one to one.

4) Determine whether $\mathbf{v_1}$=(1,2,3), $\mathbf{v_2}$=(2,0,2), $\mathbf{v_3}$=(3,2,1) span the vector space R^3.

First, we must choose an arbitrary vector in R^3, $\mathbf{b} = (b_1, b_2, b_3)$, and determine if **b** can be expressed as a linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$:

$\mathbf{b} = k_1v_1 + k_2v_2 + k_3v_3$

=> $(b_1, b_2, b_3) = k_1(1,2,3) + k_2(2,0,2) + k_3(3,2,1)$

$= (k_1 + 2k_2 + 3k_3,\ 2k_1 + 0k_2 + 2k_3,\ 3k_1 + 2k_2 + k_3)$

=>
$$\begin{aligned} k_1 + 2k_2 + 3k_3 &= b_1 \\ 2k_1 + 0k_2 + 2k_3 &= b_2 \\ 3k_1 + 2k_2 + k_3 &= b_3 \end{aligned}$$

=>
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

what's the connection between these 2 ideas?

Solve for the det (A) : det (A) = 0 + 12 + 12 – (4 + 4 + 0) = 16. Because A has a nonzero determinant, $\mathbf{v_1}$=(1,2,3), $\mathbf{v_2}$=(2,0,2), $\mathbf{v_3}$=(3,2,1) span the vector space R^3.

5) Determine whether or not the set is a vector space. If not, give at least two reasons why that is the case:

The set of all 2x2 matrices of the form:

$$\begin{bmatrix} c & c+d \\ 1 & 0 \end{bmatrix}$$

No, this set is not a vector space.

1. If **u** and **v** were objects in V, (**u** + **v**) would not be in V.

2. There does not exsist a –**u** in V for each **u** in V such that **u** +(-**u**) = (-**u**) + **u** = **0.**

3. If k is any scalar and **u** is any object in V, then k**u** would not necessarily be in V also.

(15 pts.) 6. Determine whether the vectors form a linearly dependent set or a linearly independent set.

a) $\mathbf{v}_1 = (1,3,5,7)$, $\mathbf{v}_2 = (6,7,1,3)$, $\mathbf{v}_3 = (4,3,8,5)$ (3pts)
b) $\mathbf{v}_1 = (1,3,8)$, $\mathbf{v}_2 = (6,2,9)$, $\mathbf{v}_3 = (5,-1,1)$ (3pts)
c) $\mathbf{v}_1 = (2,3,-4)$, $\mathbf{v}_2 = (9,5,-2)$, $\mathbf{v}_3 = (-3,4,-14)$ (7pts)

Answers:

a) Linearly dependent because $r \nless n$. ~~Theorem 5.3.3~~ – wrong!
b) Linearly dependent because $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. ($\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1$)
c) Linearly independent:
$k_1 * \mathbf{v}_1 + k_2 * \mathbf{v}_2 + k_3 * \mathbf{v}_3 = \mathbf{0}$
$k_1(2,3,-4) + k_2(9,5,-2) + k_1(-3,4,-14) = (0,0,0)$

$$\begin{aligned} 2k_1 + 9k_2 - 3k_3 &= 0 \\ 3k_1 + 5k_2 + 4k_3 &= 0 \\ -4k_1 - 2k_2 - 14k_3 &= 0 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 2 & 9 & -3 & 0 \\ 3 & 5 & 4 & 0 \\ -4 & -2 & -14 & 0 \end{array}\right) = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 9 & -3 & 0 \\ 3 & 5 & 4 & 0 \\ -4 & -2 & -14 & 0 \end{array}\right) \text{ row3} - 2\text{row1} \quad \left(\begin{array}{ccc|c} 2 & 9 & -3 & 0 \\ 3 & 5 & 4 & 0 \\ 0 & 16 & -8 & 0 \end{array}\right) \quad 2\text{row2} - 2\text{row1}$$

$$\begin{pmatrix} 2 & 9 & -3 & 0 \\ 0 & -17 & -17 & 0 \\ 0 & 16 & -8 & 0 \end{pmatrix} \begin{array}{l} \text{factor -17 from row 2} \\ \text{factor 8 from row 3} \end{array} \quad \left(\begin{array}{ccc|c} 2 & 9 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right) \text{ row3} - 2\text{row1}$$

$$\left(\begin{array}{ccc|c} 2 & 9 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right) \begin{array}{l} \text{factor 2 from row 1} \\ \\ \text{factor -3 from row3} \end{array} \quad \left(\begin{array}{ccc|c} 1 & 9/2 & -3/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$k_3 = 0$, $k_2 = -k_3 = 0$, $k_1 = -9/2k_2 + 3/2k_3 = 0 + 0 = 0$ so, $k_1 = 0, k_2 = 0, k_3 = 0$;
The system has the trivial solution and no vector is expressible as a linear combination of the other vectors. Therefore, the set of vectors is linearly independent.

(15 pts.) 7. Find the coordinate vector of p relative to the basis S ={ $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$}

$\mathbf{p} = 6 + 2x - 4x^2$; $\mathbf{p}_1 = 2 + -3x + x^2$, $\mathbf{p}_2 = 3 + 9x + 7x^2$, $\mathbf{p}_3 = -8 - x + x^2$

Answer: $\mathbf{p} = c_1 \mathbf{p}_1 + c_2 \mathbf{p}_2 + c_3 \mathbf{p}_3$

$$\left(\begin{array}{ccc|c} 2 & 3 & -8 & 6 \\ -3 & 9 & -1 & 2 \\ 1 & 7 & 1 & -4 \end{array}\right) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

– matrix comes from where?

$$\begin{array}{r} \\ \text{row2} + \text{row1} \\ \text{row3} - \text{row1} \end{array} \left(\begin{array}{ccc|c} 2 & 3 & -8 & 6 \\ -1 & 12 & -9 & 10 \\ -1 & 4 & 9 & -10 \end{array}\right) \text{ row2 - row 3}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -8 & 6 \\ 0 & 9 & -18 & -10 \\ -1 & 4 & 9 & -10 \end{array}\right) \begin{array}{l} \\ \text{factor 9 from row 2} \\ \text{2row3 + row1} \end{array} \left(\begin{array}{ccc|c} 2 & 3 & -8 & 6 \\ 0 & 1 & -2 & -10/9 \\ 0 & 11 & -10 & -14 \end{array}\right) \text{ row3 – 11row2}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -8 & 6 \\ 0 & 1 & -2 & -10/9 \\ 0 & 0 & 12 & -16/9 \end{array}\right) \begin{array}{l} \\ \text{factor 2 from row 1} \\ \text{factor 12 from row3} \end{array} \left(\begin{array}{ccc|c} 1 & 3/2 & -4 & 3 \\ 0 & 1 & -2 & -10/9 \\ 0 & 0 & 1 & -4/27 \end{array}\right) \begin{array}{l} c_1 = 122/27 \\ c_2 = -38/27 \\ c_3 = -4/27 \end{array}$$

So, $\mathbf{p} \neq (122/27) - (38/27)\,x + (-4/27)\,x^2 \neq 6 + 2x - 4x^2$ (!)

(15pts) 8. Determine whether the following functions are linearly dependent or linearly independent using the Wronskian of the functions.

$\mathbf{f}_1 = 2x^3$, $\mathbf{f}_2 = x^3$

Answer:

$$W(x) = \begin{pmatrix} 2x^3 & x^3 \\ 6x^2 & 3x^2 \end{pmatrix} \quad \det(W(x)) = 6x^5 - 6x^5 = 0 \text{ for all x.}$$

No conclusion can be drawn. from the Wronskian, but...

#9). Find bases for the row and column spaces of

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Because we know that row operations don't change the row space of a matrix, we can row reduce A to find a basis for the row space of A.

$$A_{(\text{row reduced})} = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since we know that the non zero row vectors of A form a basis for the row space of A the basis vectors are:

$$R^1 = [1 \quad -3 \quad 4 \quad -2 \quad 5 \quad 4]$$
$$R^2 = [0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6]$$
$$R^3 = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5]$$

To find the column spaces we look at the columns that contained leading ones in the row reduced from of A, this was columns 1, 3, and 5. So we look at the first, third, and fifth columns of A to get the column basis vectors, which are:

$$C_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} \qquad C_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix} \qquad C_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

#10). Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

The Row Reduced echelon form of A is

$$A_{(\text{row reduced})} = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are only two rows with leading 1's the rank of A = 2. In order to get the null space of A we have to solve the equation A**x** = 0. So one way to solve it is to augment the original matrix A with a row of 0's and row reduce this gives us.

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 & 0 \\ 3 & -7 & 2 & 0 & 1 & 4 & 0 \\ 2 & -5 & 2 & 4 & 6 & 1 & 0 \\ 4 & -9 & 2 & -4 & -4 & 7 & 0 \end{bmatrix}$$

And when row reduced goes to

$$A_{(\text{row reduced})} = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 & 0 \\ 0 & 1 & -2 & -12 & -16 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this matrix we have

$x_1 -4x_3 - 28x_4 - 37x_5 - 13x_6 = 0$
$x_2 -2x_3 - 12x_4 - 15x_5 + 5x_6 = 0$

Thus if we substitute in we can se that

$x_1 = 4r + 28s + 37t - 13u$
$x_2 = 2r + 12s + 16t -5u$
$x_3 = r$
$x_4 = s$
$x_5 = t$
$x_6 = u$

We can see that there are four parameters so the nullity (A) = 4. This can be seen easier if we rewrite the solution as

$$\mathbf{X} = r\begin{bmatrix}44\\2\\1\\0\\0\\0\end{bmatrix} + s\begin{bmatrix}28\\12\\0\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-13\\16\\0\\0\\1\\0\end{bmatrix} + u\begin{bmatrix}-13\\-5\\0\\0\\0\\1\end{bmatrix}$$

The four column vectors on the right hand side of the equation form a basis for the solution and so our null space for A = 4

Group Members:
Mike Miner, Brook Bromiley, Michael Gardener, Tyler Lefevor, Stephen Atkinson

1. For $u = (2, 1, 2, 0), v = (-4, 0, -3, 0), w = (1, 2, 2, 4)$
   Evaluate the following.

   a) $\|4u + 3v - 2w\|$

   b) $\|2u\| + 5\|v + w\|$ $= 2\|u\| + \ldots$

   c) $\left\|\frac{6}{\|u\|} w\right\|$ $= \frac{6}{\|u\|} \|w\|$

SOLUTION

a) $4\mathbf{u} + 3\mathbf{v} - 2\mathbf{w} = (-6, 0, -5, -8)$

$\sqrt{(-6)^2 + 0^2 + (-5)^2 + (-8)^2} = 5\sqrt{5}$

b) $2\mathbf{u} = (4, 2, 4, 0)$ $\quad \mathbf{v} + \mathbf{w} = (-3, 2, -1, 4)$

$\sqrt{4^2 + 2^2 + 4^2 + 0^2} + 5 * \sqrt{(-3)^2 + 2^2 + (-1)^2 + 4^2} = 6 + 5\sqrt{30}$

c)

$\|u\| = 3 \rightarrow \frac{6}{\|u\|} = \frac{6}{3} = 2 \rightarrow 2\,w = (2, 4, 4, 8)$

$\sqrt{2^2 + 4^2 + 4^2 + 8^2} = 10$

10 pts

2. Let $\mathbf{u} = (3, 4, 2)$,Find the magnitude of the orthogonal projection of $\mathbf{u}$ onto the xy Plane.

SOLUTION

$\mathbf{u} = (3, 4, 2)$

[T] for an orthogonal projection onto the xy plane = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$ The resulting matrix $\mathbf{w}$ is the orthogonal projection on the xy plane.

$\mathbf{w} = (3, 4, 0)$

The magnitude of $\mathbf{w}$ is expressed $\|\mathbf{w}\| = \sqrt{(3)^2 + (4)^2}$

$\|\mathbf{w}\| = 5$

10 pts

3. In R^3, rotate the vector $(\sqrt{3},2,6)$ 30 degrees about the z-axis and then reflect it about the origin.

SOLUTION

Rotating a vector about the z-axis is the same as multiplying the vector by the standard matrix for rotation about the z-axis.

Standard matrix is $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Right multiply vector into standard matrix with degrees.

$$\begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3\sqrt{3}}{2} \\ 6 \end{bmatrix}$$

Now take this vector $(\frac{1}{2},\frac{3\sqrt{3}}{2},6)$ and reflect it about the origin by right multiplying it into the standard matrix for reflection about the origin in R^3.

Standard matrix is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{3\sqrt{3}}{2} \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ \frac{-3\sqrt{3}}{2} \\ -6 \end{bmatrix}$$

The image of the vector $(\sqrt{3},2,6)$ rotated 30 degrees about the z-axis and then reflected about the origin is $(\frac{-1}{2},\frac{-3\sqrt{3}}{2},-6)$

15 pts

4. Determine if the linear operator $T: R^3 \rightarrow R^3$ for the following equations is one-to-one; if it is find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2, w_3)$.

a) $$\begin{aligned} w_1 &= 2\mathbf{x}_1 + 3\mathbf{x}_2 + 2\mathbf{x}_3 \\ w_2 &= 3\mathbf{x}_1 + 3\mathbf{x}_2 + 4\mathbf{x}_3 \\ w_3 &= 1\mathbf{x}_1 + 1\mathbf{x}_2 + 1\mathbf{x}_3 \end{aligned}$$

b) $$\begin{aligned} w_1 &= 1\mathbf{x}_1 + 3\mathbf{x}_2 - 1\mathbf{x}_3 \\ w_2 &= 0\mathbf{x}_1 + 2\mathbf{x}_2 - 1\mathbf{x}_3 \\ w_3 &= 1\mathbf{x}_1 + 1\mathbf{x}_2 + 0\mathbf{x}_3 \end{aligned}$$

c) $$\begin{aligned} w_1 &= 4\mathbf{x}_1 + 0\mathbf{x}_2 - 3\mathbf{x}_3 \\ w_2 &= 1\mathbf{x}_1 + 1\mathbf{x}_2 + 1\mathbf{x}_3 \\ w_3 &= -3\mathbf{x}_1 + 0\mathbf{x}_2 + 2\mathbf{x}_3 \end{aligned}$$

SOLUTIONS

a)

$$\begin{pmatrix} 2 & 3 & 2 \\ 3 & 3 & 4 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \rightarrow \begin{pmatrix} 2 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 4 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 6 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 & 6 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{pmatrix}$$

one-to-one, $T^1(w_1, w_2, w_3) = (-1w_1 - 1w_2 + 6w_3,\ 1w_1 - 2w_3,\ 1w_2 - 3w_3)$

b)

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & -1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \rightarrow \begin{pmatrix} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{pmatrix} \rightarrow \textbf{not invertable}$$

not one-to-one

c)

$$\begin{pmatrix} 4 & 0 & -3 \\ 1 & 1 & 1 \\ -3 & 0 & 2 \end{pmatrix}^{-1} \rightarrow \begin{pmatrix} 4 & 0 & -3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ -3 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -2 & 0 & -3 \\ 0 & 1 & 0 & 5 & 1 & 7 \\ 0 & 0 & 1 & -3 & 0 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 0 & -3 \\ 5 & 1 & 7 \\ -3 & 0 & -4 \end{pmatrix}$$

one-to-one, $T^1(w_1, w_2, w_3) = (-2w_1 - 3w_3,\ 5w_1 + 1w_2 + 7w_3,\ -3w_1 - 4w_3)$

15 pts

5. Determine if the set of all triples of real numbers (x, y, z) with the following definitions of addition and matrix multiplication is a vector space V.

(x, y, z) + (x', y', z') = (x+x'+1, y+y'+1, z+z'+1) and k(x, y, z) = (2kx, 2ky, 2kz)

SOLUTION

False…Due to axioms 9 and 10

Let **u** = (x, y, z), **v** = (a, b, c) and **w** = (d, e, f)

Axiom 1: **u + v is in vector space V** True:

$(x, y, z) + (a, b, c) = (x+a+1, y+b+1, z+c+1)$ is in V.

Axiom 2: **u + v = v + u** True:

$(x, y, z) + (a, b, c) = (x+a+1, y+b+1, z+c+1)$

$(a, b, c) + (x, y, z) = (a+x+1, b+y+1, c+z+1)$

Axiom 3: **u + (v + w) = (u + v) + w** True:

$(x, y, z) + ((a, b, c) + (d, e, f)) = ((a+d+1)+x+1, (b+e+1)+y+1, (c+f+1)+z+1) = (a+d+x+2, b+e+y+2, c+f+z+2)$

$((x, y, z) + (a, b, c)) + (d, e, f) = ((x+a+1)+d+1, (y+b+1)+e+1, (z+c+1)+f+1) = (x+a+d+2, y+b+e+2, z+c+f+2)$

$\Rightarrow (a+d+x+2, b+e+y+2, c+f+z+2) = (x+a+d+2, y+b+e+2, z+c+f+2)$

Axiom 4: **u + 0 = 0 + u = u** True:

$(x, y, z) + (-1, -1, -1) = (x, y, z)$

$(-1, -1, -1) + (x, y, z) = (x, y, z)$

Axiom 5: **u +(-u) = (-u)+u = 0** True:

$(x, y, z) + (-x-1, -y-1, -z-1) = (0, 0, 0)$

$(-x-1, -y-1, -z-1) + (x, y, z) = (0, 0, 0)$

Axiom 6: **ku is in the vector space V** True:

$k(x, y, z) = (2kx, 2ky, 2kz)$ is in V.

Axiom 7: k(**u** + **v**) = k**u** + k**v** True:

k**u** = (2kx, 2ky, 2kz)

k**v** = (2ka, 2kb, 2kc)

k**u** + k**v** = (2kx+2ka, 2ky+2kb, 2kz+2kc)

k(**u** + **v**) = (2k(x+a), 2k(y+b), 2k(z+c)) = (2kx+2ka, 2ky+2kb, 2kz+2kc)

Axiom 8: (k + m)**u** = k**u** + m**u** True:

(k + m)**u** = (2(k+m)x, 2(k+m)y, 2(k+m)z) = (2kx + 2mx, 2ky + 2my, 2kz +2mz)

k**u** + m**u** = (2kx, 2ky, 2kz) + (2mx, 2my, 2mz) = (2kx+2mx, 2ky+2my, 2kz+2mz)

Axiom 9: k(m**u**) = (km)**u** False:

m**u** = (2mx, 2my, 2mz)

k(m**u**) = (2k(2mx), 2k(2my), 2k(2mz)) = (4kmx, 4kmy, 4kmz)

(km)**u** = (2kmx, 2kmy, 2kmz)

Axiom 10: 1**u** = **u** False:

1**u** = (2x, 2y, 2z) doesn't equal **u**

<u>15 pts</u>

6. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the vectors $(\lambda, 0, 3)$, $(0, \lambda, 1)$, and $(1, 1, \lambda)$ respectively in R^3. For what values of λ do the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a linearly dependent set?

SOLUTION

The set is dependent when the equation A**x** = 0 has non-trivial solutions or in other words, when the determinant of the matrix A equals zero.

Augment matrix A with constants $\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}$. why?

$\mathbf{k_1}\,\lambda + 0\,\mathbf{k_2} + 1\,\mathbf{k_3} = 0$

$0\mathbf{k_1} + \lambda\mathbf{k_2} + 1\mathbf{k_3} = 0.$

$3\,\mathbf{k_1} + 1\,\mathbf{k_2} + \lambda\,\mathbf{k_3} = 0$

Using the shortcut to calculate 3x3 matrices and setting the determinant equal to zero,

$$\begin{vmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 1 \\ 3 & 1 & \lambda \end{vmatrix} = \lambda^3 + (0)(1)(3) + (1)(0)(1) - (\lambda)(3)(1) - (\lambda)(1)(1) - \lambda(1)(1) - (0)(0)(\lambda) =$$

$\lambda^3 - 4\lambda = 0$

$\lambda(\lambda^2 - 4) = 0$

This has solutions when λ is 0 or ± 2
This set of vectors is linearly dependent when λ is 0, 2, or -2.

15 pts

7. Show that the following form or do not form subspaces of the given set.

SOLUTIONS

a) All vectors in R^3 of the form (a, b, c, d, e, f) where a + b + c + d + e + f = 1

(a, b, c, d, e, f) + (a, b, c, d, e, f) = (2a, 2b, 2c, 2d, 2e, 2f)
2a + 2b + 2c + 2d + 2e +2f = 2
Therefore this is not a subspace
Also,
k (a, b, c, d, e, f) $\neq$ k (a + b + c + d + e + f) = k (1) — so?

b) All 3 x 3 matrices A in M_{33} such that $\det(A) \neq 0$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore not a subset

c) All f in F(-∞,∞) such that f(3) = 5

what is this?

$f_1 + f_2 = f_1(a) + f_2(a) \ldots f_1(3) + f_2(3) \ldots f_1(n) + f_2(n)$
= a' … + (5 + 5) … + n'
= a' … + 10 … + n'
Therefore not a subspace ?

d) All polynomials in P_4 of the form $a_0 + a_1\mathbf{x} + a_2\mathbf{x}^2 + a_3\mathbf{x}^3 + a_4\mathbf{x}^4$
for which $a_2 = a_4 = 0$

$$a_0 + a_1\mathbf{x} + 0\mathbf{x}^2 + a_3\mathbf{x}^3 + 0\mathbf{x}^4 + b_0 + b_1\mathbf{x} + 0\mathbf{x}^2 + b_3\mathbf{x}^3 + 0\mathbf{x}^4$$
$$= (a_0 + b_0) + (a_1 + b_1)\mathbf{x} + 0\mathbf{x}^2 + (a_3 + b_3)\mathbf{x}^3 + 0\mathbf{x}^4$$

and

$$k\,(a_0 + a_1\mathbf{x} + 0\mathbf{x}^2 + a_3\mathbf{x}^3 + 0\mathbf{x}^4)$$
$$= ka_0 + ka_1\mathbf{x} + 0\mathbf{x}^2 + ka_3\mathbf{x}^3 + 0\mathbf{x}^4$$

Therefore this is a subspace

20 pts

8. Determine the dimension and a basis for the solution space of the system

$$\begin{aligned} 1\mathbf{x}_1 - 1\mathbf{x}_2 + 1\mathbf{x}_3 &= 0 \\ 2\mathbf{x}_1 - 2\mathbf{x}_2 + 6\mathbf{x}_3 &= 0 \\ -3\mathbf{x}_1 + 3\mathbf{x}_2 - 9\mathbf{x}_3 &= 0 \end{aligned}$$

SOLUTION

$$\begin{pmatrix} 1 & -1 & 3 \\ 2 & -2 & 6 \\ -3 & 3 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$x_1 - x_2 + 3x_3 = 0$, $x_1 = x_2 - 3x_3$, $x_1 = s - 3t$

$x_2 = s$

$x_3 = t$

so

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} s - 3t \\ s \\ t \end{pmatrix} = \begin{pmatrix} s \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} -3t \\ 0 \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

so

$$v_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } v_2 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

are a basis and dimension 2

10 pts

9. Identify the following solution subspaces of the homogenous linear systems as the origin $\{\mathbf{0}\}$, a line through the origin, a plane through the origin, or all of R^3.

a) $\begin{bmatrix} -2 & -6 & 4 \\ 3 & 5 & 2 \\ 5 & 18 & -7 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -6 & 2 \\ 2 & -9 & 13 \\ -4 & 21 & 17 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 4 & 8 \\ 3 & 9 & 27 \\ 4 & 16 & 64 \end{bmatrix}$ d) $\begin{bmatrix} 9 & 3 & 12 \\ 1 & 1/3 & 4/3 \\ 3 & 1 & 4 \end{bmatrix}$

SOLUTIONS

a) $\begin{bmatrix} -2 & -6 & 4 \\ 3 & 5 & 2 \\ 5 & 18 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 2 \\ 5 & 18 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & -4 & 8 \\ 0 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 9 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ thus x=0, y=0, z=0 and the solution space is

simply the origin or **{0}**.

b) $\begin{bmatrix} 1 & -6 & 2 \\ 2 & -9 & 13 \\ -4 & 21 & -17 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -6 & 2 \\ 0 & 3 & 9 \\ 0 & -3 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -6 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$

$\begin{bmatrix} 1 & 0 & 20 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ so x= -20t, y= -3t and z=t. This is the equation of line through the origin.

c) $\begin{bmatrix} 2 & 4 & 8 \\ 3 & 9 & 27 \\ 4 & 16 & 64 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 3 & 9 & 27 \\ 4 & 16 & 64 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 15 \\ 0 & 8 & 48 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 8 & 48 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 8 \end{bmatrix} \rightarrow$

$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so x=0, y=0, z=0 and the solution space is only the

origin, in other words **{0}**.

d) $\begin{bmatrix} 9 & 3 & 12 \\ 1 & 1/3 & 4/3 \\ 3 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/3 & 4/3 \\ 1 & 1/3 & 4/3 \\ 3 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/3 & 4/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ this is a plane that goes through

the origin with $9x + 3y + 12z = 0 = 3x + y + 4z$

20 pts

ok?

10. Determine if the following are bases ~~in~~ R^3. If they are, prove them to span R^3 and be linearly independent. If not, show that they do not span R^3 or express them as linear combinations.

a) $\mathbf{v}_1 = (1, 4, 3)$ $\mathbf{v}_2 = (5, 7, 8)$ $\mathbf{v}_3 = (6, -2, 4)$

b) $\mathbf{v}_1 = (-2, 7, 4)$ $\mathbf{v}_2 = (6, -8, 3)$ $\mathbf{v}_3 = (6, 5, 18)$

c) $\mathbf{v}_1 = (6, -5, 7)$ $\mathbf{v}_2 = (1, 4, 3)$ $\mathbf{v}_3 = (-5, 9, 1)$

SOLUTIONS

a) $v_1 = (1, 4,3)$ $v_2 = (5, 7, 8)$ $v_3 = (6, -2, 4)$ → $\begin{bmatrix} 1 & 5 & 6 \\ 4 & 7 & -2 \\ 3 & 8 & 4 \end{bmatrix}$ → $\det\begin{bmatrix} 1 & 5 & 6 \\ 4 & 7 & -2 \\ 3 & 8 & 4 \end{bmatrix} = 1((7*4) -$ (-2*8)) – 5((4*4) – (-2*3)) + 6((4*8) – (7*3)) = 1(28 +16) – 5(16 +6) + 6(32 – 21) = 44 – 110 + 66 = 0. Thus, these three vectors do not span R^3.

calculations redundant

So $\begin{bmatrix} 1 & 5 & 6 \\ 4 & 7 & -2 \\ 3 & 8 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & 6 \\ 0 & -13 & -26 \\ 0 & -7 & -14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & 6 \\ 0 & 1 & 2 \\ 0 & -7 & -14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

so $v_3 = -4v_1 + 2v_2$

b) $v_1 = (-2, 7, 4)$ $v_2 = (6, -8, 3)$ $v_3 = (6, 5, 18)$ → $\det\begin{bmatrix} -2 & 6 & 6 \\ 7 & -8 & 5 \\ 4 & 3 & 18 \end{bmatrix} =$ -2((-8*18) – (5*3)) – 6((7*18) – (5*4)) + 6((7*3) – (-8*4)) = -2(-144 – 15) – 6(126 – 20) + 6(21 +32) = -2(-159) – 6(106) + 6(53) = 318 – 636 + 318 = 0 so it is not a basis because it is not independent and does not span R^3.

calculations redundant

$\begin{bmatrix} -2 & 6 & 6 \\ 7 & -8 & 5 \\ 4 & 3 & 18 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -3 \\ 7 & -8 & 5 \\ 4 & 3 & 18 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -3 \\ 0 & 13 & 26 \\ 0 & 15 & 30 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 2 \\ 0 & 15 & 30 \end{bmatrix} \rightarrow$

$\begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ so $3v_1 + 2v_2 = v_3$.

c) $v_1 = (6, -5, 7)$ $v_2 = (1, 4, 3)$ $v_3 = (-5, 9, 1)$ → $\det\begin{bmatrix} 6 & 1 & -5 \\ -5 & 4 & 9 \\ 7 & 3 & 1 \end{bmatrix} = 6((4*1) - (9*3)) - 1((-5*1) - (9*7)) + -5((-5*3) - (7*4)) = 6(4 - 27) - 1(-5 - 63) + -5(-25 - 27) = -138 + 68 + 260 = 330$ which doesn't equal 0. So the vectors do form a basis because there are three vectors and three variables and their determinant is not equal to 0.

20 pts

Problem 1

For the vectors **u** = (2, 3, 6) and **v** = (5, 1, 3)

a) Find the norms of **u** and **v.**
b) Find the Euclidean inner product of **u** and **v.**
c) Verify the Cauchy-Schwarz inequality for **u** and **v.**

Solution 1

a)
$$\|\mathbf{u}\| = \sqrt{2^2+3^2+6^2} = \sqrt{4+9+36} = \sqrt{49} = 7$$
$$\|\mathbf{v}\| = \sqrt{5^2+1^2+3^2} = \sqrt{25+1+9} = \sqrt{35}$$

b) $\mathbf{u}\cdot\mathbf{v} = (2*5)+(3*1)+(6*3) = (10+3+18) = 31$

c) Cauchy-Schwarz states: $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\| * \|\mathbf{v}\|$.

$$|31| \le 7*\sqrt{35}$$

So, $|31| \le 7*5.92$ so it does.

$$31 \le 41.41$$

Problem 2

a) Find the standard matrix for the linear operator that rotates a vector in R^3 through an angle of 270° about the z-axis. b) Then use the standard matrix from above to find the image of the vector, **v** = (5, 1, 3) after the rotation above.

Solution 2

$$\text{a)}\quad \begin{aligned} w_1 &= x\cos\theta - y\sin\theta \\ w_2 &= x\sin\theta + y\cos\theta \\ w_3 &= z \end{aligned} \Rightarrow \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$\begin{bmatrix} \cos 270^\circ & -\sin 270^\circ & 0 \\ \sin 270^\circ & \cos 270^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

b) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 3 \end{bmatrix}$ is **v** after undergoing the rotation.

Problem 3

Show whether the following transformations such that T: $R^3 \rightarrow R^2$ are linear transformations, show why or why not.

a) T(x, y, z) = (5x-z, 3y+x)
b) T(u, v, w) = (1+u, 0)
c) T(a, b, c) = (3b , c)

Solution 3

Using Theorem 4.3.2:

a) Let **u** = (1, 2, 3) and **v** = (2, 2, 2)

T(**u**+**v**) = T(**u**)+T(**v**) so

T(**u**+**v**) = T(3, 4, 5) = (15-5, 12+3) = (10, 5).

T(**u**)+T(**v**) =T(1, 2, 3)+(2, 2, 2)=(5-3, 6+1)+(10-2, 6+2) =(2, 7)+(8,8)=(10,15).

can't be specific except for counter example assuming result

Also required is for T(k**u**)=k*T(**u**).

T(k**u**)=T(k, 2k, 3k) = (2k, 7k).

k*T(**u**)= k(2, 7) = (2k, 7k).

So this is a linear transformation.

b) Using the same values for **v** and **u**.

T(**u**+**v**) = T(3, 4, 5) = (4, 0)

T(**u**)+T(**v**) =T(1, 2, 3)+(2, 2, 2)=(2, 0)+(3, 0) = (5, 0)

T(**u**+**v**) $\neq$ T(**u**)+T(**v**) so it is not a linear transformation.

o.k., since this is a counter example.

c) Using the same values for **v** and **u**.

T(**u**+**v**) = T(3, 4, 5) = (9, 5)

T(**u**)+T(**v**) =T(1, 2, 3)+(2, 2, 2)=(3, 3)+(6, 2) =(9, 5).

T(k**u**)=T(k, 2k, 3k) = (3k, 3k).

k*T(**u**)= k(3, 3) = (3k, 3k).

So this is a linear transformation.

Problem 4

Determine if the set (x, y) is a vector space with the operations $(x, y)+(x', y')=(x+x', y+y')$ and $k(x, y)=(2kx, 2ky)$, if the set is not a vector space state which axioms do not hold.

Solution 4

Not a vector space; axioms 9 and 10 do not hold.

Axiom 9: k(m**u**)=(km)(**u**), let **u** = (1, 1), k=2, m=2.

k(m**u**)=2(m**u**)=2(4,4)=(8,8).

(km)(**u**)=4(1,1)=(4,4) and (4,4)$\neq$(8,8), so this axiom fails.

Axiom 10: 1**u**=**u**. letting **u** = (1, 1) again,

(1**u**) = (2*1*1, 2*1*1) = (2, 2) $\neq$ (1, 1) = **u**, so this axiom also fails.

Problem 5

Determine whether the solution space of the system A**x** = **0** is a line through the origin, a plane through the origin, or the origin only. If it is a plane, find an equation that defines that plane; if it is a line give the parametric equations.

a) $A=\begin{bmatrix}-1 & 3 & 2\\ 2 & 1 & 4\\ -4 & 5 & 0\end{bmatrix}$ b) $B=\begin{bmatrix}-1 & 5 & 4\\ -3 & 15 & 12\\ -2 & 10 & 8\end{bmatrix}$

Solution 5

a) $A = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 4 \\ -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -2 \\ 0 & 7 & 8 \\ 0 & -7 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -2 \\ 0 & 7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 10/7 \\ 0 & 1 & 8/7 \\ 0 & 0 & 0 \end{bmatrix}$ The row of zeros means this defines a line, so setting z=t, y+8/7z=0, x+10/7z=0. $\begin{matrix} x & -10/7t \\ y = & -8/7t \\ z & t \end{matrix}$

b) $B = \begin{bmatrix} -1 & 5 & 4 \\ -3 & 15 & 12 \\ -2 & 10 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 4 \\ -1 & 5 & 4 \\ -1 & 5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Constitutes a plane, x-5y+4z=0

Problem 6

Consider the vectors **u** = (-1,2,4) and **v** = (0,4,8) in R^3. Determine whether **w** = (4,2,4) is a linear combination of **u** and **v**.

Solution 6

(4,2,4) = k_1(-1,2,4) + k_2(0,4,8)
(4,2,4) = (-k_1 + 0k_2, 2k_1 + 4k_2, 4k_1 + 8k_2)

Equating corresponding components gives

-k1 + 0k2 = 4
2k1 + 4k2 = 2
4k1 + 8k2 = 4

Solve the system using Gaussian elimination

$$\begin{bmatrix} -1 & 0 & 4 \\ 2 & 4 & 2 \\ 4 & 8 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 2 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 4 & 10 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

k_1 = -4, k_2 = 5/2

therefore

w = -4**u** + 5/2**v**

Problem 7

5.3 ?

By using the Wronskian method, determine whether the functions $\mathbf{f}_1 = 1$, $\mathbf{f}_2 = x-3$, $\mathbf{f}_3 = x^2+10$, and $\mathbf{f}_4 = \sin x$ form a linearly independent set of vectors in $C^{3}(-\infty, \infty)$.

Solution 7

$$W(x) = \begin{vmatrix} 1 & x-3 & x^2+10 & \sin x \\ 0 & 1 & 2x & \cos x \\ 0 & 0 & 2 & -\sin x \\ 0 & 0 & 0 & -\cos x \end{vmatrix} = -2\cos x$$

$\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$, and $\mathbf{f}_4$ form a linearly independent set because the Wronskian does not equal zero for all x in the interval $(-\infty, \infty)$.

Problem 8

Determine whether the vectors a = (1,-1, 3), b = (-1, 4,-6), and c = (2, 3, 1) form a linearly dependent set or a linearly independent set.

Solution 8

First form a system of equations using a, b and c.

$x - y + 2z = 0$

$-x + 4y + 3z = 0$

$x + 2y - z = 0$

Using these equations set up a matrix (the zero vector is excluded) and row reduce

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 4 & 3 \\ 3 & -6 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 5 \\ 0 & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 11/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{bmatrix}$$ Because there is a row of zeros, z will be a free variable; which means these form a linearly dependent set.

Problem 9

Let $\mathbf{v}_1 = (1, 8, 0)$, $\mathbf{v}_2 = (5, 6, 1)$, and $\mathbf{v}_3 = (4, 2, 5)$. Determine whether the set $S=\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for R^3.

Solution 9

Show that the vector $\mathbf{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination. of the v's?

$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ of the vectors in S - Is this a sentence?

$(b_1, b_2, b_3) = c_1(1,8,0) + c_2(5,6,1) + c_3(4,2,5)$

$$\begin{aligned} c_1 + 5c_2 + 4c_3 &= b_1 \\ 8c_1 + 6c_2 + 2c_3 &= b_2 \\ c_2 + 5c_3 &= b_3 \end{aligned}$$

To determine whether S is linearly independent, we must show that the only solution of $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$ is what?

$$\begin{bmatrix} 1 & 5 & 4 \\ 8 & 6 & 2 \\ 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & -34 & -30 \\ 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & 1 & 15/17 \\ 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & 1 & 15/17 \\ 0 & 0 & 70/17 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & 1 & 15/17 \\ 0 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

S is linearly independent; therefore S is a basis for R^3.

Problem 10

Determine whether b is in the column ~~vector~~ space of A, and if so, express b as a linear combination of the columns of A.

$$A = \begin{bmatrix} 1 & 2 & 4 & 15 \\ -2 & -5 & -11 & -37 \\ 2 & 4 & 17/2 & 31 \\ -1 & -1 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}.$$

Solution 10

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 15 & 1 \\ -2 & -5 & -11 & -37 & 2 \\ 2 & 4 & 17/2 & 31 & 2 \\ -1 & -10 & 0 & -3 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 4 & 15 & 1 \\ 0 & -1 & -3 & -7 & 4 \\ 0 & 0 & 1/2 & 1 & 0 \\ 0 & 1 & 4 & 12 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 9 \\ 0 & 1 & 3 & 7 & -4 \\ 0 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 9 \\ 0 & 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -13/3 \\ 0 & 1 & 0 & 0 & -20/7 \\ 0 & 0 & 1 & 0 & -16/3 \\ 0 & 0 & 0 & 1 & 8/3 \end{array}\right]$$ it is in the column ~~vector~~ space.

$$-\frac{13}{3}\begin{bmatrix}1\\2\\2\\-1\end{bmatrix}-\frac{20}{3}\begin{bmatrix}2\\5\\4\\-1\end{bmatrix}-\frac{16}{3}\begin{bmatrix}4\\-11\\17/2\\0\end{bmatrix}+\frac{8}{3}\begin{bmatrix}15\\-37\\31\\-3\end{bmatrix}=\frac{1}{3}\left(\begin{bmatrix}-13\\26\\-26\\13\end{bmatrix}-\begin{bmatrix}40\\-100\\80\\-20\end{bmatrix}-\begin{bmatrix}64\\-176\\136\\0\end{bmatrix}+\begin{bmatrix}120\\-296\\248\\-24\end{bmatrix}\right)$$

$$=\frac{1}{3}\begin{bmatrix}3\\6\\6\\9\end{bmatrix}=\begin{bmatrix}1\\2\\2\\3\end{bmatrix}$$ proves that was a linear combination of the columns of A.

1) If $\underline{u}$ and $\underline{v}$ are vectors in R^3 with the Euclidean Inner Product, then $\|\underline{u}+\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$ if and only if vectors $\underline{u}$ and $\underline{v}$ have what relationship?

Solution:
Vectors $\underline{u}$ and $\underline{v}$ must be orthogonal. So, $\underline{u} \bullet \underline{v} = 0$

2) Use vector arithmetic and Euclidean properties "to solve the find" the solutions to the following vectors given that u = (5, 7, 0, -4, 3) v = (0, -10, 2, 1, 7) w = (3, 2, 9, -1, 0) and k=1/3. — what does this "sentence" mean?

a) k(u-v) (3 points)

b) k(u·v) (3 points)

c) Euclidean distance d(u,w) (4 points)

Solutions:
a) (5/3, 17/3, -2/3, -5/3, -4/3)
b) -53/3
c) $\sqrt{157}$

Check:

a) k(u-v) = 1/3 x ((5-0), (7+10), (0-2), (-4-1), (3-7))
= 1/3 x (5, 17, -2, -5, -4)
= (5/3, 17/3, -2/3, -5/3, -4/3)

b) k(u·v) = 1/3 x ((5)(0) + (7)(-10) + (0)(2) + (-4)(1) + (3)(7))
= 1/3 x (0 + -70 + 0 + -4 + 21)
= 1/3 x (-53)
= -53/3

c) $d(u,w) = \|u-w\| = \sqrt{(5-3)^2+(7-2)^2+(0-9)^2+(-4+1)^2+(3-0)^2}$
$= \sqrt{2^2+5^2+(-9)^2+(-3)^2+3^2}$
$= \sqrt{4+25+81+9+9}$
$= \sqrt{157}$

3) A. Match the following matrices with the operation(s) they perform.

$$A=\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} \quad B=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad C=\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D=\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. Counterclockwise rotation about the positive z-axis through an angle θ.
2. Orthogonal projection on the xy-plane.
3. Reflection about the yz-plane.
4. Contraction with factor k on R^3

3) B. Find one matrix to perform all of the following linear transformations (in order): counterclockwise rotation about the positive z-axis through an angle $\pi/2$, followed by reflection about the yz plane.

Solution (A)
1. C
2. B
3. D
4. A

Solution (B)

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4) Determine the Matrix [T] which corresponds to the following transformations:

a. $T : R^2 \text{ -> } R^2$ projects a vector orthogonally along the y-axis, reflects about the x-axis, and then dilates the vector with a factor 3.

b. $T : R^3 > R^3$ projects a vector orthogonally onto the xy-plane and then reflects the vector about the xz-plane.

Solution

a) $\begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$ – show products – which don't commute in general

b) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

this looks like the solution, not just checking.

Check

a. We use theorem 4.3.3 to examine the columns of the matrix T that correspond to the unit vectors. Thus:

The x-vector given by the column matrix $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is projected orthogonally along the y-axis, which transforms the vector into $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, is then reflected about the x-axis, which transforms the vector into $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and then dilated to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The y-vector given by the column matrix $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is projected orthogonally along the y-axis, which transform the vector into $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, is then reflected about the x-axis, which transforms the vector into $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, and then dilated to $\begin{bmatrix} 0 \\ -3 \end{bmatrix}$.

So the solution [T] is given by $\left[\begin{array}{c|c} 0 & 0 \\ 0 & -3 \end{array}\right]$ or $\begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$.

Yes, this is a solution,

b. We use theorem 4.3.3 to examine the columns of the matrix T that correspond to the unit vectors. Thus:

The x-vector given by the column matrix $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, is projected orthogonally onto the xy-plane, which transforms the vector into $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, is then reflected about the xz-plane, which transforms the vector into $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$.

The y-vector given by the column matrix $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, is projected orthogonally onto the xy-plane, which transforms the vector into $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, is then reflected about the xz-plane, which transforms the vector into $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

The z-vector given by the column matrix $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, is projected orthogonally onto the xy-plane, which transforms the vector into $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, is then reflected about the xz-plane, which transforms the vector into $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

So the solution [T] is given by $\left[\begin{array}{c|c|c} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ or $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ — yes, this is a solution.

5) Determine whether the linear operators defined by the following equations are 1 to 1.

A.

$$W_1 = 16x_1 + 4x_2$$
$$W_2 = 4x_1 + x_2$$

B.

$$W_1 = 2x_1 + 4x_2$$
$$W_2 = -2x_1 + 2x_2$$

C.

$$W_1 = 10x_1 - 5x_2 + 3x_3$$
$$W_2 = 3x_1 + 2x_2 + x_3$$
$$W_3 = -20x_1 + 10x_2 - 6x_3$$

Answer

A. Not one-to-one (det = 0)
B. One-to-one
C. Not one-to-one (R.R.E. form $\neq I_n$)

Check— solution.

A. We begin by putting the operation in matrix form:
$\begin{bmatrix} 16 & 4 \\ 4 & 1 \end{bmatrix}$. By ~~definition~~ theorem!, if the determinant of the matrix = 0, the linear operations defined by the matrix aren't one-to-one. For this matrix, $\det A = (16)(1) - (4)(4) = 0$. Therefore the operations aren't one-to-one.

B. We begin by putting the operation in matrix form:
$\begin{bmatrix} 2 & 4 \\ -2 & 2 \end{bmatrix}$. Changing to reduced row-echelon form:

$\begin{bmatrix} 2 & 4 \\ -2 & 2 \end{bmatrix} \approx \begin{bmatrix} 2 & 4 \\ 0 & 6 \end{bmatrix} \approx \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore, by definition this linear operator is one-to-one since it's reduced row echelon form is equal to I_n

C. We begin by putting the operation in matrix form:

$\begin{bmatrix} 10 & -5 & 3 \\ 3 & 2 & 1 \\ -20 & 10 & -6 \end{bmatrix}$. Changing to reduced row-echelon form:

$\begin{bmatrix} 10 & -5 & 3 \\ 3 & 2 & 1 \\ -20 & 10 & -6 \end{bmatrix} \approx \begin{bmatrix} 10 & -5 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$... Since the reduced row echelon form of the matrix isn't equal to I_n, the linear operator isn't one-to-one.

6) Determine if the following are real vector spaces. If not, show why. (2 points each)

a) The set of all real with standard addition and multiplication operations.

b) The set of all real numbers > 0 with standard addition and multiplication operations.

c) The set of all 3 tuples of real numbers of the form $(0, x, x^2)$ and standard operations.

d) The set of all 3x3 matrices of the form $\begin{bmatrix} 0 & 1 & x \\ 0 & 1 & y \\ 0 & 1 & z \end{bmatrix}$ with standard matrix addition and scalar multiplication.

e) The set of all 3-tuples with the operations (a, b, c) + (d, e, f) =((a-d), (b-e), (c-f)) and k (a, b, c) = (ka, kb, c).

Solutions:

a) Yes, a vector space.
b) Not a vector space, no zero vector, no negative, and does not follow scalar rule for 0.
c) Not a vector space, does not follow addition rules.
d) Not a vector space, does not follow additive identity, zero vector property, negative identity, or scalar identity.
e) Yes, a vector space.— no way — "addition" not commutative, at least.

7) Which of the following are subspaces of R^4

a. all vectors of the form (a, b, c, d) where $a + b + c + d = 0$

b. all vectors of the form (a, 1, b, 0)

c. all vectors of the form (a, b, c, 0) where $a - b + c = 0$

d. all vectors of the form (0, 0, 0, 0)

~~Solutions~~ Answers

a) Subspace
b) Not a subspace
c) Subspace
d) Subspace

~~Check~~ Solutions

In order to be a subspace $\underline{u}, \underline{v} \in W$ and $\underline{u} + \underline{v} \in W$ and $(k)\underline{u} \in W$.

a. $\underline{u} = (a, b, c, d)$ and $\underline{v} = (f, g, h, i)$ and $a+b+c+d = f+g+h+i = 0$

Thus, $\underline{u} + \underline{v} = (a, b, c, d) + (f, g, h, i) = (a+f, b+g, c+h, d+i)$ and

$a+b+c+d+f+g+h+i = 0 = (a+f) + (b+g) + (c+h) + (d+i)$. Thus, $\underline{u} + \underline{v} \in W$, and

$\underline{u} = (a, b, c, d)$, so $(k)\underline{u} = k(a, b, c, d) = (ka, kb, kc, kd)$ and

$(ka+kb+kc+kd) = k(a+b+c+d) = k(0) = 0$, so $(k)\underline{u} \in W$. So part a is a subspace.

b. $\underline{u} = (a, 1, b, 0)$ and $\underline{v} = (c, 1, d, 0)$

Thus, $\underline{u} + \underline{v} = (a, 1, b, 0) + (c, 1, d, 0) = (a+c, 1+1, b+d, 0+0)$
$= (a+c, 2, b+d, 0)$ which is not W.
so part b is not a subspace.

c. $\underline{u} = (a, b, c, 0)$ and $\underline{v} = (d, f, g, 0)$ and $a-b+c = d-f+g = 0$

Thus, $\underline{u} + \underline{v} = (a, b, c, 0) + (d, f, g, 0) = (a+d, b+f, c+g, 0+0)$ and

$(a+d) - (b+f) + (c+g) = a - d - b - f + c + g = [(a-b) + c] + [(d - f) + g]$
$= 0 - 0 = 0$

Thus, $\underline{u} + \underline{v} \in W$, and

$\underline{u} = (a, b, c, 0)$, so $(k)\underline{u} = k(a, b, c, 0) = (ka, kb, kc, 0)$ and ka –kb +kc = k(a – b +c) =k(0) = 0, so (k)u Є W. So part c is a subspace.

d. u = (0, 0, 0, 0) and v = (0, 0, 0, 0)

Thus, u + v = (0, 0, 0, 0) + (0, 0, 0, 0) = (0 +0, 0+0, 0+0, 0+0) =

(0, 0, 0, 0) Thus, u + v Є W, and

u = (0, 0, 0, 0), so (k)u =k (0, 0, 0, 0) = (k*0, k*0, k*0, k*0) and

k(0, 0, 0, 0) = 0, so (k)u Є W. So part d is a subspace.

8) a) Determine linear independence. (2 points)

1) (3, -1), (4,5), (-4, 7) in R^2.

2) ($3+x+x^2$, $2-x + 5x^2$, $4-3x^2$) in R^3.

b) Determine linear independence using the Wronskian. (3 points)

1) (5, 8x, $15x^2$) in R^3.

2) (x, 2, 3x) in R^3.

Solutions:
a) 1) Independent
2) ?
3) Independent

b) 1) ?
2) ?

Check:
a) 1) There are three terms, if the number of terms is greater than the number of dimensions, dependence is assured.

how? standard basis of P_2.

2) This equation can be converted to the matrix: $\begin{bmatrix} 3 & 2 & 4 \\ 1 & -1 & 0 \\ 1 & 5 & -3 \end{bmatrix}$. To know dependence, we can find the determinant and if it is dependent, the determinant is <u>not</u> equal to 0. ?

?

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & -1 & 0 \\ 1 & 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 4 \\ 1 & 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 5 & 4 \\ 0 & 6 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 4/5 \\ 0 & 0 & -39/5 \end{bmatrix}$$

. Determinant of this matrix is -39/5 which is not equal to zero. This is linearly dependent. backwards.

b) 1) We can find the determinant using the Wronskian as follows:

$$\det \begin{bmatrix} 5 & 8x & 15x^2 \\ 0 & 8 & 30x \\ 0 & 0 & 30 \end{bmatrix}$$ =1200 which is not equal to 0, therefore this function is linearly independent .

2) In a~~n~~ similar method:

$$\det \begin{bmatrix} x & 2 & 3x \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$ = 0 and so this set of functions is linearly dependent. can't tell with Wronskian.

9) Show whether the following vectors are linearly independent or dependent.

a. (1, 2, 3, 4) (-2, -3, 0, 0) (3,5,6,7) (-1, -2, -3, -4)
b. (1, 2, 3) (2, 3, 4) (3, 4, 5) (0, 0, 1)
c. (1, 0, 0, 2) (0, 1, 0, 2) (0, 0, 1, 2) (0, 0, 0, 2)

Solutions
a) Dependent
b) Dependent ✓
d) Independent ✓

Check
a. $\underline{0} = (0, 0, 0, 0) = \alpha(1, 2, 3, 4) + \beta(-2, -3, 0, 0) + \gamma(3,5,6,7) + \delta(-1, -2, -3, -4)$

Which produces the augmented matrix $\begin{bmatrix} 1 & -2 & 3 & -1 & 0 \\ 2 & -3 & 5 & -2 & 0 \\ 3 & 0 & 6 & -3 & 0 \\ 4 & 0 & 7 & -4 & 0 \end{bmatrix}$ which reduces :

$$\begin{bmatrix} 1 & -2 & 3 & -1 & 0 \\ 2 & 0 & 5 & -2 & 0 \\ 3 & 0 & 6 & -3 & 0 \\ 4 & 0 & 7 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 6 & -3 & 0 & 0 \\ 0 & 8 & -5 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -2 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ so $\gamma = 0$ and $\beta - \gamma = 0$, so $\beta = 0$, and $\alpha - 2\beta + 3\gamma - \delta = 0$

So $\alpha - \delta = 0$, so $\alpha = \delta$, so these vectors are not linearly independent (they are linearly dependent). (all the coefficients are not equal to zero).

b. 4 vectors in R^3...are linearly dependent.

c. Similar to part A
$\underline{0} = (0, 0, 0, 0) = \alpha(1, 0, 0, 2) + \beta(0, 1, 0, 2) + \gamma(0, 0, 1, 2) + \delta(0, 0, 0, 2)$

Which produces the augmented matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 \end{bmatrix}$

Which is code for $\alpha = 0$
$\beta = 0$
$\gamma = 0$
$2\alpha + 2\beta + 2\gamma + 2\delta = 0 \Rightarrow 2(0) + 2(0) + 2(0) + 2\delta = 0 \Rightarrow \delta = 0$

So $\alpha = \beta = \gamma = \delta = 0$, so these vectors are linearly independent.

10) Determine whether $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for vector space R^3, where $\mathbf{v}_1 = (2,2,-1)$, $\mathbf{v}_2 = (3,2,0)$ and $\mathbf{v}_3 = (1,-2,0)$.

Answer

$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ IS a basis for vector space R^3.

Check

$$\begin{vmatrix} 2 & 2 & -1 \\ 3 & 2 & 0 \\ 1 & -2 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 0 \\ 3 & 2 & 0 \\ 2 & 2 & -1 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 0 \\ 0 & 8 & 0 \\ 0 & 6 & -1 \end{vmatrix} = -8\begin{vmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & -1 \end{vmatrix} = -8\begin{vmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = (-8)(-1) = 8 \neq 0$$

Since the determinant of the matrix is not equal to zero, $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of the vector space R^3.

1) For the 2 Euclidean Vectors:

$u = (5, -3, 0)$ and $v = (2, 4, 6)$ calculate the following:

a) $3u - 1/2\, v$
b) $u \bullet v$
c) $\|u + v\|$
d) $\|u\| + \|v\|$

Solution:

a) Scalar multiplication is accomplished by multiplying each element by the scalar, and addition is accomplished by adding each element to its corresponding element as shown below:

$3u = (3 \times 5, 3 \times -3, 0) = (15, -9, 0)$
$-1/2v = (-1/2 \times 2, -1/2 \times 4, -1/2 \times 6) = (-1, -2, -3)$
$3u - 1/2\, v = (15 - 1, -9 - 2, 0 - 3) = (14, -11, -3)$

b) To take the dot product, add the products of each element:

$u \bullet v = u(1) \times v(1) + u(2) \times v(2) + u(3) \times v(3)$
$5 \times 2 + -3 \times 4 + 0 = 10 - 12 = -2$

c) Add u and v and then calculate the length by the formula $\text{length}^2 = (u_1^2 + u_2^2 + u_3^2)$

$u + v = (5+2, -3+4, 6) = (7,1,6)$
$\text{length}^2 = 49 + 1 + 36 = 86$, so $\text{length} = \sqrt{86}$

d) Find the length of each, and then add

for u, $\text{length}^2 = 25 + 9 = 36$, so $\text{length} = 6$
for v, $\text{length}^2 = 4 + 16 + 36 = 56$, so $\text{length} = 2\sqrt{14}$

so, the length of u added to the length of v is $6 + 2\sqrt{14}$

d) no, because the dot product of u and v is not zero as shown in part b.

2) Determine whether the solution space of the system $Ax = 0$ is a line through the origin, a plane through the origin, or the origin only. If it's a plane or a line, write the equation describing it.

a) $A = \begin{bmatrix} 2 & 3\,5 \\ -2\,1\,0 & \\ 6 & 9\,10 \end{bmatrix}$ b) $A = \begin{bmatrix} 9 & 1 - 5 \\ 36 & 4 - 20 \\ 81 & 9 - 45 \end{bmatrix}$ c) $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$

Solution: Because it is a solution set for $Ax = 0$, we can solve the matrix to determine the solution to the problem by reducing the augmented matrix into row-echelon form and solving for x, y, and z.

a) By row operation, the matrix reduces:

$$\begin{bmatrix} 2 & 3 & 5 \\ -2 & 1 & 0 \\ 6 & 9 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

, which gives us the equations: $2x + 3y + 5z = 0$, $4y + 5z = 0$.

Let $z = t$, which gives us $4y + 5t = 0$, or $y = -5/4t$. So, $2x - 15/4t + 5t = 0$, and $x = -5/8t$.

These are equations of a line through the origin:

$x = -5/8\ t$
$y = -5/4\ t$
$z = t$

b) By row operations, the matrix reduces:

$$\begin{bmatrix} 9 & 1 & -5 \\ 36 & 4 & -20 \\ 81 & 9 & -45 \end{bmatrix} \Rightarrow \begin{bmatrix} 9 & 1 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This has the equation $9x + y - 5z = 0$ as its solution, which is a plane going through the origin.

c) The matrix reduces through matrix operations

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 4 \\ 0 & 5 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

From this matrix, we find that:

$-1z = 0$, so $z = 0$
$5y + 4z = 0 = 5y$, so $y = 0$
$1x + 3y + 2z = 0 = 1x + 0 + 0 = 1x = 0$, so $x = 0$
So therefore the solution set is only the origin: (0,0,0)

3) Find the coordinate vector of v relative to the basis $S = \{v_1, v_2, v_3\}$

$v = (7,1,5)$
$v_1 = (3, 10, 3)$
$v_2 = (-2, 4, 0)$
$v_3 = (5, 5, 5)$

Solution:
From these equations, we know that:

$$a\begin{bmatrix}3\\10\\3\end{bmatrix} + b\begin{bmatrix}-2\\4\\0\end{bmatrix} + c\begin{bmatrix}5\\5\\5\end{bmatrix} = \begin{bmatrix}7\\1\\5\end{bmatrix}$$

where a, b, and c are constants that will give us the coordinate vector of our basis.

We can solve for these constants by simple algebra:
$3a - 2b + 5c = 7$
$10a + 4b + 5c = 1$
$3a + 5c = 5$

– disappointing use of this "old school" method

Solve the 3rd equation for c and plug it into the first equation to solve for b
$5c = 5 - 3a$
$3a - 2b + 5 - 3a = 7$
$-2b = 2,\ b = -1$
Plug in this value to find answer for a and c
$10a - 4 + 5 - 3a = 1$
$7a + 1 = 1,\ 7a = 0,\ a = 0$
$5c = 5,\ c = 5$
So, $a = 0$, $b = -1$, and $c = 1$

Thus, our coordinate vector of v is equal to the vector (0, -1, 1)

4.) Find the rank and nullity of the matrix:

$$\begin{bmatrix}-1 & 3 & 0 & 1 & 2 & 1\\ 1 & -2 & 1 & 2 & 1 & 2\\ 2 & -5 & 1 & 1 & -1 & 1\\ -1 & 2 & -1 & -2 & -1 & -2\end{bmatrix} = A$$

Solution: First we need to reduce the matrix to reduced row echelon form as follows:

$$\begin{bmatrix} -1 & 3 & 0 & 1 & 2 & 1 \\ 1 & -2 & 1 & 2 & 1 & 2 \\ 2 & -5 & 1 & 1 & -1 & 1 \\ -1 & 2 & -1 & -2 & -1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 3 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 & 3 & 3 \\ 0 & 1 & 1 & 3 & 3 & 3 \\ 0 & 1 & 1 & 3 & 3 & 3 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} -1 & 3 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 & 8 & 7 & 8 \\ 0 & 1 & 1 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From looking at this matrix, we see that the rank(A) = 2 because there are two entries that begin with leading 1s.

This gives us the equations:

$x_1 = -3r - 8s - 7t - 8u$
$x_2 = -1 - 3s - 3t - 3u$
$x_3 = r$
$x_4 = s$
$x_5 = t$
$x_6 = u$

– too much work to answer question asked

Which gives us:

$$\begin{bmatrix} x1 \\ x2 \\ x3 \\ x4 \\ x5 \\ x6 \end{bmatrix} = r \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -8 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -8 \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Because there are 4 vectors on the right of the equation, we note that nullify(A) = 4

5) (15points) Is the vector **w** = (12, 6, 3) a linear combination of **u** = (1, 2, 4) and **v** = (2, 4, 6)?

1

Solution: Vector **w** must satisfy (5, 10, 16) = k_1(1, 2, 4) + k_2(2, 4, 6) for **w** to be a linear combination of **u** and **v**.

First you have to factor k_1 and k_2 through the two vectors to produce the system of equations of:

$$5 = k_1 + 2\,k_2$$
$$10 = 2\,k_1 + 4\,k_2$$
$$16 = 4\,k_1 + 6\,k_2$$

This in turn produces the following matrix, which will be used to solve for k_1 and k_2 using reduced row echelon form:

just 1.

$$\left|\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 4 & 10 \\ 4 & 6 & 16 \end{array}\right| \sim \left|\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & -2 & -4 \end{array}\right|$$

There appears to be infinitely many solutions since a row has zeroed out, therefore, **w** is a linear combination.

6) (15 points) Verify that the two vectors $\mathbf{V}_1$ = (1, 3) and $\mathbf{V}_2$ = (4, 2) span ~~on~~ vector space R^2.

Solution: First an arbitrary vector **b** must be assigned being equal to (b_1, b_2) and it can be expressed as the linear combination of:

$$\mathbf{b} = k_1\mathbf{V}_1 + k_2\mathbf{V}_2$$ — a bit of a detour

This produces the matrix of:

$$\left|\begin{array}{ccc} 1 & 3 & b_1 \\ 4 & 2 & b_2 \end{array}\right|$$ And the coefficient matrix: $$\left|\begin{array}{cc} 1 & 3 \\ 4 & 2 \end{array}\right|$$

The determinant of the coefficient matrix is 1 * 2 – 4 * 3 = -10, which is nonzero, therefore the two vectors span ~~onto~~ space R^2.

7) (15 points) Find a basis for the nullspace of:

$$L = \begin{vmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{vmatrix}$$

through linear combos

Solution: The basis of a nullspace is a vector ~~vector(s)~~ that produces a solution for a homogeneous system of equations. Therefore, x_1, x_2, x_3, x_4 and x_5 must first be solved ~~for~~ by row reducing

$$L = \begin{vmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{vmatrix}$$

Since **L** is a homogenous matrix simplifying to the reduced row echelon form is a simple process. The reduced echelon form becomes:

$$L = \begin{vmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Now the system of equations becomes the following:

$$\begin{aligned} x_1 + x_2 \quad\quad + x_5 &= 0 \\ x_3 \quad + x_5 &= 0 \\ x_4 \quad\quad &= 0 \end{aligned}$$

Solving for x_1, x_3, and x_4 produces:

$$x_1 = - x_2 - x_5$$
$$x_3 = - x_5$$
$$x_4 = 0$$

Now the general solution is:

$$x_1 = -a - b, \quad x_2 = a, \quad x_3 = -b, \quad x_4 = 0, \quad x_5 = b$$

These solutions written as vectors are the following:

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{vmatrix} = \begin{vmatrix} -a-b \\ a \\ -b \\ 0 \\ b \end{vmatrix} = \begin{vmatrix} -a \\ a \\ 0 \\ 0 \\ 0 \end{vmatrix} + \begin{vmatrix} -b \\ 0 \\ -b \\ 0 \\ b \end{vmatrix} = a \begin{vmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} + b \begin{vmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{vmatrix}$$

Finally, the nullspace basis is the following vectors:

$$\mathbf{V}_1 = \begin{vmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad \mathbf{V}_2 = \begin{vmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{vmatrix}$$

8) The equations:

$$
\begin{aligned}
w_1 &= 2x_1 - 3x_2 + x_3 \\
w_2 &= 4x_1 + (x_2)^2 - 2x_3 \\
w_3 &= 5x_1 - 1x_2 + 4x_3 \\
w_4 &= -5x_1 + x_2
\end{aligned}
$$

nonlinear

define a transformation $T: R^3 \rightarrow R^4$. If $V = \{ v_1, v_2, v_3, v_4\}$ where

$$
\begin{aligned}
v_1 &= (1,\ 2,\ 3) \\
v_2 &= (2,\ 4,\ 6) \\
v_3 &= (3,\ 7,\ 9) \\
v_4 &= (2,\ -6,\ 3)
\end{aligned}
$$

? o.k. – get it.

Show that the set S is or is not linearly independent in R^4 if $S = T(V)$.

The first step is to find S by transforming V using equations w_1, w_2, w_3, w_4. This is done by replacing x_1 with the first coordinant, x_2 with the second coordinant, and x_3 with the third coordinant of each vector in V.

This process produces the vector set

$$S = \{ (-1, 2, 15, -3); (-2, 12, 30, -6); (-6, 43, 36, -8); (25, 38, 28, -16) \}$$

Linear independence can then be tested for by creating a 4x4 matrix and taking its determinant.

$$
\begin{vmatrix}
-1 & -2 & 15 & 25 \\
2 & 12 & 30 & 38 \\
15 & 30 & 36 & 28 \\
-3 & -6 & -8 & -16
\end{vmatrix}
= -7027 \neq 0
$$

Therefore S is a linearly independent set of vectors.

(There are many methods for testing linear independence such as row reduction or showing that no vector is a scalar combination of the others. Because determinants were covered in the previous test the method for taking a determinant is not discussed here. It should be noted though that $\det(A) = \det(A^T)$.)

9) Linear Transformations

Transfer from $R^4 \rightarrow R^3$
If $(x_1,x_2,x_3,x_4) = (-1,0, 2, 4)$

$W_1 = 3x_1-4x_2+2x_3-x_4$

$W_2 = -2x_1+2x_2-5x_3$

$W_3 = x_1+2x_2+4x_3-3x_4$

Is there a sentence here?

Solution:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 2 & -1 \\ -2 & 2 & -5 & 0 \\ 1 & 2 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 2 & -1 \\ -2 & 2 & -5 & 0 \\ 1 & 2 & 4 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \\ -5 \end{bmatrix}$$

So $w_1= -3$, $w_2= -8$, $w_3= -5$, $(-3, -8, -5)$

10) Linear Independence

Is the following set of vectors in R^4 linearly dependent?

$V_1 = (2,1,0,-2)$ $V_2 = (1,-3,5,-2)$ $V_3 = (-3,-5,5,2)$

Solution:

what set of equations?

By solving this set of equations we find the coefficients 2, -1, and 1, which gives us the equation:

$S = 2V_1 - V_2 + V_3 = 0$

So, YES, because S = 0 for coefficient values other than 0.

1) For which values of k are **u** and **v** orthogonal?

a. $\mathbf{u} = (4, \quad 3, \quad -1), \mathbf{v} = (2, \quad 5, \quad k)$

b. $\mathbf{u} = (1, \quad -3 \quad ,-7), \mathbf{v} = (-2, \quad 4, \quad k)$

Solution:

a.
$$8 + 15 = -(-1k)$$
$$k = 23$$

b.
$$-2 - 12 = -(-7k)$$
$$-14 = 7k$$
$$k = -2$$

2) Find the standard matrix for the linear transformation T defined by the formula $T(x_1, \quad x_2) = (x_1 + 3x_2, \quad 2x_1 - x_2, \quad 3x_1 + 4x_2)$ and solve for $T(1, \quad 2)$

Solution:

T(1,2)(Standard Matrix)=Solution

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 11 \end{bmatrix}$$

3)

Let A be the matrix: $\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & 2 \end{bmatrix}$ and **b** be the column matrix: $\begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$

Is b in the column space of A? If so, express b as a combination of the ~~liner~~ column vectors of A.

Solution:

*b **is** in the column space of A. To determine this, we put A in the form Ax=b, and solve for b by Gaussian Elimination to get:*

$$x_1 = 2$$

$x_2 = -1$

$x_3 = 3$

*Because the system has a solution, **b** is in the column space of **A**, and **b** in terms of **A** is:*

$$2\begin{bmatrix}-1\\1\\2\end{bmatrix} - \begin{bmatrix}3\\2\\1\end{bmatrix} + 3\begin{bmatrix}2\\-3\\-2\end{bmatrix} = \begin{bmatrix}1\\-9\\3\end{bmatrix}$$

4) Solve both parts:

  a. Given a matrix A:

$$\begin{bmatrix}0 & 1 & 7 & 8\\1 & 3 & 3 & 8\\-2 & -5 & 1 & -8\end{bmatrix}$$

  Find the rank and nullity of A.

  b. Find the rank and nullity of A^T

Solution:

  a. *The Reduced-Row Echelon form of A is:* $\begin{bmatrix}1 & 3 & 3 & 8\\0 & 1 & 7 & 8\\0 & 0 & 0 & 0\end{bmatrix}$

  The rank of A is 2, and since rank + nullity = number of columns, the nullity is 2

  b. *Since rank(A) = rank (A^T),*

  rank(A^T) = 2

  nullity(A^T) = $\cancel{0}$ 1

5) Give at least one reason why the given set is not a vector space

  a. The set of triples of real numbers (x, y, z) under the operation

$$(x, y, z) + (x', y', z') = (x + x', y + y', z')$$
$$k(x, y, z) = (kx, ky, z)$$

  b. The set of a 2×3 matrix of the form A= $\begin{bmatrix}a & b & 1\\c & d & e\end{bmatrix}$ under standard matrix addition and multiplication rules

Solution:

a. *Let u = (u_1, u_2, u_3) and v= (v_1, v_2, v_3)*

u+v = v+u

$(u_1, u_2, u_3) + (v_1, v_2, v_3) =? (v_1, v_2, v_3) + (u_1, u_2, u_3)$

$(u_1 + v_1, u_2 + v_2, v_3) \neq (v_1 + u_1, v_2 + u_2, u_3)$

This implies the set is not a vector space under the given operation

b. *Let u =* $\begin{bmatrix} U11 & U12 & 1 \\ U21 & U22 & U23 \end{bmatrix}$ *and v=* $\begin{bmatrix} V11 & V12 & 1 \\ V21 & V22 & V23 \end{bmatrix}$

u + v = $\begin{bmatrix} U11+V11 & U12+V12 & 1+1 \\ U21+V21 & U22+V22 & U23+V23 \end{bmatrix}$

This is not an abject in the given Matrix A therefore the matrix is not a vector space under the matrix standard operations

6) Check if the set of a 3×3 matrix of the form B= $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ under standard ~~matrix~~ scalar?

multiplication and matrix addition is a vector space.

Solution:

There is no zero in B such that

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$0+\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

This implies the matrix is not a vector space under the standard matrix operations of multiplication and addition.

7) Let V_1=(2,3,4) ,V_2=(1,4,6) and V_3=(5,7,6)

Which of the following vectors are in the span of (V_1, V_2, V_3)?

(a) (8, 14, 16) (b) (15, 24, 26)

Solution:

$K_1(2,3,4) + K_2(1,4,6) + K_3(5,7,6) = (8, 14, 16)$

$K_1(2,3,4) + K_2(1,4,6) + K_3(5,7,6) = (15, 24, 26)$

$2K_1 + K_2 + 5K_3 = 8, 15$

$3K_1 + 4K_2 + 7K_3 = 14, 24$

$4K_1 + 6K_2 + 6K_3 = 16, 26$

$$\begin{bmatrix} 2 & 1 & 5 & 8 & 15 \\ 3 & 4 & 7 & 14 & 24 \\ 4 & 6 & 6 & 16 & 26 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Therefore

$1(2,3,4) + 1(1,4,6) + 1(5,7,6) = (8, 14, 16)$

$2(2,3,4) + 1(1,4,6) + 2(5,7,6) = (15, 24, 26)$

Both (a) and (b) are in the span ~~of~~ (V_1, V_2, V_3)

8) Explain whether the following span R^3:

a. $v_1 = (1,1,0), v_2 = (1,0,1)$

b. $v_1 = (1,2,1), v_2 = (1,3,3), v_3 = (2,7,10)$

c. $v_1 = (1,2,3), v_2 = (2,8,8), v_3 = (4,10,13), v_4 = (3,4,8)$

Solutions:

a. $v_1 = (1,1,0), v_2 = (1,0,1)$

This cannot span R^3. We need 3 vectors at minimum for this to span R^3, and even then it's not guaranteed. We can show why by trying to make a corresponding components matrix "that you lose the significance of the final element in the vectors."

b. $v_1 = (1,2,1), v_2 = (1,3,3), v_3 = (2,7,10)$

We can create a corresponding components matrix and get $\begin{bmatrix}1 & 1 & 2\\2 & 3 & 7\\1 & 3 & 10\end{bmatrix}$. *Add -2 row 1 to row 2 and -1 row 1 to row 3 to get* $\begin{bmatrix}1 & 1 & 2\\0 & 1 & 3\\0 & 2 & 8\end{bmatrix}$. *Add -2 row 2 to row 3* $\left(\begin{bmatrix}1 & 1 & 2\\0 & 1 & 3\\0 & 0 & 2\end{bmatrix}\right)$ *and we see that the determinant of this matrix will not be 0 at this point. This is important since this means that the vectors will be consistent. Thus, this does span* R^3.

c. $v_1 = (1,2,3), v_2 = (2,8,8), v_3 = (4,10,13), v_4 = (3,4,8)$

Again create a components matrix: $\begin{bmatrix}1 & 2 & 4 & 3\\2 & 8 & 10 & 4\\3 & 8 & 13 & 8\end{bmatrix}$. *Add -2 row 1 to row 2 and -3 row 1 to row three to get* $\begin{bmatrix}1 & 2 & 4 & 3\\0 & 4 & 2 & -2\\0 & 2 & 1 & -1\end{bmatrix}$. *We can stop right here. We can see that row 2 and row 3 are multiples of each other. This being the case, we cannot get a leading 1 for one of the rows. Thus, this does not span* R^3.

9) For which real values of λ do the following vectors create a linearly dependent set in R^3?

$v_1 = (\lambda,1,1), v_2 = (1,\lambda,1), v_3 = (1,1,\lambda)$

<u>Solution:</u>

We can create a matrix by equating corresponding components to get $\begin{bmatrix}\lambda & 1 & 1\\1 & \lambda & 1\\1 & 1 & \lambda\end{bmatrix}$. *To make things a little easier, swap rows 1 and 3* $\left(\begin{bmatrix}1 & 1 & \lambda\\1 & \lambda & 1\\\lambda & 1 & 1\end{bmatrix}\right)$.

Now add -1 time row 1 to row 2 and -λ *times row 1 to row 3 to get a leading 1 in* *column 1:* $\begin{bmatrix} 1 & 1 & \lambda \\ 0 & \lambda - 1 & 1 - \lambda \\ 0 & 1 - \lambda & 1 - \lambda^2 \end{bmatrix}$. *If we add row 2 to row 3, we get*

$\begin{bmatrix} 1 & 1 & \lambda \\ 0 & \lambda - 1 & 1 - \lambda \\ 0 & 0 & 2 - \lambda - \lambda^2 \end{bmatrix}$. *In finding linear dependence, we need to show that "only the trivial solution will yield a 0." We could have created an augmented matrix, but adding a column of 0's would not have done much. However, each row is equal to 0 in this matrix. Since we have reduced the third row to a single element, we will express it as equaling to zero:* $2 - \lambda - \lambda^2 = 0$. *We can then solve for this and get*

which matrix?

$$\begin{aligned} 2 - \lambda - \lambda^2 &= -(\lambda^2 + \lambda - 2) \\ &= -(\lambda + 2)(\lambda - 1) \\ &= 0 \end{aligned}$$

also this term is important in general.

This shows that our values of λ *should be*

$$\lambda = -2, 1$$

Which will make the given set of vectors linearly dependent.

10) Find a basis for the following solutions:

$$2x_1 + x_2 + x_3 = 0$$
$$-x_1 + x_2 - x_3 = 0$$

Solution:

If we solve for these two systems, we can get a vector which will represent possible solutions. First, we make a coefficient matrix to get

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}.$$

If we swap the rows $\left(\begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}\right)$, *and then multiply row 1 by -1 and add -2 times row 1 to row 2. This gives us the matrix* $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \end{bmatrix}$. *Multiply row 2 by* $\frac{1}{3}$ *and then add row 2 to row 1 to get* $\begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & -\frac{1}{3} \end{bmatrix}$. *This is code for*

$$\begin{aligned} x_1 + \frac{2}{3}x_3 &= 0 \\ x_2 - \frac{1}{3}x_3 &= 0 \end{aligned}$$

. *If we solve the two equations letting* $x_3 = t$, *we get*

$x_1 = -\frac{2}{3}t, x_2 = \frac{1}{3}t, x_3 = t$. *Our basis would then be* $\left(-\frac{2}{3}t, \frac{1}{3}t, t\right)$ for some value of $t \neq 0$. *To find a specific basis, choose a t such as 3, which gives* $(-2,1,3)$ *as a basis.*