KEY

Math 343 Midterm 2
Instructor: Scott Glasgow
Sections: 1, 6 and 8
Dates: October 26th and 27th, 2005

Instructions: You will be docked points for *irrelevant* developments as well as patently wrong ones. So organize your efforts on two sets of papers, one set called “scratch” containing whatever *meanderings* you needed to get things clear in your mind, and which set I do not ever want to see, the other set on which you put your name and section number containing only ***relevant*** calculations and discussions leading ***directly*** to (hopefully) correct answers, and which you hand in to the testing center staff (so that I may see and grade them). Related to this idea is the fact that you will also be docked for handwriting and organization that is so poor that either I cannot determine your development, or I find it painful to determine it. In summary, your work that I ultimately see and grade should be a (logical) work of art.

343 Midterm 2 Fall 2005

1. Demonstrate the validity of the Cauchy-Schwarz inequality for $\mathbf{u} = (3,4)$ and $\mathbf{v} = (5,12)$. In particular, calculate $\|\mathbf{u}\|\|\mathbf{v}\| - |\mathbf{u}\cdot\mathbf{v}|$

7 pts

Solution:

The Cauchy-Schwarz inequality is (for now) that for any $\mathbf{u}$ and $\mathbf{v}$ in a Euclidean space $\mathbb{R}^n$,

$$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|.$$

Here we have that

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{(3,4)\cdot(3,4)} = \sqrt{3^2+4^2} = \sqrt{5^2} = 5,$$
$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{(5,12)\cdot(5,12)} = \sqrt{5^2+12^2} = \sqrt{13^2} = 13,$$

and indeed

$$|\mathbf{u}\cdot\mathbf{v}| = |(3,4)\cdot(5,12)| = |3\cdot 5 + 4\cdot 12| = 15+48 = 63 \le 65 = 5\cdot 13 = \|\mathbf{u}\|\|\mathbf{v}\|.$$

Specifically we find that $\|\mathbf{u}\|\|\mathbf{v}\| - |\mathbf{u}\cdot\mathbf{v}| = 65-63 = 2$.

2. What does the Wronskian tell you about the following sets of vectors in $C^3(\mathbb{R})$?
a) $\{1, \sin x, \cos x\}$, b) $\{1, x-3, 2x^2+4\}$, c) $\{x, 3x+2, 9x-5\}$.

15 pts

Solution

We have
a)

$$W\{1,\sin x,\cos x\} = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = \begin{vmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{vmatrix} = -\begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix}$$
$$= -\left(\cos^2 x + \sin^2 x\right) = -1 \ne 0$$

for some (all) value(s) of x, whence the set is linearly independent,

b)

$$W\{1, x-3, 2x^2+4\} = \begin{vmatrix} 1 & x-3 & 2x^2+4 \\ 0 & 1 & 4x \\ 0 & 0 & 4 \end{vmatrix} = 4 \neq 0$$

for some (all) value(s) of x, whence the set is linearly independent, and

c)

$$W\{x, 3x+2, 9x-5\} = \begin{vmatrix} x & 3x+2 & 9x-5 \\ 1 & 3 & 9 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

for all values of x, whence use of the Wronskian is inconclusive (but it is clear that the set is linearly dependent directly from the definition since $33x - 5(3x+2) - 2(9x-5) = 0$ for every value of x).

3. The vector $\mathbf{b} = (1, -9, -3)$ is in the column space of

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}.$$

So write $\mathbf{b}$ as a specific linear combination of A's columns by solving $A\mathbf{x} = \mathbf{b}$.

<u>15 pts</u>

<u>Solution</u>

As indicated, $\mathbf{b}$ is in $\mathrm{Col}A$ iff there is an $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$. So we attempt to solve the latter for $\mathbf{x}$, here row reducing the augmented matrix $\left[A \middle| \mathbf{b}\right]$ =

$$A = \left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 1 & 2 & -3 & -9 \\ 2 & 1 & -2 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 0 & 5 & -1 & -8 \\ 0 & 7 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 0 & 5 & -1 & -8 \\ 0 & 2 & 3 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 0 & 1 & -7 & -22 \\ 0 & 2 & 3 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 0 & 1 & -7 & -22 \\ 0 & 0 & 17 & 51 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 0 & 1 & -7 & -22 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 3 & 0 & -5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

Thus

$$\mathbf{b}=\begin{bmatrix}1\\-9\\-3\end{bmatrix}=2\begin{bmatrix}-1\\1\\2\end{bmatrix}+(-1)\begin{bmatrix}3\\2\\1\end{bmatrix}+3\begin{bmatrix}2\\-3\\-2\end{bmatrix}.$$

4. Given that the set $S=\{(2,3,0),(3,2,1),(1,0,1)\}$ is a basis for $\mathbb{R}^3$, find
a) the coordinate vector of $\mathbf{v}=(2,2,2)$ with respect to S,
b) a vector $\mathbf{v}'$ in $\mathbb{R}^3$ whose coordinate vector with respect to S is $(2,2,2)$.

15 and 5 pts

Solution

a) By definition, the coordinate vector $(\mathbf{v})_S=(c_1,c_2,c_3)$ that we seek has the property that

$$\mathbf{v}=(2,2,2)=c_1(2,3,0)+c_2(3,2,1)+c_3(1,0,1),$$

whence

$$\begin{bmatrix}2&3&1\\3&2&0\\0&1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix},$$

the augmented matrix for which being

$$\left[\begin{array}{ccc|c}2&3&1&2\\3&2&0&2\\0&1&1&2\end{array}\right]\sim\left[\begin{array}{ccc|c}2&3&1&2\\1&-1&-1&0\\0&1&1&2\end{array}\right]\sim\left[\begin{array}{ccc|c}0&5&3&2\\1&-1&-1&0\\0&1&1&2\end{array}\right]\sim\left[\begin{array}{ccc|c}0&0&-2&-8\\1&-1&-1&0\\0&1&1&2\end{array}\right]\sim\left[\begin{array}{ccc|c}0&0&1&4\\1&-1&-1&0\\0&1&1&2\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}0&0&1&4\\1&-1&0&4\\0&1&0&-2\end{array}\right]\sim\left[\begin{array}{ccc|c}0&0&1&4\\1&0&0&2\\0&1&0&-2\end{array}\right],$$

so that $(\mathbf{v})_S=(c_1,c_2,c_3)=(2,-2,4)$.

b) If $(\mathbf{v}')_S=(2,2,2)$ then

$$\begin{aligned}\mathbf{v}'&=2(2,3,0)+2(3,2,1)+2(1,0,1)=(2\cdot2+2\cdot3+2\cdot1,2\cdot3+2\cdot2+2\cdot0,2\cdot0+2\cdot1+2\cdot1)\\&=(12,10,4).\end{aligned}$$

5. Find the coordinate vector $(\mathbf{p})_S\in\mathbb{R}^3$ of $\mathbf{p}=-3+5x+9x^2\in P_2$ with respect to the basis $S=\{2-3x+x^2,3+9x+7x^2,-8-x+x^2\}$ for P_2.

15 pts

Solution

By definition, the coordinate vector $(\mathbf{p})_S = (c_1, c_2, c_3)$ that we seek has the property that

$$\mathbf{p} = -3 + 5x + 9x^2 = c_1\left(2 - 3x + x^2\right) + c_2\left(3 + 9x + 7x^2\right) + c_3\left(-8 - x + x^2\right).$$

Since $S' = \{1, x, x^2\}$ is a basis for P_2, the vector equation above is the same as the scalar equations

$$2c_1 + 3c_2 - 8c_3 = -3, -3c_1 + 9c_2 - 1c_3 = 5, 1c_1 + 7c_2 + 1c_3 = 9,$$

or the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 3 & -8 & -3 \\ -3 & 9 & -1 & 5 \\ 1 & 7 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & -11 & -10 & -21 \\ 0 & 30 & 2 & 32 \\ 1 & 7 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & -11 & -10 & -21 \\ 0 & 8 & -18 & -10 \\ 1 & 7 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & -11 & -10 & -21 \\ 0 & 4 & -9 & -5 \\ 1 & 7 & 1 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & -3 & -28 & -31 \\ 0 & 4 & -9 & -5 \\ 1 & 7 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & -3 & -28 & -31 \\ 0 & 1 & -37 & -36 \\ 1 & 7 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 0 & -139 & -139 \\ 0 & 1 & -37 & -36 \\ 1 & 7 & 1 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 1 & -37 & -36 \\ 1 & 7 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 7 & 0 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}\right],$$

so that $(\mathbf{p})_S = (c_1, c_2, c_3) = (1,1,1)$.

6. For each of parts a), b), and c) below, either determine a nontrivial linear relationship among the set of 3 vectors in $\mathbb{R}^3$, or conclude that the set is a basis of $\mathbb{R}^3$ by showing the set is linearly independent:
a) $\mathbf{v}_1 = (1,4,3)$, $\mathbf{v}_2 = (5,7,8)$, and $\mathbf{v}_3 = (6,-2,4)$, b) $\mathbf{v}_1 = (-2,7,4)$, $\mathbf{v}_2 = (6,-8,3)$, and $\mathbf{v}_3 = (6,5,18)$, and c) $\mathbf{v}_1 = (6,-5,7)$, $\mathbf{v}_2 = (1,4,3)$, and $\mathbf{v}_3 = (-5,9,1)$.

18 pts

Solution

a) Linear relationships among vectors can be detected by row reducing matrices whose columns are those vectors:

$$\left[\mathbf{v}_1\middle|\mathbf{v}_2\middle|\mathbf{v}_3\right]=\begin{bmatrix}1&5&6\\4&7&-2\\3&8&4\end{bmatrix}\sim\begin{bmatrix}1&5&6\\0&-13&-26\\0&-7&-14\end{bmatrix}\sim\begin{bmatrix}1&5&6\\0&1&2\\0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&-4\\0&1&2\\0&0&0\end{bmatrix}=:\left[\mathbf{w}_1\middle|\mathbf{w}_2\middle|\mathbf{w}_3\right].$$

So since it is clear that $\mathbf{w}_3=-4\mathbf{w}_1+2\mathbf{w}_2$, it follows that $\mathbf{v}_3=-4\mathbf{v}_1+2\mathbf{v}_2$.

b)

$$\left[\mathbf{v}_1\middle|\mathbf{v}_2\middle|\mathbf{v}_3\right]=\begin{bmatrix}-2&6&6\\7&-8&5\\4&3&18\end{bmatrix}\sim\begin{bmatrix}1&-3&-3\\7&-8&5\\4&3&18\end{bmatrix}\sim\begin{bmatrix}1&-3&-3\\0&13&26\\0&15&30\end{bmatrix}\sim\begin{bmatrix}1&-3&-3\\0&1&2\\0&0&0\end{bmatrix}$$
$$\sim\begin{bmatrix}1&0&3\\0&1&2\\0&0&0\end{bmatrix}=:\left[\mathbf{w}_1\middle|\mathbf{w}_2\middle|\mathbf{w}_3\right].$$

So since it is clear that $\mathbf{w}_3=3\mathbf{w}_1+2\mathbf{w}_2$, it follows that $\mathbf{v}_3=3\mathbf{v}_1+2\mathbf{v}_2$.

c)

$$\left[\mathbf{v}_1\middle|\mathbf{v}_2\middle|\mathbf{v}_3\right]=\begin{bmatrix}6&1&-5\\-5&4&9\\7&3&1\end{bmatrix}\sim\begin{bmatrix}6&1&-5\\1&5&4\\1&2&6\end{bmatrix}\sim\begin{bmatrix}0&-29&-29\\1&5&4\\0&-3&2\end{bmatrix}\sim\begin{bmatrix}0&1&1\\1&5&4\\0&0&5\end{bmatrix}\sim\begin{bmatrix}0&1&1\\1&5&4\\0&0&1\end{bmatrix}$$
$$\sim\begin{bmatrix}0&1&0\\1&5&0\\0&0&1\end{bmatrix}\sim\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=:\left[\mathbf{w}_1\middle|\mathbf{w}_2\middle|\mathbf{w}_3\right].$$

So since it is clear then that $\mathbf{w}_1, \mathbf{w}_2$, and $\mathbf{w}_3$ are independent and form a basis for $\mathbb{R}^3$, it follows that $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are independent and form a basis for $\mathbb{R}^3$.

7. Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be linear, and let

$$T\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix},\text{ and } T\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}0\\4\end{pmatrix}.$$

Find $T\begin{pmatrix}1\\0\end{pmatrix}$, and, if possible, find $\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2$ such that $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}8\\8\end{pmatrix}$. Also, does there exist $\mathbf{u},\mathbf{v}\in\mathbb{R}^2$, $\mathbf{u}\neq\mathbf{v}$, such that $T\mathbf{u}=T\mathbf{v}$? Finally, can you find a $\mathbf{u}\in\mathbb{R}^2$, $\mathbf{u}\neq\mathbf{0}$, such that $T\mathbf{u}=\lambda\mathbf{u}\neq 1\mathbf{u}$? If so, give such a scalar $\lambda\,(\neq 1)$.

20 pts

Solution

Note that by linearity

$$\begin{pmatrix}8\\0\end{pmatrix} = 4\begin{pmatrix}2\\3\end{pmatrix} - 3\begin{pmatrix}0\\4\end{pmatrix} = 4T\begin{pmatrix}2\\3\end{pmatrix} - 3T\begin{pmatrix}3\\4\end{pmatrix} = T\left(4\begin{pmatrix}2\\3\end{pmatrix} - 3\begin{pmatrix}3\\4\end{pmatrix}\right)$$

$$= T\left(\begin{pmatrix}-1\\0\end{pmatrix}\right) = T\left(-1\begin{pmatrix}1\\0\end{pmatrix}\right) = -1T\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow$$

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-8\\0\end{pmatrix} = -8\begin{pmatrix}1\\0\end{pmatrix},$$

which answers the first question (and the last). Note also that

$$\begin{pmatrix}8\\8\end{pmatrix} = 4\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}0\\4\end{pmatrix} = 4T\begin{pmatrix}2\\3\end{pmatrix} - T\begin{pmatrix}3\\4\end{pmatrix} = T\left(4\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}3\\4\end{pmatrix}\right)$$

$$= T\begin{pmatrix}5\\8\end{pmatrix},$$

which answers the second question. Note the next question regards whether T is one-to-one, which holds for a linear mapping precisely when $T\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}$, i.e. when T, and so $[T]$, is invertible (since T maps a space into itself). To answer that question, construct $[T]$ by finding $T\begin{pmatrix}0\\1\end{pmatrix}$: Note that

$$\begin{pmatrix}6\\1\end{pmatrix} = 3\begin{pmatrix}2\\3\end{pmatrix} - 2\begin{pmatrix}0\\4\end{pmatrix} = 3T\begin{pmatrix}2\\3\end{pmatrix} - 2T\begin{pmatrix}3\\4\end{pmatrix} = T\left(3\begin{pmatrix}2\\3\end{pmatrix} - 2\begin{pmatrix}3\\4\end{pmatrix}\right)$$

$$= T\begin{pmatrix}0\\1\end{pmatrix},$$

so that now, by theorem,

$$[T] = \left[T\mathbf{e}_1 \middle| T\mathbf{e}_2\right] = \left[T\begin{pmatrix}1\\0\end{pmatrix}\middle| T\begin{pmatrix}0\\1\end{pmatrix}\right] = \left[\begin{pmatrix}-8\\0\end{pmatrix}\middle|\begin{pmatrix}6\\1\end{pmatrix}\right] = \begin{bmatrix}-8 & 6\\0 & 1\end{bmatrix},$$

from which it is obvious that $[T]$ (and so T) is invertible, giving no distinct vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ such that $T\mathbf{u} = T\mathbf{v}$. From this form it is also obvious that another eigenvalue (besides 1) is -8, which is the desired λ, and which gives $T\mathbf{u} = -8\mathbf{u}$ for all $\mathbf{u} \in Span\{(1,0)\}$.

8. Let $V=\{\underline{x}\,|\,x>0\}$ (the set of positive numbers written underscored), $S=\mathbb{R}$, and let vector addition $"+"$ and scalar multiplication $"\bullet"$ be defined through

$$\text{for } \underline{x},\underline{y}\in V, \quad \underline{x}"+"\underline{y}:=\underline{xy}$$
$$\text{for } k\in S, \underline{x}\in V, \quad k"\bullet"\underline{x}:=\underline{x^k}$$

(Here $\underline{xy}$ denotes an underscored number, the number being xy—meaning x times y—and $\underline{x^k}$ denotes an underscored number, the number being x^k—meaning x raised to the k^{th} power.) Determine whether the two sets together with the two binary operations constitute a vector space, by demonstrating each satisfied axiom, as well as demonstrating each unsatisfied axiom. (You must demonstrate satisfaction or failure of each of the ten axioms.)

20 pts

Solution

1. If $\underline{x},\underline{y}\in V$, then $x,y>0$, so that $xy>0$, yielding $\underline{x}"+"\underline{y}:=\underline{xy}\in V$ as required.
2. For $\underline{x},\underline{y}\in V$, $\underline{x}"+"\underline{y}:=\underline{xy}=\underline{yx}=:\underline{y}"+"\underline{x}$, as required.
3. For $\underline{x},\underline{y},\underline{z}\in V$, $\underline{x}"+"\left(\underline{y}"+"\underline{z}\right):=\underline{x}"+"\underline{yz}:=\underline{x(yz)}=\underline{(xy)z}=:\underline{xy}"+"\underline{z}$
   $=:\left(\underline{x}"+"\underline{y}\right)"+"\underline{z}$, as required.
4. Note that $\underline{1}\in V$ has the property that, for any $\underline{x}\in V$,
   $\underline{1}"+"\underline{x}=\underline{1\cdot x}=\underline{x}=\underline{x\cdot 1}=\underline{x}"+"\underline{1}$, so that $\underline{1}\in V$ is the "zero vector", or "additive identity", as required.
5. Note that since $x>0\Rightarrow x^{-1}>0$, for each $\underline{x}\in V$ there exists $\underline{x^{-1}}\in V$, having the property that $\underline{x^{-1}}"+"\underline{x}=\underline{x^{-1}\cdot x}=\underline{1}=\underline{x\cdot x^{-1}}=\underline{x}"+"\underline{x^{-1}}$, so that we can take the "negative" (or additive inverse) to be $-\underline{x}:=\underline{x^{-1}}$, as required.
6. Since $k\in\mathbb{R}$, $x>0\Rightarrow x^k>0$, we have that, for $k\in S,\underline{x}\in V$, $k"\bullet"\underline{x}:=\underline{x^k}\in V$, as required.
7. We have $k"\bullet"\left(\underline{x}"+"\underline{y}\right):=k"\bullet"\underline{xy}:=\underline{(xy)^k}=\underline{x^k y^k}=:\underline{x^k}"+"\underline{y^k}$
   $=:k"\bullet"\underline{x}"+"k"\bullet"\underline{y}"$, as required.
8. We have $(k+m)"\bullet"\underline{x}:=\underline{x^{k+m}}=\underline{x^k x^m}=:\underline{x^k}"+"\underline{x^m}=:k"\bullet"\underline{x}+m"\bullet"\underline{x}$, as required.
9. We have $k"\bullet"(m"\bullet"\underline{x}):=k"\bullet"(\underline{x^m}):=\underline{\left(x^m\right)^k}=\underline{x^{(km)}}=:(km)"\bullet"\underline{x}$, as required.
10. We have $1"\bullet"\underline{x}:=\underline{x^1}=\underline{x}$, as required.

9. Using the axioms recalled in problem 8, prove that, in any vector space, $0\mathbf{u}=\mathbf{0}$ for any $\mathbf{u}$. Here be sure to use that a) the 0 here is the usual 0, namely the additive identity in the scalars (generally called "the field"), but that b) $\mathbf{0}$ is the additive identity in the

vectors. You will be given points for each relevant use of an axiom (and docked for each irrelevant use). (And you do not have to remember the axioms in the same order the book lists them, but you do have to be clear which one you are using.)

6 pts

Solution

Since 0 is the additive identity in the scalars, we have that $0 = 0 + 0$, and then that for any $\mathbf{u}$, $0\mathbf{u} = (0+0)\mathbf{u}$. But then by axiom 8 we have

$$0\mathbf{u} = (0+0)\mathbf{u} = 0\mathbf{u} + 0\mathbf{u}\,. \tag{0.1}$$

Now by axiom 5 there exists $-0\mathbf{u}$ such that $0\mathbf{u} + (-0\mathbf{u}) = \mathbf{0}$. Adding such to each side of (0.1) gives

$$0\mathbf{u} + (-0\mathbf{u}) = (0\mathbf{u} + 0\mathbf{u}) + (-0\mathbf{u}),$$

or

$$\mathbf{0} = (0\mathbf{u} + 0\mathbf{u}) + (-0\mathbf{u})\,. \tag{0.2}$$

Using axiom 3, the latter becomes

$$\mathbf{0} = 0\mathbf{u} + (0\mathbf{u} + (-0\mathbf{u}))$$

or, using axiom 5 again,

$$\mathbf{0} = 0\mathbf{u} + \mathbf{0}\,. \tag{0.3}$$

Using axiom 4, the latter is $\mathbf{0} = 0\mathbf{u} + \mathbf{0} = 0\mathbf{u}$, i.e. $0\mathbf{u} = \mathbf{0}$.

10. If

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the standard matrices for counter-clock-wise rotations of θ radians in $\mathbb{R}^3$ about the indicated coordinate axes, then find the standard matrix $[T]$ for the linear operators giving a) a counter-clock-wise rotation of $\pi/2$ radians about the z-axis, followed by a counter-clock-wise rotation of $\pi/2$ radians about the y-axis, followed by a counter-clock-wise rotation of $\pi/2$ radians about the x-axis, and b) a reflection about the $y\ z$ plane, followed by a dilation of $1/2$. (Recall that one need not actually use any of the matrices above, but rather all one has to do is find out what the linear mappings do to unit vectors along the (standard) coordinate axes.)

9 and 5 pts

Solution

a) By theorem

$$[T]=\left[T\mathbf{e}_1\middle|T\mathbf{e}_2\middle|T\mathbf{e}_3\right]=\left[T\begin{pmatrix}1\\0\\0\end{pmatrix}\middle|T\begin{pmatrix}0\\1\\0\end{pmatrix}\middle|T\begin{pmatrix}0\\0\\1\end{pmatrix}\right].$$

Now under these three mappings, $\mathbf{e}_1$ first gets sent to $\mathbf{e}_2$ by the first rotation, and then $\mathbf{e}_2$ does not change under the second, and finally $\mathbf{e}_2$ gets sent to $\mathbf{e}_3$ by the third. Thus $T\mathbf{e}_1=\mathbf{e}_3$. On the other hand, $\mathbf{e}_2$ first gets sent to $-\mathbf{e}_1$ by the first rotation, and then $-\mathbf{e}_1$ gets mapped to $\mathbf{e}_3$ under the second, and finally $\mathbf{e}_3$ gets sent to $-\mathbf{e}_2$ by the third. Thus $T\mathbf{e}_2=-\mathbf{e}_2$. Lastly $\mathbf{e}_3$ is unchanged by the first rotation, and then $\mathbf{e}_3$ gets mapped to $\mathbf{e}_1$ under the second, and finally $\mathbf{e}_1$ is unchanged by the third. Thus $T\mathbf{e}_3=\mathbf{e}_1$, and we now have that

$$[T]=\left[T\mathbf{e}_1\middle|T\mathbf{e}_2\middle|T\mathbf{e}_3\right]=\left[\mathbf{e}_3\middle|-\mathbf{e}_2\middle|\mathbf{e}_1\right]=\left[\begin{pmatrix}0\\0\\1\end{pmatrix}\middle|-\begin{pmatrix}0\\1\\0\end{pmatrix}\middle|\begin{pmatrix}1\\0\\0\end{pmatrix}\right]=\begin{bmatrix}0&0&1\\0&-1&0\\1&0&0\end{bmatrix}$$

b) Reflecting about the $y\ z$ plane sends $\mathbf{e}_1$ to $-\mathbf{e}_1$, and leaves $\mathbf{e}_2$ and $\mathbf{e}_3$ unchanged. Thus, since the subsequent dilation is rather trivial, we have

$$[T]=\left[T\mathbf{e}_1\middle|T\mathbf{e}_2\middle|T\mathbf{e}_3\right]=\frac{1}{2}\left[-\mathbf{e}_1\middle|\mathbf{e}_2\middle|\mathbf{e}_3\right]=\frac{1}{2}\left[-\begin{pmatrix}1\\0\\0\end{pmatrix}\middle|\begin{pmatrix}0\\1\\0\end{pmatrix}\middle|\begin{pmatrix}0\\0\\1\end{pmatrix}\right]=\frac{1}{2}\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$