Problem 1

1. Find a system of equations that corresponds to each augmented matrix:

a. $\begin{bmatrix} 1 & 3 & 0 \\ 6 & 7 & 8 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 3 & 7 & 6 \\ 5 & 9 & 2 & 8 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

c. $\begin{bmatrix} -6 & 2 & 8 & 3 & 0 \\ 0 & -3 & 27 & 1 & 16 \end{bmatrix}$

Problem 2

2. Solve the following system of equations by Gauss-Jordan elimination.

a. $\begin{array}{rrrcr} 3x_1 & -2x_2 & 5x_3 & = & 17 \\ -x_1 & & +2x_3 & = & 5 \\ & 4x_2 & -x_3 & = & -3 \end{array}$

Problem 3

3. If matrix $A = \begin{bmatrix} 3 & 7 & 3 \\ 2 & 1 & 4 \\ -8 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ 6 & 9 \\ 3 & 4 \end{bmatrix}$, what is $\frac{1}{2}(AB)^T$?

Problem 4

4. Simplify the following:

$P = A^2 + 6A + 9I$

where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Problem 5

5. Find the inverses of the following matrices:

a. $\begin{bmatrix} 3 & 6 \\ 7 & 9 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 4 \\ -6 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Problem 6

6. Solve these three systems of equations *simultaneously*:

a. $\begin{array}{rrrcr} x_1 & -x_2 & +3x_3 & = & 7 \\ 2x_1 & & -x_3 & = & 4 \\ -x_1 & 5x_2 & 2x_3 & = & 9 \end{array}$

b. $\begin{array}{rrrcr} x_1 & -x_2 & +3x_3 & = & -1 \\ 2x_1 & & -x_3 & = & 5 \\ -x_1 & 5x_2 & 2x_3 & = & 3 \end{array}$

c. $\begin{array}{rrrcr} x_1 & -x_2 & +3x_3 & = & 11 \\ 2x_1 & & -x_3 & = & 7 \\ -x_1 & 5x_2 & 2x_3 & = & 15 \end{array}$

Problem 7

7. If $AB=C$, fill in the blank by placing something on either side of A:

a. C^{-1}=____A^{-1}____

b. C^T=______A^T______

Problem 8

8. Consider the following matrix A:

$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

a. Find the inverse of A (A^{-1}) by adjoining an identity matrix and performing row operations. conceptually

b. Express A^{-1} as a product of elementary matrices E_n such that $A^{-1} = I \cdot E_1E_2E_3$. Eeek!

c. If $I = AA^{-1}$, is A^{-1} a left-hand or right-hand inverse? Why? *Hint:* this is a trick question.

Problem 9

9. Let A and B equal a matrix of an unspecified size. Consider matrices A and B and nonzero products C and D:

$$A \times B = C$$

$$B \times A = D$$

   a. What conditions must be met for both C and D to exist? Why or why not?
   b. If B is invertible and $AB^{-1} = B^{-1}A = I$, does A^{-1} exist? Why or why not?
   c. If C exists and is invertible, are A and/or B invertible? Which one(s) and why or why not?

Problem 10

10. Express $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$ as a product of elementary matrices. *Hint*: There are five factors.

Solutions

Problem 1

1. Solution:

a. $$\begin{aligned} 1x_1 + 3x_2 &= 0 \\ 6x_1 + 7x_2 &= 8 \end{aligned}$$

b. $$\begin{aligned} 2x_1 + 3x_2 + 7x_3 &= 6 \\ 5x_1 + 9x_2 + 2x_3 &= 8 \\ 1x_3 &= 7 \end{aligned}$$

c. $$\begin{aligned} -6x_1 + 2x_2 + 8x_3 3x_4 &= 0 \\ -3x_2 + 27x_3 + 1x_4 &= 16 \end{aligned}$$

Problem 2

2. Solution:

$$\begin{bmatrix} 3 & -2 & 5 & 17 \\ -1 & 0 & 2 & 5 \\ 0 & 4 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -5 \\ 3 & -2 & 5 & 17 \\ 0 & 4 & -1 & -3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -2 & -5 \\ 0 & -2 & 11 & 32 \\ 0 & 0 & 21 & 61 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 11/2 & 16 \\ 0 & 0 & 1 & 61/21 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 17/21 \\ 0 & 1 & 0 & -1/42 \\ 0 & 0 & 1 & 61/21 \end{bmatrix}$$

Therefore, $x_1 = 17/21, \quad x_2 = -1/42, \quad x_3 = 61/21.$

Problem 3

3. Solution:

$$AB = \begin{bmatrix} (15+42+9) & (6+63+12) \\ (10+6+24) & (4+9+16) \\ (-40+6) & (-16+8) \end{bmatrix} = \begin{bmatrix} 66 & 81 \\ 40 & 29 \\ -34 & -8 \end{bmatrix}$$

$$\Rightarrow (AB)^T = \begin{bmatrix} 66 & 40 & -34 \\ 81 & 29 & -8 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2}(AB)^T = \begin{bmatrix} 33 & 20 & -17 \\ 81/2 & 29/2 & -4 \end{bmatrix}$$

Problem 4

4. Solution:

$$A^2 + 6A + 9I = (A + 3I)^2$$

$$(A+3I)^2 = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right)^2$$

$$\begin{bmatrix} 4 & 2 \\ 3 & 7 \end{bmatrix}^2 = \begin{bmatrix} 22 & 22 \\ 33 & 55 \end{bmatrix}$$

Problem 5

5. Solution:

a.

$$\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 7 & 9 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1/3 & 0 \\ 7 & 9 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & 1/3 & 0 \\ 0 & -5 & -7/3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1/3 & 0 \\ 0 & 1 & 7/15 & -1/5 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -3/5 & 2/5 \\ 0 & 1 & 7/15 & -1/5 \end{array}\right]$$

$$\therefore \text{The inverse is: } \begin{bmatrix} -3/5 & 2/5 \\ 7/15 & -1/5 \end{bmatrix}$$

b.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 24 & 6 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -4 \\ 0 & 1 & 0 & 6 & 1 & -24 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\therefore \text{The inverse is: } \begin{bmatrix} 1 & 0 & -4 \\ 6 & 1 & -24 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 6

6. Solution:

$$\left[\begin{array}{ccc|c|c|c} 1 & -1 & 3 & 7 & -1 & 11 \\ 2 & 0 & -1 & 4 & 5 & 7 \\ -1 & 5 & 2 & 9 & 3 & 15 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c|c|c} 1 & -1 & 3 & 7 & -1 & 11 \\ 0 & 2 & -7 & -10 & 7 & -15 \\ 0 & 4 & 5 & 16 & 2 & 26 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c|c|c} 1 & -1 & 3 & 7 & -1 & 11 \\ 0 & 1 & -7/2 & -5 & 7/2 & -15/2 \\ 0 & 0 & 19 & 36 & -12 & 56 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c|c|c} 1 & 0 & -1/2 & 2 & 5/2 & 7/2 \\ 0 & 1 & 0 & 31/19 & 49/38 & 107/38 \\ 0 & 0 & 1 & 36/19 & -12/19 & 56/19 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 59/19 & 71/38 & 245/38 \\ 0 & 1 & 0 & 31/19 & 49/38 & 107/38 \\ 0 & 0 & 1 & 36/19 & -12/19 & 56/19 \end{array}\right]$$

Therefore, the variables x_1, x_2, x_3 are "expressed for each equation in the matrix:"

$$\begin{bmatrix} 59/19 & 71/38 & 245/38 \\ 31/19 & 49/38 & 107/38 \\ 36/19 & -12/19 & 56/19 \end{bmatrix}$$

Problem 7

7. Solution:
$C^{-1}=B^{-1}A^{-1}$. The same property holds for transposes; the inverse or transpose of a product is the product of the inverses/transposes in reverse order.

Problem 8

8. Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 4 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

a.
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 4 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1/4 & -3/4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

b.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c. A^{-1} is a both a right-hand and left-hand inverse. If B (a left-hand inverse) and C (a right-hand inverse) are both inverses of A, then $B=C$. Hence, inverses commute.

Problem 9

9. Solution:

a. Because matrix multiplication depends on the *size* of matrices, factors do not obey the commutative law of multiplication. In other words, AB may exist, but BA may not if the sizes are not equal. Therefore, matrices A and B must be square and of the same size.

b. Yes, A^{-1} exists. Because the two factors commute, they are the same size. Because their product is the identity matrix I, B^{-1} is the inverse of A. Therefore, $B^{-1} = A^{-1}$.

c. Yes, both A and B are invertible. If the product of two (another) matrix factors are invertible, the factors themselves are also invertible. This is the converse of the theorem that states that invertible factors produce invertible products.

Problem 10

10. Solution:

$$\begin{bmatrix} 1 & 3 & 5 \\ 3 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\cdot\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Show how you got these — by row reduction.

1) Solve this system of linear equations using the row reduction method:

$$4x_1 + 7x_2 - 6x_3 = 4$$
$$8x_1 - x_2 = 5$$
$$2x_1 + 5x_2 - 7x_3 = 10$$

Solution:

[4 7 -6 4]
[8 -1 0 5] ~
[2 5 -7 10]

[1 (7/4) (-3/2) 1]
[8 -1 0 5] ~
[2 5 -7 10]

[1 (7/4) (-3/2) 1]
[0 -15 12 -3]
[0 (3/2) -4 8]

[1 (7/4) (-3/2) 1]
[0 1 (-4/5) 1]
[0 (3/2) -4 8]

[1 0 (1/5) (27/20)]
~ [0 1 (-4/5) (-1/5)]
[0 0 (-14/5) (77/10)]

[1 0 (1/5) (27/20)]
~ [0 1 (-4/5) (1/5)]
[0 0 1 (-11/4)]

[1 0 0 (3/8)]
[0 1 0 -2]
[0 0 1 (-11/4)]

$x_1 = 3/8$
$x_2 = -2$
$x_3 = -11/4$

2) Explain the difference between row-echelon form and reduced row-echelon form.

Solution:

Row-echelon form has the following three characteristics:
1 – The first nonzero in a row is a 1.
2 – Any rows of zeros are grouped at the bottom of the matrix.
3 – With any two successive rows which are not zeros, the leading 1 in lower row is to the right from the leading 1 of the upper row.

Reduced row-echelon form contains all three of the preceding characteristics, but adding a fourth:
4 – Every column with a leading 1 contains zeros everywhere else.

3) Find the inverse of the matrix A:

[1 0 3]
[0 -2 -3]
[1 -1 -1]

Solution:

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -3 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -2 & -3 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & -1/2 & 0 \\ 0 & -1 & -4 & -1 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & -1/2 & 0 \\ 0 & 0 & -4 & -1 & -1/2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 2/5 & 1/5 & -2/5 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4/15 & -19/30 & -4/15 \\ 0 & 0 & 1 & 2/5 & 1/5 & -2/5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/5 & -3/5 & 6/5 \\ 0 & 1 & 0 & -4/15 & -19/30 & -4/15 \\ 0 & 0 & 1 & 2/5 & 1/5 & -2/5 \end{bmatrix}$$

Therefore, the inverse of matrix A is:

$$\begin{bmatrix} -1/5 & -3/5 & 6/5 \\ -4/15 & -19/30 & -4/15 \\ 2/5 & 1/5 & -2/5 \end{bmatrix}$$

4) Calculate the determinant of the following matrix

$$\begin{bmatrix} 3 & -1 & 7 \\ 1 & 2 & 0 \\ -1 & 6 & 5 \end{bmatrix}:$$

Solution:

$$\begin{vmatrix} 3 & -1 & 7 \\ 1 & 2 & 0 \\ -1 & 6 & 5 \end{vmatrix}$$

by what method?

$$\begin{aligned} &= 3(2)(5) + (-1)(0)(-1) + 7(1)(6) \\ &\quad - 3(0)(6) - (-1)(1)(5) - 7(2)(-1) \\ &= 30 + 0 + 42 - 0 + 5 + 14 \\ &= 91 \end{aligned}$$

5) Show how to multiply a 3 x 3 matrix (A) with a 3 x 1 column vector (B). This means you make up your own matrix A and column vector B with the given dimensions.

Solution:

$$A = \begin{pmatrix} 1 & 6 & 7 \\ 2 & 5 & 8 \\ 3 & 4 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

AB = (1)(2) + (6)(0) + (7)(1) = 9
(2)(2) + (5)(0) + (8)(1) = 12
(3)(2) + (4)(0) + (9)(1) = 15

$$AB = \begin{pmatrix} 9 \\ 12 \\ 15 \end{pmatrix}$$

6) Solve the following system of equations by inverting the coefficient matrix and using the appropriate theorem:

$$\begin{array}{llll} x_1 & +2x_2 & -x3 & =1 \\ 2x_1 & -3x_2 & +x_3 & =5 \\ 2x_1 & -x_2 & +3x_3 & =-4 \end{array}$$

Solution:

$$\begin{array}{llll} x_1 & +2x_2 & -x3 & =1 \\ 2x_1 & -3x_2 & +x_3 & =5 \\ 2x_1 & -x_2 & +3x_3 & =-4 \end{array} \Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 2 & -3 & 1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

Let A be the coefficient matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & -3 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

Then find A^{-1}:

$$[A|I] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -7 & 3 & -2 & 1 & 0 \\ 0 & -5 & 5 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{-3}{7} & \frac{2}{7} & \frac{-1}{7} & 0 \\ 0 & -5 & -5 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{-1}{7} & \frac{3}{7} & \frac{2}{7} & 0 \\ 0 & 1 & \frac{-3}{7} & \frac{2}{7} & \frac{-1}{7} & 0 \\ 0 & 0 & \frac{20}{7} & \frac{-4}{7} & \frac{-5}{7} & 1 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{-1}{7} & \frac{3}{7} & \frac{2}{7} & 0 \\ 0 & 1 & \frac{-3}{7} & \frac{2}{7} & \frac{-1}{7} & 0 \\ 0 & 0 & 1 & \frac{-1}{5} & \frac{-1}{4} & \frac{7}{20} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & \frac{1}{4} & \frac{1}{20} \\ 0 & 1 & 0 & \frac{1}{5} & \frac{-1}{4} & \frac{3}{20} \\ 0 & 0 & 1 & \frac{-1}{5} & \frac{-1}{4} & \frac{7}{20} \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{4} & \frac{1}{20} \\ \frac{1}{5} & \frac{-1}{4} & \frac{3}{20} \\ \frac{-1}{5} & \frac{-1}{4} & \frac{7}{20} \end{bmatrix} . Ax = b \Rightarrow x = A^{-1} b \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_{¿} \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{4} & \frac{1}{20} \\ \frac{1}{5} & \frac{-1}{4} & \frac{3}{20} \\ \frac{-1}{5} & \frac{-1}{4} & \frac{7}{20} \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{29}{20} \\ \frac{-33}{20} \\ \frac{-57}{20} \end{bmatrix} \Rightarrow \begin{matrix} x_1 = \frac{29}{20} \\ x_2 = \frac{-33}{20} \\ x_3 = \frac{-57}{20} \end{matrix}$$

7) What do the following two symbols stand for and what is the difference between the operations they denote to perform on the matrix A?

A^T & tr (A)

Solution:

A^T stands for the transpose of the matrix A
tr (A) stands for the trace of the matrix A

The transpose of A is found by interchanging the rows with the columns of a matrix.
The trace of A is found by adding up the main diagonals of the matrix. It is undefined if the matrix is not a square matrix.

8) Find $(AB)^T$ if A = $\begin{pmatrix} 4 & 2 \\ 7 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 8 & 1 \\ 0 & 9 \end{pmatrix}$

Solution:

Multiply the two matrices together and then take their transpose:

$$\begin{pmatrix} 4 & 2 \\ 7 & 3 \end{pmatrix} \text{ x } \begin{pmatrix} 8 & 1 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 32 & 24 \\ 56 & 28 \end{pmatrix} \text{^T} = \begin{pmatrix} 32 & 56 \\ 24 & 28 \end{pmatrix}$$

9) Put the following augmented matrix in row-echelon form:

$$\begin{pmatrix} 4 & 20 & 32 \\ -2 & 5 & 14 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 4 & 20 & 32 \\ -2 & 5 & 14 \end{pmatrix}$$ Row one times (1/4)

$$\begin{pmatrix} 1 & 5 & 8 \\ -2 & 5 & 14 \end{pmatrix}$$ Row two + 2 times row one

$$\begin{pmatrix} 1 & 6 & 8 \\ 0 & 15 & 30 \end{pmatrix}$$ Row two time (1/15)

$$\begin{pmatrix} 1 & 6 & 8 \\ 6 & 1 & 2 \end{pmatrix}$$ This is now in row-echelon form.

10) Is this statement true or false for the following matrices? Explain

AB = BA

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 4 & 3 & 1 \\ 8 & 7 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 7 & 6 \\ 1 & 9 & 3 \\ 10 & 0 & 1 \end{pmatrix}$$

Solution:

The statement is false, because the rule for multiplying matrices is not always commutative.

$$AB = \begin{pmatrix} 29 & 23 & 17 \\ 29 & 55 & 34 \\ 89 & 119 & 74 \end{pmatrix} \quad \text{but} \quad BA = \begin{pmatrix} 84 & 67 & 45 \\ 62 & 49 & 26 \\ 28 & 17 & 25 \end{pmatrix}$$

Grading Key

Problem 1 (20 points total)
a) 10 points
b) 10 points

Problem 2 (15 points total)
a) 3 points
b) 3 points
c) 3 points
d) 3 points
e) 3 points

Problem 3 (20 points total)
a) 5 points
b) 5 points
c) 5 points
d) 5 points

Problem 4 (20 points total)
a) 5 points
b) 5 points
c) 5 points
d) 5 points

Problem 5 (10 points total)

Problem 6 (10 points total)

Problem 7 (10 points total)

Problem 8 (15 points total)
- 6 points - proper setup of matrix to solve simultaneously

a) 3 points
b) 3 points
c) 3 points

Problem 9 (15 points total)
a) 3 points
b) 3 points
c) 3 points
d) 3 points
e) 3 points

Problem 10 (15 points total)
a) 10 points
b) 5 points

Problem 1

1.Given the system of equations

x + y + 2z = 8
-x - 2y + 3z = 1
3x - 7y + 4z = 10

a.) Solve using Gaussian Elimination.

b.) Solve using Guass-Jordan Elimination.

Solution 1
The augmented matrix for this system of linear equations is

a)

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$ Add first row to second row

$$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$ Add the first row multiplied by -3 to the third row

$$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$ Multiply the second row by -1

$$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$ Add the second row multiplied by 10 the the third row

$$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix}$$ Multiply third row by $-\frac{1}{52}$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ Add the second row multiplied by 10 the the third row

The resulting equations are
$x + y + 2z = 8$
$y - 5z = -9$
$z = 2$
Now solve by back substitution
$z = 2$ so substitute 2 in for z in equation $y - 5z = -9$ so $y - 10 = -9$ or $y = 1$
now put $z = 2$ and $y = 1$ in equation $x + y + 2z = 8$ so $x + 1 + 4 = 8$ or $x = 3$
So $x = 3$, $y = 1$ and $z = 2$

b) For the solution to this part, we will continue the row reduction of the augmented matrix where we left off with part a)

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ Add the third row multiplied by 5 to the the second row

$$\sim\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ Add the third row multiplied by -2 the the first row

$$\sim\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ Add the second row multiplied by -1 the the first row

$$\sim\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ Add the second row multiplied by 10 the the third row

From the resulting matrix, we see that $x = 3$, $y = 1$ and $z = 2$

Problem 2

2. Given the following Matrices $A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -2 & 4 \\ 2 & 0 & 3 \\ -3 & 1 & -1 \end{bmatrix}$,

a.) Find $((A)^T)^T$.

b.) Find $(A+B)^T$.

c.) Find $(A-B)^T$.

d.) Find $(kA)^T$.

e.) Find $(AB)^T$.

Solution 2

a.) Find $((A)^T)^T$.

Step 1................Compute $(A)^T = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 5 \\ -3 & -1 & -6 \end{bmatrix}$ by replacing column n with row n.

Step 2................ Compute $((A)^T)^T = \begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix}$ by replacing column n with row n again.

b.) Find $(A+B)^T$

Step 1..................Compute $(A+B) =$

$$\begin{bmatrix} 1+5 & 0+(-2) & (-3)+4 \\ 2+2 & (-4)+0 & (-1)+3 \\ (-3)+(-3) & 5+1 & (-6)+(-1) \end{bmatrix} = \begin{bmatrix} 6 & -2 & 1 \\ 4 & -4 & 2 \\ -6 & 6 & -7 \end{bmatrix}$$

Step 2..................Compute $(A+B)^T = \begin{bmatrix} 6 & 4 & -6 \\ -2 & -4 & 6 \\ 1 & 2 & -7 \end{bmatrix}$ by replacing column *n* with row *n*.

c.) Find $(A-B)^T$. =

Step 1....................Compute (A-B) =

$$\begin{bmatrix} 1-5 & 0-(-2) & (-3)-4 \\ 2-2 & (-4)-0 & (-1)-3 \\ (-3)-(-3) & 5-1 & (-6)-(-1) \end{bmatrix} = \begin{bmatrix} -4 & 2 & -7 \\ 0 & -4 & -4 \\ 0 & 4 & -5 \end{bmatrix}$$

Step 2..................Compute $(A-B)^T = \begin{bmatrix} -4 & 0 & 0 \\ 2 & -4 & 4 \\ -7 & -4 & -5 \end{bmatrix}$ by replacing column *n* with row *n*.

d.) Find $(kA)^T$. =

Step 1....................Compute Matrix (*k*A) by multiplying A by scalar *k*. $kA =$

$$\begin{bmatrix} k & 0 & -3k \\ 2k & -4k & -k \\ -3k & 5k & -6k \end{bmatrix}$$

Step 2.................... Replace column *n* with row *n* to compute $(kA)^T = \begin{bmatrix} k & 2k & -3k \\ 0 & -4k & 5k \\ -3k & -k & -6k \end{bmatrix}$

e.) Find $(AB)^T$ =

Step 1.............Compute AB = $\begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix} \times \begin{bmatrix} 5 & -2 & 4 \\ 2 & 0 & 3 \\ -3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & -5 & 7 \\ 5 & -5 & -3 \\ 13 & 0 & 9 \end{bmatrix}$

Step 2............. Compute $(AB)^T = \begin{bmatrix} 14 & 5 & 13 \\ -5 & -5 & 0 \\ 7 & -3 & 9 \end{bmatrix}$ by replacing column *n* with row *n*.

Problem 3

3. Using the given matricies, apply the Properties of Matrix Arithmetic.

$$A=\begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix} \qquad B=\begin{bmatrix} 5 & -2 & 4 \\ 2 & 0 & 3 \\ -3 & 1 & -1 \end{bmatrix} \qquad C=\begin{bmatrix} 1 & 2 & 2 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

a.) Find A (B + C).

b.) Find (B + C) A.

c.) Find A (B - C).

d.) Find (B - C) A.

Solution 3

a.) Find A (B + C).

According to Theorem 1.4.1 we know that A (B + C) = AB + AC. Therefore

$$A(B+C)=\begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix}\left(\begin{bmatrix} 5 & -2 & 4 \\ 2 & 0 & 3 \\ -3 & 1 & -1 \end{bmatrix}+\begin{bmatrix} 1 & 2 & 2 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right)$$

A (B + C) = AB + AC =

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix}\begin{bmatrix} 5 & -2 & 4 \\ 2 & 0 & 3 \\ -3 & 1 & -1 \end{bmatrix}+\begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A(B+C)=AB+AC=\begin{bmatrix} 14 & -5 & 7 \\ 5 & -5 & -3 \\ 13 & 0 & 9 \end{bmatrix}+\begin{bmatrix} 1 & -1 & -1 \\ 6 & -9 & -1 \\ -8 & 3 & -7 \end{bmatrix}$$

$$A(B+C) = AB + AC = \begin{bmatrix} 15 & -6 & 6 \\ 11 & -14 & -4 \\ 5 & 3 & 2 \end{bmatrix}$$

b) Find $(B+C)A$.

According to Theorem 1.4.1 we know that $(B+C)A = BA + CA$. Therefore

$$(B+C)A = \left(\begin{bmatrix} 5 & -2 & 4 \\ 2 & 0 & 3 \\ -3 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 2 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix}$$

$(B+C)A = BA + CA =$

$$\begin{bmatrix} 5 & -2 & 4 \\ 2 & 0 & 3 \\ -3 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 2 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix}$$

$$(B+C)A = BA + CA = \begin{bmatrix} -11 & 28 & -37 \\ -7 & 15 & -24 \\ 2 & -9 & 14 \end{bmatrix} + \begin{bmatrix} -1 & 2 & -17 \\ 2 & -7 & -6 \\ -1 & 1 & -7 \end{bmatrix}$$

$$(B+C)A = BA + CA = \begin{bmatrix} -12 & 30 & -54 \\ -5 & 8 & -30 \\ 1 & -8 & 7 \end{bmatrix}$$

c.) Find $A(B-C)$.

According to Theorem 1.4.1 we know that $A(B-C) = AB - AC$. Therefore

$$A(B+C) = \begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix} \left(\begin{bmatrix} 5 & -2 & 4 \\ 2 & 0 & 3 \\ -3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 2 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right)$$

$A(B-C) = AB - AC =$

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix}\begin{bmatrix} 5 & -2 & 4 \\ 2 & 0 & 3 \\ -3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A(B-C) = AB - AC = \begin{bmatrix} 14 & -5 & 7 \\ 5 & -5 & -3 \\ 13 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 1 & -1 & -1 \\ 6 & -9 & -1 \\ -8 & 3 & -7 \end{bmatrix}$$

$$A(B-C) = AB - AC = \begin{bmatrix} 13 & -4 & 8 \\ -1 & 4 & -2 \\ 21 & -3 & 16 \end{bmatrix}$$

d) Find $(B - C)A$.

According to Theorem 1.4.1 we know that $(B - C)A = BA - CA$. Therefore

$$(B-C)A = \left(\begin{bmatrix} 5 & -2 & 4 \\ 2 & 0 & 3 \\ -3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 2 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right)\cdot\begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix}$$

$(B - C)A = BA - CA =$

$$\begin{bmatrix} 5 & -2 & 4 \\ 2 & 0 & 3 \\ -3 & 1 & -1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 2 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 & -3 \\ 2 & -4 & -1 \\ -3 & 5 & -6 \end{bmatrix}$$

$$(B-C)A = BA - CA = \begin{bmatrix} -11 & 28 & -37 \\ -7 & 15 & -24 \\ 2 & -9 & 14 \end{bmatrix} - \begin{bmatrix} -1 & 2 & -17 \\ 2 & -7 & -6 \\ -1 & 1 & -7 \end{bmatrix}$$

$$(B-C)A = BA - CA = \begin{bmatrix} -10 & 26 & -20 \\ -9 & 22 & -18 \\ 3 & -10 & 21 \end{bmatrix}$$

Problem 4

4. Given $A = \begin{bmatrix} 4 & 2 & 5 \\ 3 & -6 & 7 \\ 1 & 9 & 8 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 3 & 2 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 5 & -2 \\ 2 & 4 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

Determine if the following matrices are possible. If so, compute.

a) AB b) BC c) AC d) CD

Solution 4

In order to multiply matrices, the number of rows on the left nust equal the number of columns on the right.

For example : $(n \times r)$ and $(r \times m)$

For matrix multiplication, we must multiply corresponding entries from the row of the matrix on the left by the column on the right, and add the products together to form the new entries.

a) $AB = \begin{bmatrix} 4 & 2 & 5 \\ 3 & -6 & 7 \\ 1 & 9 & 8 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 3 & 2 \\ 2 & 3 \end{bmatrix}$

$(4 \cdot -1) + (2 \cdot 3) + (5 \cdot 2) = 12$ $\quad (4 \cdot -1) + (2 \cdot 2) + (5 \cdot 3) = 23$

$(3 \cdot -1) + (-6 \cdot 3) + (7 \cdot 2) = -7$ $\quad (3 \cdot 1) + (-6 \cdot 2) + (7 \cdot 3) = 13$

$(1 \cdot -1) + (9 \cdot 3) + (8 \cdot 2) = 32$ $\quad (1 \cdot 1) + (9 \cdot 2) + (8 \cdot 3) = 43$

$$AB = \begin{bmatrix} 12 & 23 \\ -7 & 13 \\ 32 & 43 \end{bmatrix}$$

b) $BC = \begin{bmatrix} -1 & 1 \\ 3 & 2 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 & -2 \\ 2 & 4 & 3 \end{bmatrix}$

$(-1 \cdot 1) + (1 \cdot 2) = 1$ $\quad (-1 \cdot 5) + (1 \cdot 4) = -1$ $\quad (-1 \cdot -2) + (1 \cdot 3) = 5$

$(3 \cdot 1) + (2 \cdot 2) = 7 \qquad (3 \cdot 5) + (2 \cdot 4) = 23 \qquad (3 \cdot 2) + (2 \cdot 3) = 12$

$(2 \cdot 1) + (3 \cdot 2) = 8 \qquad (2 \cdot 5) + (3 \cdot 4) = 22 \qquad (2 \cdot -2) + (3 \cdot 3) = 5$

$$BC = \begin{bmatrix} 1 & -1 & 5 \\ 7 & 23 & 12 \\ 8 & 22 & 5 \end{bmatrix}$$

c) $AC = \begin{bmatrix} 4 & 2 & 5 \\ 3 & -6 & 7 \\ 1 & 9 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 & -2 \\ 2 & 4 & 3 \end{bmatrix}$

AC is undefined because there are more ~~rows~~ colums in A than ~~columns~~ rows in C.

d) $CD = \begin{bmatrix} 1 & 5 & -2 \\ 2 & 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \\ 4 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

$(1 \cdot 1) + (5 \cdot 4) + (2 \cdot 3) = 27 \qquad (1 \cdot 0) + (5 \cdot 2) + (-2 \cdot 1) = 8 \qquad (1 \cdot 2) + (5 \cdot 1) + (-2 \cdot 2) = 3$

$(2 \cdot 1) + (4 \cdot 4) + (3 \cdot 3) = 27 \qquad (2 \cdot 0) + (4 \cdot 2) + (3 \cdot 1) = 11 \qquad (2 \cdot 2) + (4 \cdot 1) + (3 \cdot 2) = 14$

$$CD = \begin{bmatrix} 27 & 8 & 3 \\ 27 & 11 & 14 \end{bmatrix}$$

Problem 5

5. What conditions for b_1, b_2 and b_3 make the following system of equations consistent.

$x_1 + x_3 = b_1$
$3x_1 + x_2 + x_3 = b_2$
$4x_1 + x_2 + 2x_3 = b_3$

Solution 5

Put the coefficient values into an augmented matrix and reduce to row-echelon form

$$\begin{bmatrix} 1 & 0 & 1 & b_1 \\ 3 & 1 & 1 & b_2 \\ 4 & 1 & 2 & b_3 \end{bmatrix}$$ Add the first row multiplied by -3 to the second row and multiplied by -4 to the third.

$$\begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 1 & -2 & b_2 - 3b_1 \\ 0 & 1 & -2 & b_3 - 4b_1 \end{bmatrix}$$ Add the second row multiplied by -1 to the third

$$\begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 1 & -2 & b_2 - 3b_1 \\ 0 & 1 & -2 & b_3 - 4b_1 \end{bmatrix}$$ Add the second row multiplied by -1 to the third

$$\begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 1 & -2 & b_2 - 3b_1 \\ 0 & 0 & 0 & -b_2 + 3b_1 + b_3 - 4b_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 1 & -2 & b_2 - 3b_1 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

Notice that $b_3 - b_2 - b_1$ must equal zero for this matrix to be consistent,
So $b_3 = b_2 + b_1$ will make the matrix consistent.

Problem 6

6. Let A+B be square Marices with the same size, show that $(A+B)^2 \neq A^2 + 2AB + B^2$. What is a matrix identing that is valid for all choices of A+B which satisfies.

$$(A+B)^2 = A^2 + B^2 + \_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$

Solution 6

$(A+B)^2 = (A+B)\times(A+B) = AA+AB+BA+BB$. $AA = A^2$, $BB = B^2$, however, since not every matrix is satisfied with the rule $AB = BA$, $(A+B)^2$ cannot always equal $A^2 + 2AB + B^2$

Let $A+B = C$

therefore $(A+B)^2 = (A+B)C$

by using right distributive law, $(A+B)C = AC + BC$

substitute A+B for C. $A(A+B) + B(A+B) = AA+AB+BA+BB$. $AA = A^2$, $BB = B^2$

$(A+B)^2 = A^2 + B^2 + AB + BA$.

Problem 7

7. Given p(x) $= x^2 - 3x + 2$ and A $= \begin{bmatrix} 5 & 2 \\ -3 & 1 \end{bmatrix}$, Give p(A)

Solution 7

$p(A) = A^2 - 3A + 2I$

$$= \begin{bmatrix} 5 & 2 \\ -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 2 \\ -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 5 & 2 \\ -3 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 12 \\ -18 & -5 \end{bmatrix} - \begin{bmatrix} 15 & 6 \\ -9 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ -9 & -6 \end{bmatrix}$$

Problem 8

8. Consider the system of linear equations.

$$x_1 + 2x_2 + x_3 = b_1$$
$$x_1 - x_2 + x_3 = b_2$$
$$x_1 + x_2 = b_3$$

Solve simultaneously given the following values of **b**.

a.) $b_1 = -1, \quad b_2 = 3, \quad b_3 = 4$

b.) $b_1 = 5, \quad b_2 = 0, \quad b_3 = 0$

c.) $b_1 = -1, \quad b_2 = -1, \quad b_3 = 3$

Solution 8

The system of linear equations can be augmented in the form Ax = b. This problem is using Theorem 1.6.2 and Section 1.6 example 2.
(Solution1)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \mathbf{b}_a = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \quad \mathbf{b}_b = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{b}_c = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

Step 1............. Set the equations to solve simultaneouly = $\left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & -1 & 5 & -1 \\ 1 & -1 & 1 & 3 & 0 & -1 \\ 1 & 1 & 0 & 4 & 0 & 3 \end{array}\right]$

Step 2.............. Convert the first submatrix to the I_3 using elementary row operations=

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & -1 & 5 & -1 \\ 1 & -1 & 1 & 3 & 0 & -1 \\ 1 & 1 & 0 & 4 & 0 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & -1 & 5 & -1 \\ 1 & -1 & 1 & 3 & 0 & -1 \\ 1 & 1 & 0 & 4 & 0 & 3 \end{array}\right]$$ Multiply row 1 by -1 and add to row2 and row3.

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & -1 & 5 & -1 \\ 0 & -3 & 0 & 4 & -5 & 0 \\ 0 & -1 & -1 & 5 & -5 & 4 \end{array}\right]$$ Multiply row3 by -3 and add to row 2.

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & -1 & 5 & -1 \\ 0 & -3 & 0 & 4 & -5 & 0 \\ 0 & 0 & 3 & 11 & 10 & -12 \end{array}\right]$$ Multiply row1 by -3 and add to row 3.

$$\left[\begin{array}{ccc|c|c|c} -3 & -6 & 0 & -8 & -5 & -9 \\ 0 & -3 & 0 & 4 & -5 & 0 \\ 0 & 0 & 3 & 11 & 10 & -12 \end{array}\right]$$ Multiply row2 by -2 and add to row1.

$$\left[\begin{array}{ccc|c|c|c} -3 & 0 & 0 & 16 & 5 & -9 \\ 0 & -3 & 0 & 4 & -5 & 0 \\ 0 & 0 & 3 & 11 & 10 & -12 \end{array}\right]$$ Multiply rows1&2 by $-\frac{1}{3}$ and row 3 by $\frac{1}{3}$.

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & \frac{16}{3} & -\frac{5}{3} & 3 \\ 0 & 1 & 0 & -\frac{4}{3} & \frac{5}{3} & 0 \\ 0 & 0 & 1 & \frac{11}{3} & \frac{10}{3} & -4 \end{array}\right]$$

Therefore the solutions of x are:

a.) $x_1 = \frac{16}{3}, \quad x_2 = -\frac{4}{3}, \quad x_3 = -\frac{11}{3},$

b.) $x_1 = \frac{-5}{3}, \quad x_2 = \frac{5}{3}, \quad x_3 = -\frac{10}{3},$

c.) $x_1 = 3, \quad x_2 = 0, \quad x_3 = -4,$

Problem 9

9. By inspection, determine whether the given matrix is invertible. If so what is its inverse?
(Note: Those problems with inverses may need to be computed...or...may be obvious by inspection)

a.) $= \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$

b.) $= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

c.) $= \begin{bmatrix} 4 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

d.) $= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

e.) $= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 5 & 0 \end{bmatrix}$

Solution 9

a.) $= \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$ Yes. $= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}$

b.) $= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ Yes. $= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

$$c.) = \begin{bmatrix} 4 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$ No.

There is a zero on the main diagonal of this upper triangular matrix and by Theorem 1.7.1c it is invalid.

$$d.) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$ Yes. $= \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$e.) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 5 & 0 \end{bmatrix}$$ No.

There is a zero on the main diagonal of this lower triangular matrix and is invalid

Problem 10

10. Given $A = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$

a) What is the determinant of A?
b) Based on the determinant, does A have an inverse?

Solution 10

a) Write the matrix, and copy the first two columns to the right of the matrix. Examine the diagonals indicated, and for the diagonals which go to the right, multiply the terms and add. For the diagonals which go to the left, multiply the terms and subtract.

$$\begin{matrix} 4 & 5 & 6 & 4 & 5 \\ 1 & 2 & 3 & 1 & 2 \\ 7 & 8 & 9 & 7 & 8 \end{matrix}$$
\- - -   + + +

$(-6\cdot2\cdot7) - (4\cdot3\cdot8) - (5\cdot1\cdot9) + (4\cdot2\cdot9) + (5\cdot3\cdot7) + (6\cdot1\cdot8) = 0$

b) Since det(A) = 0, no inverse exists.

1. Solve the following system by Gauss-Jordan elimination:

$$\begin{aligned} 2x_1 \quad &+2x_2 \quad +2x_3 \quad =0 \\ -2x_1 \quad &+5x_2 \quad +2x_3 \quad =1 \\ 8x_1 \quad &+x_2 \quad +4x_3 \quad =-1 \end{aligned}$$

Solution:

Matrix Format:

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$

Divide top row by 2:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$

Multiply the top row by 2 and add to the middle row:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$

Divide the middle row by 7:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$

Multiply the top row by -8 and add to bottom row:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & -7 & -4 & -1 \end{bmatrix}$$

Divide the bottom row by -7:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 1 & 4/7 & 1/7 \end{bmatrix}$$

Multiply middle row by -1 and add to bottom row:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Subtract second row from first row:

$$\begin{bmatrix} 1 & 0 & 3/7 & -1/7 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_3 &= s \\ x_2 &= 1/7 \quad -4/7\, s \\ x_1 &= -1/7 \quad -3/7\, s \end{aligned}$$

2. Consider the matrices:

$$A=\begin{bmatrix}1 & 0\\ -2 & 3\\ 4 & 1\end{bmatrix}\quad B=\begin{bmatrix}3 & -2\\ 1 & 4\end{bmatrix}\quad C=\begin{bmatrix}2 & 3 & -1\\ 1 & 0 & 5\end{bmatrix}\quad D=\begin{bmatrix}2 & 0 & 1\\ 1 & 2 & 3\\ -1 & 4 & 2\end{bmatrix}\quad E=\begin{bmatrix}1 & 4 & 2\\ -3 & 1 & 4\\ 2 & 0 & 1\end{bmatrix}$$

Compute the following (where possible):

a) $D+E$
b) $D-E$
c) $3A$
d) $-6C$
e) $3B+C$
f) $5E-3D$
g) $3A^T+2C$
h) $(D+E)^T$

Solutions:

a) $D+E=\begin{bmatrix}3 & 4 & 3\\ -2 & 3 & 7\\ 1 & 4 & 3\end{bmatrix}$

b) $D-E=\begin{bmatrix}1 & -4 & -1\\ 4 & 1 & -1\\ -3 & 4 & 1\end{bmatrix}$

c) $3A=\begin{bmatrix}3 & 0\\ -6 & 9\\ 12 & 3\end{bmatrix}$

d) $-6C=$
$\begin{bmatrix}-12 & -18 & 6\\ -6 & 0 & -30\end{bmatrix}$

e) $3B+C=$ *Not Possible*

f) $5E-3D=\begin{bmatrix}-1 & 20 & 7\\ -18 & -1 & 11\\ 13 & -12 & -1\end{bmatrix}$

g) $3A^T+2C=\begin{bmatrix}7 & 0 & 10\\ 2 & 9 & 13\end{bmatrix}$

h) $(D+E)^T=\begin{bmatrix}1 & 4 & -3\\ -4 & 1 & 4\\ -1 & -1 & 1\end{bmatrix}$

3. Please fill in the missing parts.

a. $(A-B)^2=A^2+B^2$ ______________

b. $(A+B)^2=$ ______________

c. $(A+B)(A-B)=$ ______________

d. $(A-B)(A+B)\overset{?}{=}(A+B)(A-B)$ **explain your reasoning:**

Solution:

a. $(A-B)^2 = A^2 + B^2 - AB - BA$

b. $(A+B)^2 = A^2 + B^2 + AB + BA$

c. $(A+B)(A-B) = A^2 - B^2 - AB + BA$

d. $(A-B)(A+B) \stackrel{?}{=} (A+B)(A-B)$ explain your reasoning:

Order matters in matrix multiplication, so this would not be equal. The left side of this equation produces a AB-BA whereas the right side produces a BA-AB which are not the same value.

4. Show whether or not the following resultant matrices (R) are symmetric: (A and B are not necessarily symmetric)

a. $R = (A)(A^T)$

b. $R = B + B^T$

c. Are all symmetric matrices invertible?

Solution:

a. $R = (A)(A^T)$

$$R^T = ((A)(A^T))^T = (A^{TT})(A^T) = (A)(A^T)$$

By definition if $R = R^T$ than it is symmetric

b. $R = B + B^T$

$$R^T = (B + B^T)^T = (B^T + B^{TT}) = B^T + B = B + B^T$$

Again by definition R is symmetric. Because matrix addition is commutative, this ends up equal.

c. Are all symmetric matrices invertible?

No, the matrix 1x1 matrix [0] is symmetric, but not invertible.

5) Find the A^{-1} of A, solving with the identity matrix I.

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 0 & 6 \\ 1 & 3 & 2 & 7 \\ 3 & 6 & 0 & 17 \\ 2 & 0 & -7 & 16 \end{bmatrix}$$

Solution:

We want to reduce A using row operations and simultaneously apply the same row operations to the identity matrix to produce A^{-1}.

$$\text{Let } A \mid I = \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 6 & 1 & 0 & 0 & 0 \\ 1 & 3 & 2 & 7 & 0 & 1 & 0 & 0 \\ 3 & 6 & 0 & 17 & 0 & 0 & 1 & 0 \\ 2 & 0 & -7 & 16 & 0 & 0 & 0 & 1 \end{array}\right]$$

Add -1 times the first row to the second,
-3 times the first row to the third,
and -2 times the first row to the fourth to produce:

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 6 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -3 & 0 & 1 & 0 \\ 0 & -4 & -7 & 4 & -2 & 0 & 0 & 1 \end{array}\right]$$

Add -4 times the second row to the fourth row,
Multiply the third row by -1,
Switch the third and fourth rows to produce:

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 6 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8 & -6 & 4 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 & -1 & 0 \end{array}\right]$$

Add -8 times the fourth row to the third row,
-1 times the fourth row to the second row,
-6 times the fourth row to the first row,

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & -17 & 0 & 6 & 0 \\ 0 & 1 & 2 & 0 & -4 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -30 & -4 & 8 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 & -1 & 0 \end{array}\right]$$

Add -2 times the third row to the second row to produce:

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & -17 & 0 & 6 & 0 \\ 0 & 1 & 0 & 0 & 56 & 7 & -15 & -2 \\ 0 & 0 & 1 & 0 & -30 & 4 & 8 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 & -1 & 0 \end{array}\right]$$

and -2 times the second row to the first row

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -129 & 14 & 36 & 4 \\ 0 & 1 & 0 & 0 & 56 & -7 & -15 & -2 \\ 0 & 0 & 1 & 0 & -30 & 4 & 8 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 & -1 & 0 \end{array}\right]$$

which is the inverse of A

6. For A<u>x</u> = <u>b</u>, solve for <u>x</u>.

Let: $A=\begin{bmatrix} 2 & -1 & 6 & -1 \\ 1 & 1 & 0 & 4 \\ -1 & 2 & -5 & 4 \\ 2 & -1 & 8 & -1 \end{bmatrix}$ $\underline{b}=\begin{bmatrix} 6 \\ 12 \\ 3 \\ 4 \end{bmatrix}$

Solution:

$$A^{-1} = \begin{bmatrix} 17/6 & -2/3 & 1 & -3/2 \\ 9/6 & -7/3 & 3 & -1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 3/2 & 1 & -1 & 1/2 \end{bmatrix}$$

$$A\underline{x} = \underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b} \ , \ \underline{x} = \begin{bmatrix} 6 \\ -2 \\ -1 \\ 2 \end{bmatrix}$$

7. Which of the following are invertible?

a) $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 3 \\ 5 & 1 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 & 3 \\ 3 & 0 & 7 \\ 2 & -1 & 4 \end{bmatrix}$ d) $\begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Solution:

a, b, c

8. Find the cofactor and minor matrices of A.

Let $A = \begin{bmatrix} 3 & 2 & 3 \\ 7 & 1 & 3 \\ 1 & 4 & 2 \end{bmatrix}$

Solution:

$$C = \begin{bmatrix} -10 & -11 & 27 \\ 8 & 3 & -10 \\ 3 & 12 & -9 \end{bmatrix} \qquad M = \begin{bmatrix} -10 & 11 & 27 \\ -8 & 3 & 10 \\ 3 & -12 & -9 \end{bmatrix}$$

9. a) Find det(A).

Let A= $\begin{bmatrix} 2 & 4 & 1 \\ 1 & 4 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

Solution:

det(A) = 2

b) Find A^{-1} using det(A).

Solution:

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

$$adj(A) = C^T = \begin{bmatrix} -4 & 4 & -6 \\ 2 & -2 & 4 \\ 4 & -3 & 4 \end{bmatrix}^T = \begin{bmatrix} -4 & 2 & 4 \\ 4 & -2 & -3 \\ -6 & 4 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -4 & 2 & 4 \\ 4 & -2 & -3 \\ -6 & 4 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ 2 & -1 & -3/2 \\ -3 & 2 & 2 \end{bmatrix}$$

10. Prove that $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Solution:

assuming result — False

$\det(A^{-1}) = \dfrac{1}{\det(A)} \neq \det(A^{-1})\det(A\ ) = 1$ when you multiply it out.

Then since we know that $\det(A)\det(B)$=det(AB), so

$\det((A)(A^{-1})) = \det(I) = 1$ Which is true, the determinant for the Identity matrix is 1.

1. Solve the following system of equations using Gauss-Jordan elimination:

$$\begin{aligned} w + x - y + 2z &= -4 \\ 2w + y + z &= 4 \\ 3w + 2x + z &= -1 \\ 3x + 2y - 4z &= -3. \end{aligned}$$

Solution:

The system of equations

$$\begin{aligned} w + x - y + 2z &= -4 \\ 2w + y + z &= 4 \\ 3w + 2x + z &= -1 \\ 3x + 2y - 4z &= -3 \end{aligned}$$

has the augmented matrix $\begin{bmatrix} 1 & 1 & -1 & 2 & -4 \\ 2 & 0 & 1 & 1 & 4 \\ 3 & 2 & 0 & 2 & -2 \\ 0 & 3 & 2 & -4 & -3 \end{bmatrix}$.

Using row operations we reduce the matrix to reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 & -1 & 2 & -4 \\ 2 & 0 & 1 & 1 & 4 \\ 3 & 2 & 0 & 2 & -2 \\ 0 & 3 & 2 & -4 & -3 \end{bmatrix} \begin{matrix} \\ R2 = R2 - 2R1 \\ R3 = R3 - 3R1 \\ \to \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 2 & -4 \\ 0 & -2 & 3 & -3 & 12 \\ 0 & -1 & 3 & -4 & 10 \\ 0 & 3 & 2 & -4 & -3 \end{bmatrix}$$

$$\begin{matrix} R2 \to -R2 \\ R2 \leftrightarrow R3 \\ \to \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 2 & -4 \\ 0 & 1 & -3 & -4 & -10 \\ 0 & -2 & 3 & -3 & 12 \\ 0 & 3 & 2 & -4 & -3 \end{bmatrix}$$

$$\begin{matrix} R3 = R3 + 2R2 \\ R4 = R4 - 3R2 \\ \to \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 2 & -4 \\ 0 & 1 & -3 & -4 & -10 \\ 0 & 0 & -3 & 5 & -8 \\ 0 & 0 & 11 & -16 & 27 \end{bmatrix}$$

$$R3 \to -1/3R3 \quad \to \quad \begin{bmatrix} 1 & 1 & -1 & 2 & -4 \\ 0 & 1 & -3 & -4 & -10 \\ 0 & 0 & 1 & -5/3 & 8/3 \\ 0 & 0 & 11 & -16 & 27 \end{bmatrix}$$

$$R4 = R4 - 11R3 \quad \to \quad \begin{bmatrix} 1 & 1 & -1 & 2 & -4 \\ 0 & 1 & -3 & -4 & -10 \\ 0 & 0 & 1 & -5/3 & 8/3 \\ 0 & 0 & 0 & 7/3 & -7/3 \end{bmatrix}$$

$$R4 \to 3/7R4 \quad \to \quad \begin{bmatrix} 1 & 1 & -1 & 2 & -4 \\ 0 & 1 & -3 & -4 & -10 \\ 0 & 0 & 1 & -5/3 & 8/3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{aligned} R1 &= R1 - 2R4 \\ R2 &= R2 - 4R4 \\ R3 &= R3 + 5/3R4 \end{aligned} \quad \to \quad \begin{bmatrix} 1 & 1 & -1 & 0 & -2 \\ 0 & 1 & -3 & 0 & -6 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{aligned} R1 &= R1 + R3 \\ R2 &= R2 + 3R3 \end{aligned} \quad \to \quad \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$R1 = R1 - R2 \quad \to \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Therefore w = 1, x = -3, y = 1, and z = -1.

2. If

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & -1 \\ -2 & 6 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix},$$

Find $(AB)^T$ and $(BA)^T$.

Solution: Performing the requested operations we see that

$$(AB)=\begin{bmatrix} 4 & -3 & 4 \\ 4 & 5 & 0 \\ 9 & 20 & 8 \end{bmatrix} \qquad (AB)^T=\begin{bmatrix} 2 & -2 & 2 \\ 5 & 18 & -3 \\ -5 & 24 & -3 \end{bmatrix}$$

$$(BA)=\begin{bmatrix} 4 & 4 & 9 \\ -3 & 5 & 20 \\ 4 & 0 & 8 \end{bmatrix} \qquad (BA)^T=\begin{bmatrix} 2 & 5 & -5 \\ -2 & 18 & 24 \\ 2 & -3 & -3 \end{bmatrix}.$$

3. Compute the product of

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 6 & 3 & 1 & 2 & 5 \\ 2 & 1 & -1 & 3 & 0 \\ 3 & 4 & 5 & 7 & -6 \\ -1 & 1 & 2 & -2 & 3 \end{bmatrix}=$$

Solution: Because the first matrix is a diagonal matrix, the solution comes simply as

$$\begin{bmatrix} 24 & 12 & 4 & 8 & 20 \\ -2 & -1 & 1 & -3 & 0 \\ 6 & 8 & 10 & 14 & -12 \\ -3 & 3 & 6 & -6 & 9 \end{bmatrix}.$$

4. Solve these two systems simultaneously

$$\text{a.}\quad \begin{aligned} x-y+2z&=5 \\ 2x+y-4z&=-6 \\ x+3y+z&=3 \end{aligned} \qquad \text{b.}\quad \begin{aligned} 2x-2y+4z&=-2 \\ 2x+y-4z&=9 \\ x+3y+z&=4 \end{aligned}$$

Solution: After dividing the first equation of the second system of equations by two we see that these two systems have the same coefficient matrix. So we can solve them at the same time by augmenting the coefficient matrix with the columns of constants on the right side of the equations to get

$$\left[\begin{array}{ccc|c|c} 1 & -1 & 2 & 5 & -1 \\ 2 & 1 & -4 & -6 & 9 \\ 1 & 3 & 1 & 3 & 4 \end{array}\right].$$

We then reduce the matrix to reduced row-echelon form by using row operations

$$\left[\begin{array}{ccc|c|c} 1 & -1 & 2 & 5 & -1 \\ 2 & 1 & -4 & -6 & 9 \\ 1 & 3 & 1 & 3 & 4 \end{array}\right] \begin{array}{l} R2 = R2 - 2R1 \\ R3 = R3 - R1 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c|c} 1 & -1 & 2 & 5 & -1 \\ 0 & 3 & -8 & -16 & 11 \\ 0 & 4 & -1 & -2 & 5 \end{array}\right]$$

$$\begin{array}{l} R2 \rightarrow 1/3R2 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c|c} 1 & -1 & 2 & 5 & -1 \\ 0 & 1 & -8/3 & -16/3 & 11/3 \\ 0 & 4 & -1 & -2 & 5 \end{array}\right]$$

$$\begin{array}{l} R3 = R3 - 4R2 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c|c} 1 & -1 & 2 & 5 & -1 \\ 0 & 1 & -8/3 & -16/3 & 11/3 \\ 0 & 0 & 29/3 & -58/3 & -29/3 \end{array}\right]$$

$$\begin{array}{l} R3 \rightarrow 3/29R3 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c|c} 1 & -1 & 2 & 5 & -1 \\ 0 & 1 & -8/3 & -16/3 & 11/3 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$\begin{array}{l} R1 = R1 - 2R3 \\ R2 = R2 + 8/3R3 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$\begin{array}{l} R1 = R1 + R2 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right].$$

So for a. x = 1, y = 0, and z = 2,
and for b. x = 2, y = 1, and z = -1.

5. Find (A + B)C if

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 0 \\ 3 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 4 \\ 7 & -1 \\ 0 & -1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 4 & 1 & -1 & 0 \end{bmatrix}.$$

Solution: Fortunately we can see based on the size of the matrices that the involved functions are possible, so the solution is as follows:

$$\left(\begin{bmatrix} 2 & 4 \\ -1 & 0 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 7 & -1 \\ 0 & -1 \end{bmatrix}\right)\begin{bmatrix} 1 & 2 & 3 & -2 \\ 4 & 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 6 & -1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & -2 \\ 4 & 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 35 & 14 & 1 & -6 \\ 2 & 11 & 19 & -12 \\ -9 & 3 & 12 & -6 \end{bmatrix}.$$

6. Compute the following determinants:

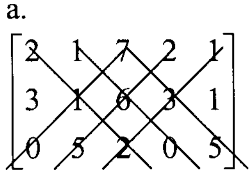
   a. Det (A) when A= $\begin{bmatrix} 2 & 1 & 7 \\ 3 & 1 & 6 \\ 0 & 5 & 2 \end{bmatrix}$

   b. Det (B) when B= $\begin{bmatrix} 1 & 3 & 1 \\ 3 & 6 & 2 \\ 5 & 4 & 2 \end{bmatrix}$

Solutions:

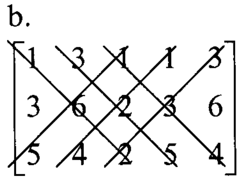
For these problems we can use the method of summing the products of the rightward lines and then subtracting the products of the leftward lines.

a.



Det (A)= (2*1*2)+(7*2*5)+(1*6*0)-(0*1*7)-(5*6*2)-(2*3*1)=4+70+0-0-60-6=8

b.



Det (B)= (1*6*2)+(3*2*5)+(1*3*4)-(5*6*1)-(2*4*1)-(3*3*2)=12+30+12-30-8-18= -2

7. If possible, find the inverse of A using $[A|\ I]$.

$$A = \begin{bmatrix} 4 & -5 & 3 \\ 1 & 2 & -2 \\ -2 & 3 & 5 \end{bmatrix}.$$

Solution: We set the given matrix next to an identity matrix of the same size. Using elementary row operations to solve for the identity matrix on the right we get

$$A^{-1}\left[\begin{array}{ccc|ccc} 4 & -5 & 3 & 1 & 0 & 0 \\ 1 & 2 & -2 & 0 & 1 & 0 \\ -2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 0 & 1 & 0 \\ 4 & -5 & 3 & 1 & 0 & 0 \\ -2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 0 & 1 & 0 \\ 0 & -37 & 11 & 1 & -4 & 0 \\ 0 & 19 & 1 & 0 & 2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 0 & 1 & 0 \\ 0 & 1 & -11/37 & -1/37 & 4/37 & 0 \\ 0 & 19 & 1 & 0 & 2 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 0 & 1 & 0 \\ 0 & 1 & -11/37 & -1/37 & 4/37 & 0 \\ 0 & 0 & 246/37 & 19/37 & -2/37 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 0 & 1 & 0 \\ 0 & 1 & -11/37 & -1/37 & 4/37 & 0 \\ 0 & 0 & 1 & 19/246 & -1/123 & 37/246 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 19/123 & 121/123 & 37/123 \\ 0 & 1 & 0 & -1/246 & 503/4551 & 11/246 \\ 0 & 0 & 1 & 19/246 & -1/123 & 37/246 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 20/123 & 1157/1517 & 26/123 \\ 0 & 1 & 0 & -1/246 & 503/4551 & 11/246 \\ 0 & 0 & 1 & 19/246 & -1/123 & 37/246 \end{array}\right].$$

8. Determine the determinant of A if

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 2 & -1 \\ 2 & 1 & -2 \end{bmatrix}.$$

Solution:

detA = (-4) +(-6)+(-4)-(16)-(6)-(-1)= -45

9. Calculate det (A) using row opporations.

$$A = \begin{matrix} 3 & 6 & 12 \\ 3 & 3 & 1 \\ 1 & 3 & 2 \end{matrix}$$

Solution:

Start with A: actions on Det.

3	6	12		
3	3	1	switch R1 and R3	-1
1	3	2		

1	3	2		
3	3	1	R2-3R1 and R3-3R1	1
3	6	12		

1	3	2		
0	-6	-5	R3-1/2R2	1
0	-3	6		

$$\begin{matrix} 1 & 3 & 2 \\ 0 & -6 & -5 \\ 0 & 0 & 17/2 \end{matrix}$$

Now we can find that the determinate of this diagonal matrix is:

1(-6)(17/2)=51

Now we take into effect the product of the numbers we wrote down to the right.

51/-1=-51

10. Use cofactor expansion along a row or column of your choice to find det(A) if

$$A = \begin{bmatrix} 0 & 3 & -2 & 4 \\ 5 & 0 & -3 & 0 \\ -2 & -5 & 3 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix}.$$

Solution: It would make sense to choose row 4 because only one of the entries is not zero. Expanding, we find that

$$\det A = (-2)\begin{vmatrix} 0 & -2 & 4 \\ 5 & -3 & 0 \\ -2 & 3 & 1 \end{vmatrix}.$$

We can then use the method that multiplies the diagonals downward, with positive products going to the right and negative products going to the left. Therefore detA = (-2)(0 + 0 + 60 – 0 –(-10) – 24) = -92.

Section 1.1

1. Find the augmented matrix (2 pts each)

a.

$$3x_1 - 4x_2 = 3$$
$$4x_1 + 3x_2 = 1$$
$$7x_1 + 2x_2 = 2$$

b.

$$2x_1 + 2x_3 = 1$$
$$7x_1 + x_2 - 4x_3 = 7$$
$$3x_1 - 2x_2 + 6x_3 = 0$$

2. For which values of the constant K does the system (2 pts for each part)

$$2x - y = 4$$
$$4x - 2y = K$$

a. have no solutions?
b. Exactly one?
c. Infinitely many?

<u>Solutions</u>

1a.

$$\begin{bmatrix} 3 & -4 & 3 \\ 4 & 3 & 1 \\ 7 & 2 & 2 \end{bmatrix}$$

1b.

$$\begin{bmatrix} 2 & 0 & 2 & 1 \\ 7 & 1 & -4 & 7 \\ 3 & -1 & 6 & 0 \end{bmatrix}$$

2a. No solution for all real numbers $K \neq 8$
2b. One solution is not possible because the lines never cross
2c. Many solutions are possible if K=8 because the equations will be similar

Section 1.2

1. Reduce these Matricies using Gauss Jordan Elimination (4 pts Each)

a)

$$A = \begin{bmatrix} 0 & 5 & 10 & 25 \\ 1 & 2 & 4 & 2 \\ 3 & 6 & 9 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 5 & 10 & 25 \\ 1 & 2 & 4 & 2 \\ 3 & 6 & 9 & 3 \end{bmatrix}$$

Swap Rows 1 & 2 to get rid of Leading 0 →

$$\begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 5 & 10 & 25 \\ 3 & 6 & 9 & 3 \end{bmatrix}$$

-3x 1^{st} Row →

$$\begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 5 & 7 & 25 \\ 0 & 0 & -3 & -3 \end{bmatrix}$$

Divide by 5
Divide by -3 →

$$\begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Subtract 4x 3^{rd} Row
Subtract 2x 2^{nd} Row →

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Subtract 2x 2^{nd} Row →

$$\begin{bmatrix} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Done!

b)

$$B = \begin{bmatrix} 0 & 1 & 7 & 0 \\ 2 & 8 & 10 & 11 \\ 1 & 4 & 0 & 8 \\ 1 & 4 & 0 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 7 & 0 \\ 2 & 8 & 10 & 11 \\ 1 & 4 & 0 & 8 \\ 1 & 4 & 0 & 8 \end{bmatrix}$$

Swap up 3^{rd} Row
Swap down 1^{st} Row
Swap down 2^{nd} Row →

$$\begin{bmatrix} 1 & 4 & 0 & 8 \\ 0 & 1 & 7 & 0 \\ 2 & 8 & 10 & 11 \\ 1 & 4 & 0 & 8 \end{bmatrix}$$

Subtract 2x First Row
Subract First Row →

$$\begin{bmatrix} 1 & 4 & 0 & 8 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 10 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Divide by 10
Drop off 0 Row →

$$\begin{bmatrix} 1 & 4 & 0 & 8 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

Subtract 7x 3^{rd} Row →

$$\begin{bmatrix} 1 & 4 & 0 & 8 \\ 0 & 1 & 0 & 7/2 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

Subtract 4x 2^{nd} Row →

$$\begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 7/2 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

Done!

c)

$$C = \begin{bmatrix} 2 & 3 & -5 & 2 & 8 \\ 4 & -7 & 3 & 7 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & -5 & 2 & 8 \\ 4 & -7 & 3 & 7 & -2 \end{bmatrix}$$ Divide By 2 →

$$\begin{bmatrix} 1 & 3/2 & -5/2 & 1 & 4 \\ 4 & -7 & 3 & 7 & -2 \end{bmatrix}$$ Subtract 4x 1st Row →

$$\begin{bmatrix} 1 & 3 & -5 & 2 & 8 \\ 0 & -13 & 13 & 3 & -18 \end{bmatrix}$$ Divide By -13 →

$$\begin{bmatrix} 1 & 3 & -5 & 2 & 8 \\ 0 & 1 & -1 & -3/13 & 50/13 \end{bmatrix}$$ Subtract -3x 2nd Row →

$$\begin{bmatrix} 1 & 0 & -2 & 35/13 & 50/13 \\ 0 & 1 & -1 & -3/13 & 18/13 \end{bmatrix}$$

2. **Define Gauss-Jordan Elimination (3 Points) –**
The process by which one changes a matrix into reduced row-echelon form. – ambiguous – change entries any way – erasing?. ∅

Section 1.3 Compute *(AB)C+C^TC (15 Points)*

$$A = \begin{bmatrix} 1 & -5 & 6 \\ -3 & 3 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 7 & 3 & -2 \\ 5 & 5 & 3 & 3 \\ 9 & 6 & -3 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -6 \\ -5 & 2 \\ 7 & 4 \\ 3 & -8 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -5 & 6 \\ -3 & 3 & 2 \end{bmatrix} x \begin{bmatrix} 4 & 7 & 3 & -2 \\ 5 & 5 & 3 & 3 \\ 9 & 6 & -3 & 1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} (1)(4)+(-5)(5)+(6)(9) & (1)(7)+(-5)(5)+(6)(6) & (1)(3)+(-5)(3)+(6)(-3) & (1)(-2)+(-5)(3)+(6)(\\ (-3)(4)+(3)(5)+(2)(9) & (-3)(7)+(3)(5)+(2)(6) & (-3)(3)+(3)(3)+(2)(-3) & (-3)(-2)+(3)(3)+(2)(\end{bmatrix}$$

$$AB = \begin{bmatrix} 13 & 18 & -30 & -11 \\ 21 & 6 & -6 & 17 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 13 & 18 & -30 & -11 \\ 21 & 6 & -6 & 17 \end{bmatrix} x \begin{bmatrix} 1 & -6 \\ -5 & 2 \\ 7 & 4 \\ 3 & -8 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} (13)(1)+(18)(-5)+(-30)(7)+(-11)(3) & (13)(-6)+(18)(2)+(-30)(4)+(-11)(-8) \\ (21)(1)+(6)(-5)+(-6)(7)+(17)(3) & (21)(-6)+(6)(2)+(-6)(4)+(17)(-8) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -320 & 82 \\ 0 & -22 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 & -5 & 7 & 3 \\ -6 & 2 & 4 & -8 \end{bmatrix}$$

$$C^T C = \begin{bmatrix} 1 & -5 & 7 & 3 \\ -6 & 2 & 4 & -8 \end{bmatrix} X \begin{bmatrix} 1 & -6 \\ -5 & 2 \\ 7 & 4 \\ 3 & -8 \end{bmatrix} =$$

$$\begin{bmatrix} (1)(1) + (-5)^2 + (7)^2 + (3)^2 & (1)(-6) + (-5)(2) + (7)(4) + (3)(8) \\ (-6)(1) + (2)(-5) + (4)(7) + (-8)(3) & (-6)^2 + (2)^2 + (4)^2 + (-8)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 84 & -12 \\ -12 & 120 \end{bmatrix}$$

(AB)C+C^TC=

$$\begin{bmatrix} -320 & 82 \\ 0 & -22 \end{bmatrix} + \begin{bmatrix} 84 & -12 \\ -12 & 120 \end{bmatrix} = \begin{bmatrix} -284 & 70 \\ -12 & 98 \end{bmatrix}$$

Section 1.4: Inverses: Rule of Matrix Arithmetic

1. Given the following matrices, perform the indicated operations: (1 point each)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 & 1 & 3 \\ 2 & -1 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

a. $A+B$

Simple matrix arithmetic provides the answer:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 7 & 1 & 3 \\ 2 & -1 & 1 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 4 \\ 2 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

b. $A+C^T$

First the transpose of C is found by interchanging the columns and rows of C,

$$C^T = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 3 & 5 & 3 \end{bmatrix}$$

then add C^T to A

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 3 & 5 & 3 \end{bmatrix}$$

and receive the sum:

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 5 & 4 \end{bmatrix}$$

c. *$(A+B)C$*

The sum of A+B has already been determined, so that matrix is left multiplied into C and the product is:

$$\begin{bmatrix} 8 & 2 & 4 \\ 2 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 5 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 46 \\ 0 & 2 & 12 \\ 2 & 0 & 20 \end{bmatrix}$$

d. C^{-1}

C^{-1} is found by forming the augmented matrix $[C \mid I]$

***and performing elementary row operation until the resulting matrix is of the form** $[I \mid B]$*

where B is equal to C^{-1}

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 3 & 1 & 0 & 0 \\ 2 & 0 & 5 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 5 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

Swap the top row with the second row

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 5 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

Multiply the bottom row by - 1 and add it to the second row, then divide the bottom row by 3

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 1 & -5/3 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

Multiply the bottom row by -5 and add it to the top row

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/2 & -5/6 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

Divide the top row by 2

2. Given A and B find $(AB)^{-1}$ (2 points)

$$A = \begin{bmatrix} 4 & 1 & -7 \\ 0 & 9 & 6 \\ 2 & -5 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 5 & -1 \\ -2 & 1 & -10 \\ 2 & 3 & -2 \end{bmatrix}$$

First the product AB must be found in order to calculate its inverse.
This product is given as:

$$AB = \begin{bmatrix} 0 & 0 & 0 \\ -6 & 27 & -102 \\ 26 & 17 & 40 \end{bmatrix}$$

since the matrix has a row of complete zeros, it is singular meaning it has no inverse.

3. Prove the theorem: (3 points)

$$if: A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

– pf by computing AA^{-1} simpler

This theorem can be proved by simply finding the inverse through the augmented matrix method

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 1 & d/c & 0 & 1/c \end{array}\right]$$ Divide the leading entries by themselves

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & cb-da/ac & 1/a & -1/c \end{array}\right]$$ Subtract the bottom row from the bottom row

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & c/cb-da & -a/cb-da \end{array}\right]$$ Multiply the bottom row by the leading entries' scalar inverse

$$\left[\begin{array}{cc|cc} 1 & 0 & -d/cb-da & b/cb-da \\ 0 & 1 & c/cb-da & -a/cb-da \end{array}\right]$$ Multiply the bottom row by $\frac{-b}{a}$ and add to the top

$$A^{-1} = \begin{bmatrix} -d/cb-da & b/cb-da \\ c/cb-da & -a/cb-da \end{bmatrix}$$ Begin to reorganize and simplify A^{-1}

$$A^{-1} = \frac{-1}{cb - da}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ Factor out $\frac{-1}{cb - da}$ because of the properties of scalar - matrix multiplication

$$A^{-1} = \frac{1}{da - cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ Multiply the scalar by $\frac{-1}{-1}$

4. Find a matrix M such a that (1) MB = MC and (2) B = C, given: (

$B \neq C$?

$$B = \begin{bmatrix} 2 & -9 & 3 \\ 6 & -2 & 7 \\ -4 & 1 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 2 & 0 \\ -8 & 6 & 6 \\ -5 & 4 & 1 \end{bmatrix}$$

Performing simple matrix operations, shows us that left multiplying both sides of equation 1 by

an inverse of M:
$M^{-1}MB = M^{-1}MC$
gives us IB = IC
or B = C. — but $B \neq C$
So as long as M is any invertible matrix (ie $\det(M) \neq 0$) both equations 1 and 2 are true.

I don't understand.

what is M?

Section 1.5

1. Which of the following are elementary matrices? (4 pts)

a. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 3 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ c. $\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 1 \\ \sqrt{3} & 0 \end{bmatrix}$

2. Find a row operation that will restore the given elementary matrix to an identity matrix. (2 pts each)

a.
$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

3. Use row operations to find the matrix inverse ([A| I]) (5 pts each)

a.
$$A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$$

b.
$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solutions

1a. you could swap row 1 and 2 to show

b?, d?

1c. you could divide row 1 by $\sqrt{3}$

2a. Swap rows 1 and 4 giving:
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2b. Multiply row 1 by -3 giving:
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3a.

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{2} & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

3b.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Section 1.6

1. True or False (3 points Each)

If A is invertible then A**x** = **b** is consistent for every n x 1 matrix **b**: **True**

– not a statement

If the reduced row-echelon form of A is I_n: **True**

If A is invertible then A**x** = **b** has infinite solutions: **False**

2. Use the Following System of Equations to Solve for x given b. (3 Points Each)

$x - 3y = b_1$
$3x + 2y = b_2$

a) $b_1 = 5 \quad b_2 = 7$
b) $b_1 = -5 \quad b_2 = 10$

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 2 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 2/11 & 3/11 \\ -3/11 & 1/11 \end{bmatrix}$$

B = ?

$A^{-1} * B =$

a)

$$\begin{bmatrix} 2/11 & 3/11 \\ -3/11 & 1/11 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 31/11 \\ -8/11 \end{bmatrix}$$

x = 31/11
y = -8/11

b)

$$\begin{bmatrix} 2/11 & 3/11 \\ -3/11 & 1/11 \end{bmatrix} \begin{bmatrix} -5 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

x = 4
y = -1

Section 1.7 (10 Points)

Q: To what purposes can a diagonal matrix be used? What is the effect of left multiplying a diagonal matrix into any matrix A?

A: A diagonal matrix can be used to multiply each row by an individual scalar. Whereas previously we could only multiply the entire matrix by a single scalar we can modify each row (or each column by right multiplying by the diagonal matrix).

Section 2.1

1. Evaluate the determinate using the diagonal method.
   (2pts each)

a.
$$\begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}$$

b.
$$\begin{bmatrix} 4 & 8 \\ 1 & 3 \end{bmatrix}$$

(4 pts each)

c.
$$\begin{bmatrix} 1 & 0 & 2 \\ 4 & 6 & 3 \\ 1 & 5 & 2 \end{bmatrix}$$

d.
$$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 3 \\ 5 & 6 & 1 \end{bmatrix}$$

<u>Solutions</u>

a. $(3*5)-(6*2)=3$

b. $(4*3)-(1*8)=4$

c. $(1*6*2)+(0*3*1)+(2*4*5)-(1*6*2)-(5*3*1)-(2*4*0)=25$

d. $(0*0*1)+(1*3*5)+(0*2*6)-(5*0*0)-(6*3*0)-(1*2*1)=13$

Section 2.2

1. Find All Minors & Cofactors for the matrix: (5 Pts)

$$A = \begin{bmatrix} 3 & 5 & -7 \\ 6 & 1 & 2 \\ 4 & 8 & -4 \end{bmatrix}$$

$$M_{11} \begin{bmatrix} 1 & 2 \\ 8 & -4 \end{bmatrix} = -20$$

$$M_{21} \begin{bmatrix} 5 & 7 \\ 8 & -4 \end{bmatrix} = 36$$

$$M_{31} \begin{bmatrix} 5 & -7 \\ 1 & 2 \end{bmatrix} = 17$$

$$M_{12} \begin{bmatrix} 6 & 2 \\ 4 & -4 \end{bmatrix} = -32$$

$$M_{22} \begin{bmatrix} 3 & -7 \\ 4 & -4 \end{bmatrix} = 16$$

$$M_{32} \begin{bmatrix} 3 & -7 \\ 6 & 2 \end{bmatrix} = 48$$

$$M_{13} \begin{bmatrix} 6 & 1 \\ 4 & 8 \end{bmatrix} = 44$$

$$M_{23} \begin{bmatrix} 3 & 5 \\ 4 & 8 \end{bmatrix} = 4$$

$$M_{33} \begin{bmatrix} 3 & 5 \\ 6 & 1 \end{bmatrix} = -27$$

$$\text{Minors} = \begin{bmatrix} -20 & -32 & 44 \\ 36 & 16 & 4 \\ 17 & 48 & -27 \end{bmatrix}$$

$$\text{Cofactors} = \begin{bmatrix} -20 & 32 & 44 \\ -36 & 16 & -4 \\ 17 & -48 & -27 \end{bmatrix}$$

2. Evaluate Det(B) with a Cofactor Expansion (5pts)

$$B = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$$

$$K(K^2 - K^2) + K(K^2 - K^2) + K(K^2 - K^2) = 0$$

Section 2.3 (15 pts)

Q: Using row operations find the Determinant of the follow matrix:

$$\begin{bmatrix} 1 & 5 & 2 & 3 \\ 3 & 8 & -1 & 6 \\ -1 & 0 & -4 & 0 \\ 5 & 2 & 1 & 9 \end{bmatrix}$$

A: Start by trying to create an upper-triangular matrix.

$$\begin{bmatrix} 1 & 5 & 2 & 3 \\ 3 & 8 & -1 & 6 \\ -1 & 0 & -4 & 0 \\ 5 & 2 & 1 & 9 \end{bmatrix} \Rightarrow 3R_1 - R_2 \Rightarrow \begin{bmatrix} 1 & 5 & 2 & 3 \\ 0 & 7 & 7 & 3 \\ -1 & 0 & -4 & 0 \\ 5 & 2 & 1 & 9 \end{bmatrix} \quad (-1)$$

$\rightarrow R_2$

(-3)

$$\Rightarrow R_1 + R_3 \rightarrow R_3 \Rightarrow \begin{bmatrix} 1 & 5 & 2 & 3 \\ 0 & 7 & 7 & 3 \\ 0 & 5 & -2 & 3 \\ 5 & 2 & 1 & 9 \end{bmatrix} \qquad (1)$$

(-1)

$$\Rightarrow 5R_1 - R_4 \rightarrow R_4 \Rightarrow \begin{bmatrix} 1 & 5 & 2 & 3 \\ 0 & 7 & 7 & 3 \\ 0 & 5 & -2 & 3 \\ 0 & 23 & 9 & 6 \end{bmatrix} \qquad ~~(-5)~~$$

(-5)

$$\Rightarrow 23R_3 - 5R_4 \rightarrow R_4 \Rightarrow \begin{bmatrix} 1 & 5 & 2 & 3 \\ 0 & 7 & 7 & 3 \\ 0 & 5 & -2 & 3 \\ 0 & 0 & -91 & 39 \end{bmatrix} \qquad ~~(23)~~(-5)$$

$$\Rightarrow\ 5R_2 - 7R_3 \rightarrow\ R_3 \Rightarrow \begin{bmatrix} 1 & 5 & 2 & 3 \\ 0 & 7 & 7 & 3 \\ 0 & 0 & 49 & 3 \\ 0 & 0 & -91 & 39 \end{bmatrix} \qquad (-5)(7)$$

$$\Rightarrow\ 91R_3 + 49R_4 \rightarrow\ R_4 \Rightarrow \begin{bmatrix} 1 & 5 & 2 & 3 \\ 0 & 7 & 7 & 3 \\ 0 & 0 & 49 & 3 \\ 0 & 0 & 0 & 2184 \end{bmatrix} \qquad (91)(49)$$

$k\det(A) = \det(B)$

$\det(B) = (1)(7)(49)(2184) = 749112$

$\det(A) = 749112/k \qquad k = (-3)(1)(-5)(23)(-5)(-5)(7)(91)(49) = 269212125$

$$\det(A) = 749112/269212125 = \frac{8}{2875}$$

Section 2.4

1. If A is an n×n matrix and A^{-1} exists, find $\det(A^{-1})$: (4 points)

We know that $A A^{-1} = I$, and also $\det(I) = 1$ (determined by multiplying all the diagonal entries) therefore taking the determinant of both sides of the equation gives us:

$\det(A A^{-1}) = \det(I) = 1$

it is also know that we can break "distribute" the determinant operator:

$\det(A)\det(A^{-1}) = 1$

since determinants are scalar values we can divide each side by det(A):

O.K.

$\det(A^{-1}) = \dfrac{1}{\det(A)}$ which is acceptable because $\det(A) \neq 0$, a fact because A^{-1} exists

2. Which of the following statements are true? (4 points)

definitions: E-E_k are defined as n×n elementary matrices

A and B are any n×n matrices

a: $\det(EB) \neq \det(E)\det(B)$

False

b: $\det(B) = \det(B^T)$

true

c: a possible solution ~~of the~~ product

$\det(E_1)\det(E_2)\ldots\det(E_K)$ is 0

false

3.a: How is the characteristic polynomial determined? (3 points)

A is an n×n matrix and x is a variable, then the c.p. is defined as:

$\det(xI - A)$

b: What is the c.p. of: (4 points)

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 6 & 9 & 7 \\ 2 & 8 & 4 \end{bmatrix}$$

set up the expression $xI - A$:

$$\begin{bmatrix} x-1 & -5 & -3 \\ -6 & x-9 & -7 \\ -2 & -8 & x-4 \end{bmatrix}$$

taking the determinant gives us

$x^3-14x^2-38x-168$

4. Which of the following is/are possible eigenvalue(s) of A? (5 points)

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

a: -1, 1

b: -1

c: $(-2\pm\sqrt{2})/2$

d: 1/2, -1

Only B is correct

7

1. **a.** Find the augmented matrix for the given system of equations.
b. Solve the system of equations using row operations.

$$
\begin{aligned}
x_1 \quad -3x_2 + \; x_3 &= -3 \\
5x_1 \quad +x_2 \quad -x_3 &= 9 \\
2x_1 + 10x_2 \; -2x_3 &= 26
\end{aligned}
$$

Answer:

a.

$$
A = \left[\begin{array}{ccc|c} 1 & -3 & 1 & -3 \\ 5 & 1 & -1 & 9 \\ 2 & 10 & -2 & 26 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & -3 \\ 0 & 16 & -6 & 24 \\ 0 & 16 & -4 & 32 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & -3 \\ 0 & 16 & -6 & 24 \\ 0 & 0 & 2 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 0 & -7 \\ 0 & 16 & 0 & 48 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]
$$

so $x_1 = 2$, $x_2 = 3$, and $x_3 = 4$.

2. List the three types of row operations and make a 3x3 elementary matrix for each one:

____________________, ____________________, and ____________________.

$$
\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \quad \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \quad \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}
$$

Answer:

Swap two rows, multiply a row by a constant, and add a multiple of one row to another

Examples:
$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Good.

3. Find the solutions using $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & -1 \\ 0 & 4 & 1 \end{bmatrix}$:

a. A+B **b.** A(A+B) **c.** tr(B)A^T **d.** $B^T A$

Solution:

a. $\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & -1 \\ 1 & 6 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & -1 \\ 1 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -6 & -2 \\ 21 & 12 & -3 \\ 14 & 14 & 0 \end{bmatrix}$

c. $2\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 6 & 4 \\ -2 & 0 & 2 \end{bmatrix}$ **d.** $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 4 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 9 & -2 \\ 6 & 11 & 4 \\ 0 & -1 & 0 \end{bmatrix}$

6.

$$A = \begin{matrix} 3 & 5 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 1 \end{matrix} \qquad B = \begin{matrix} 1 & 6 & 2 \\ 6 & 2 & 1 \\ 2 & 1 & 3 \end{matrix}$$

Do A and B commute?

Solution:

$$AB = \begin{matrix} 3 & 5 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 1 \end{matrix} \times \begin{matrix} 1 & 6 & 2 \\ 6 & 2 & 1 \\ 2 & 1 & 3 \end{matrix} = \begin{matrix} -31 & 10 & -5 \\ 5 & -25 & 15 \\ 6 & 15 & 0 \end{matrix}$$

Since AB is not symmetric A and B **do not** commute.

1.4

Find the inverse (if it exists) when $A=$

a) $\begin{bmatrix} 3 & 5 \\ 8 & 4 \end{bmatrix}$

$$\frac{1}{(3)(4)-(5)(8)}\begin{bmatrix} 4 & -5 \\ -8 & 3 \end{bmatrix} = \frac{1}{-32}\begin{bmatrix} 4 & -5 \\ -8 & 3 \end{bmatrix}$$

b) $\begin{bmatrix} 1 & 8 \\ 4 & 32 \end{bmatrix}$

Does Not Exist because (1)(32)-(8)(4)=0

1.5

Use row operations to find the inverses of the following matrices:

a) $\begin{bmatrix} 1 & 6 \\ 3 & 12 \end{bmatrix}$

$$\begin{bmatrix} 1 & 6 \\ 3 & 12 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 \\ 0 & -6 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix} \textit{ therefore the inverse is } \begin{bmatrix} -2 & 1 \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix}$$

b) $\begin{bmatrix} 1 & x^3 & 6x^2+2 \\ 0 & 0 & 0 \\ y & 3 & 46 \end{bmatrix}$

Not invertible because of the row of zero's.

c) $\begin{bmatrix} 1 & 6 & 4 \\ 0 & -4 & 0 \\ 0 & 3 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & -4 & 0 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 4 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & \frac{3}{4} & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 6 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & \frac{3}{8} & \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\frac{15}{2} & -2 \\ 0 & -\frac{1}{4} & 0 \\ 0 & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

$$\textit{therefore the inverse is } \begin{bmatrix} 1 & -\frac{15}{2} & -2 \\ 0 & -\frac{1}{4} & 0 \\ 0 & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

1.6

Solve the system by inverting the coefficient matrix and using theorem 1.6.2

$x+5y=2$

$6y+3z=7$

$-x+7y-z=-20$

$$\sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & 6 & 3 \\ -1 & 7 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -20 \end{bmatrix}$$

$$\textit{the inverse of the coefficient matrix is } \begin{bmatrix} 1 & 5 & 0 \\ 0 & 6 & 3 \\ -1 & 7 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix}1&5&0\\0&1&\frac{1}{2}\\0&2&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&\frac{1}{6}&0\\-1&0&1\end{bmatrix}\sim\begin{bmatrix}1&5&0\\0&1&\frac{1}{2}\\0&0&-2\end{bmatrix}\begin{bmatrix}1&0&0\\0&\frac{1}{6}&0\\-1&-\frac{1}{3}&1\end{bmatrix}\sim\begin{bmatrix}1&5&0\\0&1&\frac{1}{2}\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&\frac{1}{6}&0\\\frac{1}{2}&\frac{1}{6}&-\frac{1}{2}\end{bmatrix}\sim\begin{bmatrix}1&5&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-\frac{1}{4}&\frac{1}{12}&\frac{1}{4}\\\frac{1}{2}&\frac{1}{6}&-\frac{1}{2}\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\frac{9}{4}&-\frac{5}{12}&-\frac{5}{4}\\-\frac{1}{4}&\frac{1}{12}&\frac{1}{4}\\\frac{1}{2}&\frac{1}{6}&-\frac{1}{2}\end{bmatrix}\ \textit{therefore the inverse of the coefficient matrix is}\ \begin{bmatrix}\frac{9}{4}&-\frac{5}{12}&-\frac{5}{4}\\-\frac{1}{4}&\frac{1}{12}&\frac{1}{4}\\\frac{1}{2}&\frac{1}{6}&-\frac{1}{2}\end{bmatrix}$$

left multiply that by the 3x1 matrix to solve for the variable matrix

$$\begin{bmatrix}\frac{9}{4}&-\frac{5}{12}&-\frac{5}{4}\\-\frac{1}{4}&\frac{1}{12}&\frac{1}{4}\\\frac{1}{2}&\frac{1}{6}&-\frac{1}{2}\end{bmatrix}\begin{bmatrix}2\\7\\-20\end{bmatrix}=\begin{bmatrix}26.5833\\-4.9167\\12.1667\end{bmatrix}$$?

7.

$$A = \begin{matrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 5 & -3 & 2 \end{matrix}$$

Find A^{-1} using its adjoint.

Solution:

$$A^{-1} = \frac{1}{\det A} adj(A)$$

Det A = (1)(2)(2) + (3)(-3)(2) + (1)(1)(5) - (5)(2)(2) - (3)(1)(2) - (1)(-3)(1)
= 4+18+5-20-6+3
= 4

$$C_{11} = \det \begin{matrix} 2 & 1 \\ -3 & 2 \end{matrix} = 7 \qquad C_{12} = -\det \begin{matrix} 3 & 1 \\ 5 & 2 \end{matrix} = -1 \qquad C_{13} = \det \begin{matrix} 3 & 2 \\ 5 & -3 \end{matrix} = -19$$

$$C_{21} = -\det \begin{matrix} 1 & 2 \\ -3 & 2 \end{matrix} = 8 \qquad C_{22} = \det \begin{matrix} 1 & 2 \\ 5 & 2 \end{matrix} = -8 \qquad C_{23} = -\det \begin{matrix} 1 & 1 \\ 5 & -3 \end{matrix} = 8$$

$$C_{31} = \det \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} = -3 \qquad C_{32} = -\det \begin{matrix} 1 & 2 \\ 3 & 1 \end{matrix} = 5 \qquad C_{33} = \det \begin{matrix} 1 & 1 \\ 3 & 2 \end{matrix} = -1$$

$$\text{Adj } A = \begin{matrix} 7 & -1 & -19 \\ 8 & -8 & 8 \\ -3 & 5 & -1 \end{matrix}^{T} = \begin{matrix} 7 & 8 & -3 \\ -1 & -8 & 5 \\ -19 & 8 & -1 \end{matrix}$$

$$\mathbf{A}^{-1} = \begin{matrix} \frac{7}{4} & 2 & \frac{-3}{4} \\ \frac{-1}{4} & -2 & \frac{5}{4} \\ \frac{-19}{4} & 2 & \frac{-1}{4} \end{matrix}$$

8.

$$A=\begin{matrix} 2 & 3 & -4 & 1 \\ 1 & -2 & -2 & 1 \\ 3 & 5 & -6 & 2 \\ 2 & -5 & -4 & 2 \end{matrix}$$

Find the determinant of A

Solution:

Det A= 0 since by a theorem a matrix with two proportional rows or columns has a determinant of 0 and column 1 of A need only be multiplied by -2 to create column 3.

9. Evaluate the determinant of the following matrices by reducing the matrix to row-echelon form.

a) $\begin{bmatrix} 4 & 8 & 12 \\ 3 & 5 & 7 \\ 2 & 3 & 4 \end{bmatrix}$ **b)** $\begin{bmatrix} 3 & 1 & 5 & 4 \\ 2 & 6 & 2 & 8 \\ 0 & 7 & 1 & 9 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

Solution:

a) $\det\begin{bmatrix} 4 & 8 & 12 \\ 3 & 5 & 7 \\ 2 & 3 & 4 \end{bmatrix}=(4)\det\begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 2 & 3 & 4 \end{bmatrix}=(4)\det\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}=(-4)\det\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}=$

(-4)(1)(1)(0)=0.

b) $\det\begin{bmatrix} 3 & 1 & 5 & 4 \\ 2 & 6 & 2 & 8 \\ 0 & 7 & 1 & 9 \\ 1 & 2 & 3 & 4 \end{bmatrix}=(3)\det\begin{bmatrix} 1 & 1/3 & 5/3 & 4/3 \\ 0 & 16/3 & -4/3 & 16/3 \\ 0 & 7 & 1 & 9 \\ 0 & 5/3 & 4/3 & 8/3 \end{bmatrix}=(16)\det\begin{bmatrix} 1 & 1/3 & 5/3 & 4/3 \\ 0 & 1 & -1/4 & 1 \\ 0 & 0 & 11/4 & 2 \\ 0 & 0 & 7/4 & 1 \end{bmatrix}=$

$(44)\det\begin{bmatrix} 1 & 1/3 & 5/3 & 4/3 \\ 0 & 1 & -1/4 & 1 \\ 0 & 0 & 1 & 8/11 \\ 0 & 0 & 0 & -3/11 \end{bmatrix}$=(44)(1)(1)(1)(-3/11)=-12.

10. If $A=\begin{bmatrix}1 & 2 & 3\\ 4 & 3 & 1\\ 0 & 5 & 2\end{bmatrix}$, find $\det(A^{-1})$.

Solution:

$$\det\begin{bmatrix}1 & 2 & 3\\ 4 & 3 & 1\\ 0 & 5 & 2\end{bmatrix}=\det\begin{bmatrix}1 & 2 & 3\\ 0 & -5 & -11\\ 0 & 0 & -9\end{bmatrix}=(1)(-5)(-9)=45, \text{ so } \det(A^{-1})=1/\det(A)=1/45.$$

1.) (15 Points)
Find the solution set for the system of linear equations

$$\begin{aligned} 2x + 4y - 6z &= 3 \\ x + y \quad\quad &= 3 \\ 3y + 9z &= 18 \end{aligned}$$

Solution
The system of equations can be rewritten in matrix form:

$$\begin{bmatrix} 2 & 4 & -6 & 3 \\ 1 & 1 & 0 & 3 \\ 0 & 3 & 9 & 18 \end{bmatrix}$$

which row-reduces as follows

$$\begin{bmatrix} 1 & 2 & -3 & \frac{3}{2} \\ 0 & -1 & 0 & 3 \\ 0 & 3 & 9 & 18 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & \frac{9}{2} \\ 0 & 1 & -3 & -\frac{3}{2} \\ 0 & 0 & 18 & \frac{45}{2} \end{bmatrix} \sim \begin{bmatrix} 6 & 0 & 18 & 27 \\ 0 & 6 & -18 & -9 \\ 0 & 0 & 18 & \frac{45}{2} \end{bmatrix} \sim$$

$$\begin{bmatrix} 6 & 0 & 0 & \frac{9}{2} \\ 0 & 6 & 0 & \frac{27}{2} \\ 0 & 0 & 18 & \frac{45}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} \\ 0 & 1 & 0 & \frac{9}{4} \\ 0 & 0 & 1 & \frac{5}{4} \end{bmatrix}$$

Which is code for $x = 3/4$, $y = 9/4$, $z = 5/4$

2.) (10 Points)

For Matrices $A = \begin{bmatrix} 5 & 2 \\ 1 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 1 \\ 4 & 6 \\ 11 & 7 \end{bmatrix}$ $C = \begin{bmatrix} 4 & 1 & 8 \\ 5 & 2 & 1 \end{bmatrix}$

a) What is the product of AB?
b.) What is the product of BC?
c.) What is the product of CA?

Solution

a.) Undefined

b.) $\begin{bmatrix} 3 & 1 \\ 4 & 6 \\ 11 & 7 \end{bmatrix} \begin{bmatrix} 4 & 1 & 8 \\ 5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 6 & 25 \\ 46 & 16 & 38 \\ 79 & 25 & 95 \end{bmatrix}$

c.) Undefined

3.) (20 Points)

Find the determinant of the matrix A = $\begin{bmatrix} 2 & 3 & 1 & -4 & 6 \\ 4 & 9 & 4 & 1 & 9 \\ 6 & -3 & 2 & 5 & -8 \\ 2 & 7 & 10 & 6 & 4 \\ 7 & 3 & 1 & 0 & 7 \end{bmatrix}$

Solution

Using row operations, make A into an upper-triangular matrix B as follows:

$$\begin{bmatrix} 2 & 3 & 1 & -4 & 6 \\ 4 & 9 & 4 & 1 & 9 \\ 6 & -3 & 2 & 5 & -8 \\ 2 & 7 & 10 & 6 & 4 \\ 7 & 3 & 1 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 1 & -4 & 6 \\ 0 & 3 & 2 & 9 & -3 \\ 0 & -12 & -4 & 17 & -26 \\ 0 & 4 & 9 & 10 & -2 \\ 14 & 6 & 2 & 0 & 14 \end{bmatrix} \sim$$

$$\begin{bmatrix} 2 & 3 & 1 & -4 & 6 \\ 0 & 3 & 2 & 9 & -3 \\ 0 & -12 & -4 & 17 & -26 \\ 0 & 4 & 9 & 10 & -2 \\ 14 & 6 & 2 & 0 & 14 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 1 & -4 & 6 \\ 0 & 3 & 2 & 9 & -3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 12 & 18 & 30 & -6 \\ 0 & -15 & -5 & 28 & 35 \end{bmatrix} \sim$$

$$\begin{bmatrix} 2 & 3 & 1 & -4 & 6 \\ 0 & 3 & 2 & 9 & -3 \\ 0 & 0 & 10 & -6 & 6 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 5 & 73 & 50 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 1 & -4 & 6 \\ 0 & 3 & 2 & 9 & -3 \\ 0 & 0 & 5 & -3 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 76 & 47 \end{bmatrix} \sim$$

$$\begin{bmatrix} 2 & 3 & 1 & -4 & 6 \\ 0 & 3 & 2 & 9 & -3 \\ 0 & 0 & 5 & -3 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 199 \end{bmatrix}$$

det(A) = det(B)/C where C = the product of the numbers written on the side of the matrix reduction correlating to the operations used (swap rows = -1, multiply row by nonzero constant k = k, add the multiple of one row to another = 1). Using this formula we get
det(A) = (2*3*5*1*199)/(1^9*2*3*-1*1/2) = -1990

4.) (20 Points)

For the following system of equations:

$$\begin{aligned} -3y + 2z &= 8 \\ 3x - 5y + z &= 2 \\ x - y &= 4 \end{aligned}$$

a.) Set up the corresponding augmented matrix
b.) Use Gaussian elimination to determine the solution, if possible, for x, y, and z.

Solution

a.)

$$\begin{bmatrix} 0 & -3 & 2 & 8 \\ 3 & -5 & 1 & 2 \\ 1 & -1 & 0 & 4 \end{bmatrix}$$

b.)

$$\begin{bmatrix} 0 & -3 & 2 & 8 \\ 3 & -5 & 1 & 2 \\ 1 & -1 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 4 \\ 0 & -3 & 2 & 8 \\ 3 & -5 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 4 \\ 0 & 1 & -2/3 & -8/3 \\ 3 & -5 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 4 \\ 0 & 1 & -2/3 & -8/3 \\ 0 & -2 & 1 & -10 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 0 & 4 \\ 0 & 1 & -2/3 & -8/3 \\ 0 & 0 & -1/3 & -46/3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 4 \\ 0 & 1 & -2/3 & -8/3 \\ 0 & 0 & 1 & 46 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 4 \\ 0 & 1 & 0 & 28 \\ 0 & 0 & 1 & 46 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 32 \\ 0 & 1 & 0 & 28 \\ 0 & 0 & 1 & 46 \end{bmatrix} \sim$$

Which is code for:

$x = 32$

$y = 28$

$z = 46$

5.) (10 Points)

Compute det(A) using the definition that the determinant of A is the sum of all the signed elementary products of A, when:

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix}$$

Solution

There are only two permutations: $a_{11}a_{22}$ and $a_{12}a_{21}$. The first is even and the second is odd, therefore: det(A) = 4(5) + (-1)(2)(3) = 14

6.) (20 Points)

Find the inverse of A using Gaussian elimination method, when:

$$A= \begin{bmatrix} 1 & 6 & 2 \\ 3 & 4 & 0 \\ 5 & 8 & 4 \end{bmatrix}$$

Solution

[A | I] =

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 5 & 8 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 6 & 2 & 1 & 0 & 0 \\ 0 & -14 & -6 & -3 & 1 & 0 \\ 0 & -22 & -6 & -5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 6 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3/7 & 3/14 & -1/14 & 0 \\ 0 & 0 & 24/7 & -2/7 & -11/7 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3/7 & 3/14 & -1/14 & 0 \\ 0 & 0 & 1 & -1/12 & -11/24 & 7/24 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 6 & 0 & 7/6 & 11/12 & -7/12 \\ 0 & 1 & 0 & 1/4 & 1/8 & -1/8 \\ 0 & 0 & 1 & -1/12 & -11/24 & 7/24 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 1/6 & 1/6 \\ 0 & 1 & 0 & 1/4 & 1/8 & -1/8 \\ 0 & 0 & 1 & -1/12 & -11/24 & 7/24 \end{array}\right]$$

Thus,

$$A^{-1} = \begin{bmatrix} -1/3 & 1/6 & 1/6 \\ 1/4 & 1/8 & -1/8 \\ -1/12 & -11/24 & 7/24 \end{bmatrix}$$

7.) (10 Points)

Is the product of a symmetric and skew-symmetric matrix symmetric, skew symmetric, or neither? Show why.

Solution

A is symmetric and B skew-symmetric if

$A^T = A$, $B^T = -B$

So, $(AB)^T = B^T * A^T = (-B)(A) = -BA$

Therefore, AB will be skew-symmetric iff A and B commute.

8.) (10 Points)
Find a nonzero 3x3 matrix A, such that A^T =A, and a nonzero 3x3 matrix B, such that B^T= -B.

Solution

A= [1 1 1]
[1 1 1]
[1 1 1]

A^T= [1 1 1]
[1 1 1]
[1 1 1]

✓ A = A^T

B= [0 -1 -1]
[1 0 -1]
[1 1 0]

B^T= [0 1 1]
[-1 0 1]
[-1 -1 0]

–B= [0 1 1]
[-1 0 1]
[-1 -1 0]

✓ –B = B^T

9.) (15 Points)
Find the solution of this system using A^-1. Hint(x=A^-1* b)

$$\begin{aligned} 2x + 4y + z &= 5 \\ 9x + 2y &= 3 \\ x + 3y + z &= 17 \end{aligned}$$

Solution

A= [2 4 1] det(A)=5
[0 9 2]
[1 3 1]

B= [5]
[3]
[17]

The method of finding the inverse

[2 4 1|1 0 0] [1 3 1 | 0 0 1]
[0 9 2|0 1 0] ~ [0 9 2 | 0 1 0] ~
[1 3 1|0 0 1] [0 2 -1| 1 0 -2]

[1 3 1 | 0 0 1] [1 3 1 | 0 0 1]
[0 1 2/9 | 0 1/9 0] ~ [0 1 2/9 | 0 1/9 0] ~
[0 0 -5/9 | 1 2/9 -2] [0 0 1 |-9/5 -2/5 18/5]

[1 3 0 | 9/5 2/5 -13/5] [1 0 0 | 3/5 -1/5 -1/5]
[0 1 0 | 2/5 1/5 -4/5] ~ [0 1 0 | 2/5 1/5 -4/5] ~
[0 0 1 | -9/5 -2/5 -18/5] [0 0 1 | -9/5 -2/5 18/5]

A^-1= [3/5 -1/5 -1/5]
[2/5 1/5 -4/5]
[-9/5 -2/5 18/5]

$A^{-1}b$?

x= -1
y=-11
z=51

10.) (20 Points)
Find the determinant of the following matrix D by reducing to row-echelon form.

$$D = \begin{bmatrix} 1 & 5 & 2 & 4 & 3 \\ -3 & 1 & 0 & -3 & 1 \\ 0 & 0 & 2 & 5 & -6 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 & 2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 5 & 2 & 4 & 3 \\ 0 & 16 & 6 & 9 & 10 \\ 0 & 0 & 2 & 5 & -6 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 2 & 4 & 3 \\ 0 & 16 & 5 & 9 & 10 \\ 0 & 0 & 2 & 5 & -6 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -4 & 8 \end{bmatrix} \sim$$

(-)

$$\begin{bmatrix} 1 & 5 & 2 & 4 & 3 \\ 0 & 16 & 6 & 9 & 10 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 5 & -6 \\ 0 & 0 & 0 & -4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 2 & 4 & 3 \\ 0 & 16 & 6 & 9 & 10 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -1 & -8 \\ 0 & 0 & 0 & -4 & -8 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 5 & 2 & 4 & 3 \\ 0 & 16 & 6 & 9 & 10 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 40 \end{bmatrix}$$

missing an operation changing the det.

det(d) = 1*16*1*1*40 = 640

5pts

1. Consider the matrices

$$A=\begin{bmatrix}3 & 0\\ -1 & 2\\ 1 & 1\end{bmatrix}\quad B=\begin{bmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{bmatrix}\quad C=\begin{bmatrix}6 & 1 & 3\\ -1 & 1 & 2\\ 4 & 1 & 3\end{bmatrix}.$$

Compute the following:

a) 5(B-C)

b) -3(B+2C)

c) $(BA)^T$

d) $tr(BB^T)$

e) $2C^T + 3B^T$

Solution

a)

Since 5 is a scalar we take $B-C=\begin{bmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{bmatrix}-\begin{bmatrix}6 & 1 & 3\\ -1 & 1 & 2\\ 4 & 1 & 3\end{bmatrix}=\begin{bmatrix}-5 & 4 & -1\\ 0 & -1 & -1\\ -1 & 1 & 1\end{bmatrix}$.

5 times $B-C=\begin{bmatrix}-25 & 20 & -5\\ 0 & -5 & -5\\ -5 & 5 & 5\end{bmatrix}$.

b) $-3(B+2C)=-3\left(\begin{bmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{bmatrix}+2\begin{bmatrix}6 & 1 & 3\\ -1 & 1 & 2\\ 4 & 1 & 3\end{bmatrix}\right)$.

This yields: $-3(B+2C)=-3\left(\begin{bmatrix}13 & 7 & 8\\ -3 & 2 & 5\\ 11 & 4 & 10\end{bmatrix}\right)=\begin{bmatrix}-39 & -21 & -24\\ 9 & -6 & -15\\ -33 & -12 & -30\end{bmatrix}$.

c)first we take $BA = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} * \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ -2 & 1 \\ 11 & 8 \end{bmatrix}$.

Then taking the transverse we get $(BA)^T = \begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$.

d) $tr(BB^T)$

first we find $B^T = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix}$. Then

$$BB^T = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} * \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 30 & 1 & 21 \\ 1 & 2 & 1 \\ 21 & 1 & 29 \end{bmatrix}.$$

Then the trace = 30 + 2 + 29 = 61.

e) $2C^T + 3B^T$

We get that $C^T = \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$, and from the previous problem we get $B^T = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix}$.

Therefore $2C^T + 3B^T = \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix} = \begin{bmatrix} 15 & -5 & 17 \\ 17 & 2 & 8 \\ 12 & 7 & 18 \end{bmatrix}$.

10pts

2.)Find if the matrix has an inverse using the form [A|I] to get $\left[I|A^{-1}\right]$. If the matrix is invertible, check your answer by multiplication.

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Solution:

We start with

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ adding -1 times row 1 to row three gives

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$ adding -1 times row 2 to row 3 gives

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right]$ multiplying row 3 by -1/2 gives

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array}\right]$ adding -1 times row 3 to row 1 gives

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array}\right]$ therefore the inverse is

$\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$.

Checking this with multiplication gives $AA^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} * \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

15pts

3. Solve the systems simultaneously.

$$\begin{aligned} x_1 - 5x_2 &= b_1 \\ 3x_1 + 2x_2 &= b_2 \end{aligned} \quad \text{where} \quad \begin{aligned} a)b_1 &= 1, b_2 = 4 \\ b)b_1 &= -2, b_2 = 5 \end{aligned}$$

Solution

We start by creating the augmented matrix for the systems and putting the ~~solutions~~ r.h.s's side by side with the matrix.

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 3 & 2 & 4 & 5 \end{array}\right]$$ adding -3 times row 1 to row 2 gives

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 0 & 17 & 1 & 11 \end{array}\right]$$ multiplying row 2 by 1/17 yields

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 0 & 1 & \frac{1}{17} & \frac{11}{17} \end{array}\right]$$ adding 5 times row 2 to row 1 produces

$$\left[\begin{array}{cc|c|c} 1 & 0 & \frac{22}{17} & \frac{21}{17} \\ 0 & 1 & \frac{1}{17} & \frac{11}{17} \end{array}\right]$$

therefore

a) $x_1 = 22/17$, $x_2 = 1/17$

b) $x_1 = 21/17$, $x_2 = 11/17$

10pts

4. Solve the following system of equations by using the inverted coefficient matrix:

$$5x_1 + 3x_2 + 2x_3 = 4$$
$$3x_1 + 3x_2 + 2x_3 = 2$$
$$x_2 + \ x_3 = 5$$

Solution: This system can be written as $\mathbf{Ax} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

We can calculate the inverse of A:

$$\left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{4}{5} & -\frac{3}{5} & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{2} & \frac{5}{6} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{2} & \frac{5}{6} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{2} & \frac{5}{6} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{5}{2} & -2 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 5 & 3 & 0 & -2 & 5 & -6 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{5}{2} & -2 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & \frac{5}{2} & -\frac{5}{2} & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{5}{2} & -2 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{5}{2} & -2 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{bmatrix}$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix}$$

Therefore,

$$x_1 = 1, \ x_2 = -11, \ x_3 = 16.$$

20pts

5. Find the conditions that the b's must satisfy for the system to be consistent:

$$x_1 - x_2 + 3x_3 + 2x_4 = b_1$$
$$-2x_1 + x_2 + 5x_3 + x_4 = b_2$$
$$-3x_1 + 2x_2 + 2x_3 - x_4 = b_3$$
$$4x_1 - 3x_2 + x_3 + 3x_4 = b_4$$

Solution: All we need to do is set up a partitioned matrix and solve:

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ -2 & 1 & 5 & 1 & b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1+b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1+b_2 \\ 0 & -1 & 11 & 5 & 3b_1+b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1+b_2 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1+b_2 \\ 4 & -3 & 1 & 3 & b_4 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1+b_2 \\ 0 & 1 & -11 & 5 & -4b_1+b_4 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1+b_2 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_4 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1+b_2 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_4 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \end{array}\right]$$

We can see that the system will be consistent if and only if:

$$\begin{array}{l} 0=-2b_1+b_2+b_4 \\ 0=b_1-b_2+b_3 \end{array}, \text{ or re-written: } \begin{array}{l} b_1=b_3+b_4 \\ b_2=2b_3+b_4 \end{array}$$

10pts

6. Solve the following system of linear equations using Gaussian elimination and back substitution.

$$\begin{array}{l} w+x+y-z=2 \\ 2w-3x-2y+z=2 \\ w+2x-3y+4z=0 \\ w \quad\quad +y-z=3 \end{array}$$

Solution:

First, we find the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 2 & -3 & -2 & 1 & 2 \\ 1 & 2 & -3 & 4 & 0 \\ 1 & 0 & 1 & -1 & 3 \end{array}\right]$$

Next, we reduce the matrix to row-eschelon form.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 0 & -5 & -4 & 3 & -2 \\ 0 & 1 & -4 & 5 & -2 \\ 0 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & -4 & 5 & -2 \\ 0 & -5 & -4 & 3 & -2 \\ 0 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & -4 & 5 & -2 \\ 0 & 0 & -24 & 28 & -12 \\ 0 & 0 & -4 & 5 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & -4 & 5 & -2 \\ 0 & 0 & 1 & (-7)/6 & ½ \\ 0 & 0 & -4 & 5 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & -4 & 5 & -2 \\ 0 & 0 & 1 & (-7)/6 & ½ \\ 0 & 0 & 0 & ⅓ & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & -4 & 5 & -2 \\ 0 & 0 & 1 & (-7)/6 & ½ \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

This augmented matrix yields the following system of linear equations:

$w + x + y - z = 2$
$x - 4y + 5z = -2$
$y - 7/6z = 1/2$
$z = 3$

Using back substitution, we replace $z = 3$ into the equation above:

$y - (7/6) * 3 = 1/2 \quad \sim \quad y - 7/2 = 1/2 \quad \sim \quad y = 4$

We then replace $y = 4$ and $z = 3$ into the equation above that:

$x - 4 * 4 + 5 * 3 = -2 \quad \sim \quad x - 16 + 15 = -2 \quad \sim \quad x = -1$

We then replace $x = -1$, $y = 4$, and $z = 3$ into the equation above that:

$w - 1 + 4 - 3 = 2 \quad \sim \quad w = 2$

Thus we have the solution set $\{w = 2, x = -1, y = 4, z = 3\}$.

25pts

7. You have the following equation: $y = ax^3 + bx^2 + cx + d$.
By inspecting the graph, you see that points (0,4), (1,3), (2,6) and (-1,3) lie on the graph.
Solve for a, b, c, and d.

Solution: To begin, create 4 linear equations, placing the (x,y) values of the points into the equation:

$4 = a(0)^3 + b(0)^2 + c(0) + d$
$3 = a(1)^3 + b(1)^2 + c(1) + d$
$3 = a(-1)^3 + b(-1)^2 + c(-1) + d$
$6 = a(2)^3 + b(2)^2 + c(2) + d$

The first thing to notice is the first equation, which says $d = 4$.
Replace d with the value "4" in the other three equations, and compute to get:

$a + b + c = -1$
$-a + b - c = -1$
$8a + 4b + 2c = 2$

Creat an augmented matrix for the system of linear equations:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 8 & 4 & 2 & 2 \end{array}\right]$$ Add the first row to the second: $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 2 & 0 & -2 \\ 8 & 4 & 2 & 2 \end{array}\right]$$

Reduce: $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 2 & 0 & -2 \\ 0 & -4 & -6 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -6 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

From the last two equations we have b = -1 and c = -1
Placing these values in equation #1, we have a - 1 - 1 = -1 or a = 1.
So a = 1, b = -1, c = -1, and d = 4. Placing these values in the original equation, our solution is: $y = x^3 - x^2 - x + 4$

Good

10pts 8. Find the determinant of the matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & -1 \\ 2 & 0 & -3 \end{bmatrix}$ using the definition of a determinant.

Solution: The determinant for matrix A = Σ signσ * (A(1),σ(1))*((A(2),σ(2))*…*(A(n),σ(n)) for all permutations of σ in S(n).

In this particular matrix, n = 3. We will list the permutations for σ.

123	A(1,1)*A(2,2)*A(3,3)	=1*4*-3	=-12	sign = 1 (0 d.o. pairs)
132	A(1,1)*A(2,3)*A(3,2)	=1*-1*0	=0	sign = -1 (1 d.o. pairs)
213	A(1,2)*A(2,1)*A(3,3)	=3*-1*-3	=9	sign = -1 (1 d.o. pairs)
231	A(1,2)*A(2,3)*A(3,1)	=3*-1*2	=-6	sign = 1 (2 d.o. pairs)
312	A(1,3)*A(2,1)*A(3,2)	=2*-1*0	=0	sign = 1 (2 d.o. pairs)
321	A(1,3)*A(2,2)*A(3,1)	=2*4*2	=16	sign = -1 (3 d.o. pairs)

Now we take the sum of the values multiplied by their respective signs:
-12(1) + 0(-1) + 9(-1) + -6(1) + 0(1) + 16(-1) = -19
Thus the determinant of A is -19.

Good

30pts

9. What is the determinant of the matrix AB, where A is an upper triangular matrix and B is a lower triangular matrix? det(AB) = detA detB = adf · gil.

Solution:

Let matrix A = $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ and matrix B = $\begin{bmatrix} g & 0 & 0 \\ h & i & 0 \\ j & k & l \end{bmatrix}$

Then the matrix AB =
AB = $\begin{bmatrix} ag+bh+cj & bi+ck & cl \\ dh+ej & di+ek & el \\ fj & fk & fl \end{bmatrix}$

To take the determinant of this matrix, which is 3x3, rewrite the two left rows to the right of the matrix, as follows: $\begin{bmatrix} ag+bh+cj & bi+ck & cl \\ dh+ej & di+ek & el \\ fj & fk & fl \end{bmatrix}$ $\begin{matrix} ag+bh+cj \\ dh+ej \\ fj \end{matrix}$

The determinant will be the sum of the diagonals multiplied together.
If the diagonal is going down and to the right, it has a positive sign.
If the diagonal is going up and to the right, it has a negative sign.

det(AB) = (ag+bh+cj)(di+ek)(fl) + (bi+ck)(el)(fj) + (cl)(dh+ej)(fk) -
[(cl)(di+ek)(fj) + (ag+bh+cj)(el)(fk) + (bi+ck)(dh+ej)(fl)]

= (adgi+bdhi+cdij+aegk+behk+cejk)(fl) + befilj + cefjkl + cdfhkl + cefjkl -
[cdfijl + cefjkl + aefgkl + befhkl + cefjkl + (bdhi+beij+cdhk+cejk)(fl)]

= adfgil + bdfhil + cdfijl + aefgkl + befhkl + cefjkl + befijl + cefjkl + cdfhkl + cefjkl -
(cdfijl + cefjkl + aefgkl + befhkl + cefjkl + bdfhil + befijl + cdfhkl + cefjkl)

= adfgil + (bdfhil - bdfhil) + (cdfijl - cdfijl) + (aefgkl - aefgkl) + (befhkl - befhkl) +
(cefjkl - cefjkl) + (befijl - befijl) + (cefjkl - cefjkl) + (cdfhkl - cdfhkl) + (cefjkl - cefjkl)

= adfgil + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0

= adfgil

But adfgil is just the entries of the two main diagonals multiplied together.

Thus the determinant of the product of an upper triangular matrix and a lower triangular matrix is the product of the entries in the main diagonal of both matrices.

15pts

10. Find A^{-1} using its adjoint:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

Solution:

The cofactors of A are

$C_{11} = 12$ $\quad C_{12} = 6$ $\quad C_{13} = -16$

$C_{21} = 4$ $\quad C_{22} = 2$ $\quad C_{23} = 16$

$C_{31} = 12$ $\quad C_{32} = -10$ $\quad C_{33} = 16$

So the matrix of cofactors is

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

and the adjoint of A is

$$\text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}.$$

The determinant of A is

$$\det(A) = \begin{vmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ 6 & 3 \end{vmatrix} - (-4)\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix}$$

$$= 2(12) - (-4)(10) = 64.$$

Thus

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & \frac{-10}{64} \\ \frac{-16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}.$$

1. For the following systems of equations, find the augmented matrix if the system is linear. If the system is not linear, identify it as such.

a) $\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 6x_2 &= 5 \\ 7x_1 + 3x_2 &= 0 \end{aligned}$

b) $\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 4 \end{aligned}$

c) $\begin{aligned} x_1^2 + 3x_2 + x_3 &= 1 \\ \sqrt{x_1} + 7x_2 - 4x_3 &= 5 \\ -4x_1 + 9x_2 &= 2 \end{aligned}$

Solution:

a) $\begin{bmatrix} 3 & -2 & -1 \\ 4 & 6 & 5 \\ 7 & 3 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

c) Non-linear

2. Find the inverse of A using elementary row operations $[A \mid I] \rightarrow [I \mid A]$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & -5 & -1 & 0 & 1 \end{array}\right]$ *Add −2 times the first row to the second row and −1 times the first row to the third row*

$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$ *Add 2 times the second row to the third row*

$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$ *Multiply row three by −1*

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \quad \textit{Add 3 times the third row to the second row and } -3 \textit{ times the third row to the first row}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \quad \textit{Add } -2 \textit{ times the second row to the first row}$$

3. Given matrices [A], [B], and [C], compute the following if possible:

$$A = \begin{bmatrix} 1 & 8 & 3 \\ 6 & 4 & 9 \\ 2 & 7 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 10 & 0 & 5 \\ 3 & 8 & 6 \\ 1 & 7 & 9 \end{bmatrix} \qquad C = \begin{bmatrix} 45 & 3 & 18 & 5 \\ 3 & 7 & 5 & 9 \\ 10 & 12 & 21 & 3 \\ 5 & 10 & 3 & 45 \end{bmatrix}$$

a) $6(A-4B)$ b) $\frac{1}{2}(AC-B^T)-(CA-B^T)$ c) $2(BA+A^T)^T$

Solution:

a)

$$6\left(\begin{bmatrix} 1 & 8 & 3 \\ 6 & 4 & 9 \\ 2 & 7 & 5 \end{bmatrix} - 4\begin{bmatrix} 10 & 0 & 5 \\ 3 & 8 & 6 \\ 1 & 7 & 9 \end{bmatrix}\right) = 6\left(\begin{bmatrix} 1 & 8 & 3 \\ 6 & 4 & 9 \\ 2 & 7 & 5 \end{bmatrix} - \begin{bmatrix} 40 & 0 & 20 \\ 12 & 32 & 24 \\ 4 & 28 & 36 \end{bmatrix}\right)$$

$$= 6\left(\begin{bmatrix} -39 & 8 & -17 \\ -6 & -28 & -15 \\ -2 & -21 & -31 \end{bmatrix}\right) = \begin{bmatrix} -239 & 48 & -102 \\ -36 & -168 & -90 \\ -12 & -126 & -186 \end{bmatrix}$$

b) Undefined

c)

$$2\left(\begin{bmatrix} 10 & 0 & 5 \\ 3 & 8 & 6 \\ 1 & 7 & 9 \end{bmatrix}\begin{bmatrix} 1 & 8 & 3 \\ 6 & 4 & 9 \\ 2 & 7 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 8 & 3 \\ 6 & 4 & 9 \\ 2 & 7 & 5 \end{bmatrix}^T\right)^T = 2\left(\begin{bmatrix} 20 & 115 & 55 \\ 63 & 98 & 111 \\ 61 & 99 & 111 \end{bmatrix} + \begin{bmatrix} 1 & 6 & 2 \\ 8 & 4 & 7 \\ 3 & 9 & 5 \end{bmatrix}\right)^T$$

$$= 2\left(\begin{bmatrix} 21 & 121 & 57 \\ 71 & 102 & 118 \\ 64 & 108 & 116 \end{bmatrix}\right)^T = 2\left(\begin{bmatrix} 21 & 71 & 64 \\ 121 & 102 & 108 \\ 57 & 118 & 116 \end{bmatrix}\right) = \begin{bmatrix} 42 & 142 & 128 \\ 242 & 204 & 216 \\ 114 & 236 & 232 \end{bmatrix}$$

– worded badly

4. If A and B are matrices such that $AB = BA$ then A and B are said to commute. Determine which of the following commute. Then, using simple row operations, find an upper triangular matrix ($\underset{..}{A}$) of the symmetric product.

$$\text{a)} \begin{bmatrix} 6 & 8 & 1 \\ 8 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 42 & 16 & 7 \\ 56 & 0 & 14 \\ 7 & 4 & 35 \end{bmatrix} \qquad \text{b)} \begin{bmatrix} 2 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 9 & 9 \\ 9 & 10 & 9 \\ 9 & 9 & 10 \end{bmatrix}$$

Solution:

'a' does not commute, 'b' does. To solve for an upper triangular matrix:

$$\begin{bmatrix} 10 & 9 & 9 \\ 9 & 10 & 9 \\ 9 & 9 & 10 \end{bmatrix} \sim \text{Subtract the third row from the second to get} \sim \begin{bmatrix} 10 & 9 & 9 \\ 9 & 10 & 9 \\ 0 & 1 & -1 \end{bmatrix} \sim \text{Multiply the}$$

$$\text{third row by -10 and add to the second row} \sim \begin{bmatrix} 10 & 9 & 9 \\ 9 & 10 & 9 \\ 0 & 0 & 19 \end{bmatrix} \sim \text{Multiply the first row by 9, the}$$

$$\text{second row by -10 and add together} \sim \begin{bmatrix} 10 & 9 & 9 \\ 0 & -19 & -9 \\ 0 & 0 & 19 \end{bmatrix}$$

$$\text{Upper triangular matrix } \underset{..}{A} = \begin{bmatrix} 10 & 9 & 9 \\ 0 & -19 & -9 \\ 0 & 0 & 19 \end{bmatrix}$$

5. Find A^{-1} by using its adjoint.

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} \qquad \textit{By inspection}: \ \det A = 1x2x3 = 6$$

$$C_{11} = \begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} = 6 \qquad C_{12} = -\begin{vmatrix} 0 & 4 \\ 0 & 3 \end{vmatrix} = 0 \qquad C_{13} = \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{21} = -\begin{vmatrix} 3 & 5 \\ 0 & 3 \end{vmatrix} = -9 \qquad C_{22} = \begin{vmatrix} 1 & 5 \\ 0 & 3 \end{vmatrix} = 3 \qquad C_{23} = -\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{31} = \begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} = 2 \qquad C_{32} = -\begin{vmatrix} 1 & 5 \\ 0 & 4 \end{vmatrix} = -4 \qquad C_{33} = \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = 2$$

$$adj(A) = \begin{bmatrix} 6 & 0 & 0 \\ -9 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix}^T = \begin{bmatrix} 6 & -9 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} adj(A) = \frac{1}{6}\begin{bmatrix} 6 & -9 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

6. Find the inverse (A^{-1}), then the inverse of the transpose $(A^T)^{-1}$, followed by the transpose of the inverse $(A^{-1})^T$:

$$\begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$$

Solution:

We can find the inverse by using the theorem demonstrated in 1.4.5 of the text. To find A^{-1}, we calculate

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} \frac{2}{(3)(2)-(4)(5)} & -\frac{5}{(3)(2)-(4)(5)} \\ -\frac{4}{(3)(2)-(4)(5)} & \frac{3}{(3)(2)-(4)(5)} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{7} & \frac{5}{14} \\ \frac{2}{7} & -\frac{3}{14} \end{bmatrix}$$ is the A^{-1} of the given matrix.

Performing the shift and then applying the same theorem as in the above statement, we can calculate the inverse of the transpose as so:

$A^T = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$, now to take the inverse, by $(A^T)^{-1} =$

$$\begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} \frac{2}{(3)(2)-(4)(5)} & -\frac{4}{(3)(2)-(4)(5)} \\ -\frac{5}{(3)(2)-(4)(5)} & \frac{3}{(3)(2)-(4)(5)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} & \frac{2}{7} \\ \frac{5}{14} & -\frac{3}{14} \end{bmatrix}.$$

Now to finish the explanation, we will calculate the transpose of the inverse,

$$A^{-1} = \begin{bmatrix} -\frac{1}{7} & \frac{5}{14} \\ \frac{2}{7} & -\frac{3}{14} \end{bmatrix} = (A^{-1})^T = \begin{bmatrix} -\frac{1}{7} & \frac{2}{7} \\ \frac{5}{14} & -\frac{3}{14} \end{bmatrix}.$$

We can determine that both $(A^T)^{-1}$ and $(A^{-1})^T$ are equivalent.

7. Find the determinant of the following matrices:

a.) A = $\begin{bmatrix} 5 & 8 \\ 2 & 7 \end{bmatrix}$

b.) B = $\begin{bmatrix} 4 & 5 & 6 \\ 3 & 8 & 9 \\ 1 & 2 & 0 \end{bmatrix}$

Solution:

For part (a) we start by applying methodology of section 2.1.3. This states that the determinant of a 2X2 matrix can be found by the equation:

$$\det(A) = a_{11}a_{22} + (-1)a_{12}a_{21} = ad - bc.$$

By inserting the various entries from the matrix we can find the determinant. Shown as,

$$\det(A) = (5)(7) - (8)(2) = 35 - 16 = 19, \text{ 19 being our solution.}$$

For part (b) we will use the format of solving a 3X3 matrix demonstrated in 2.1.4 of the hand-out. Using the given theorem we can find the determinant, keeping in mind that each variable describes an entry in matrix B:

$$\det(B) = aei - afh + bfg - bdi + cdh - ceg.$$

Now we insert the entries from B that will satisfy this equation,

$$\det(B) = (4)(8)(0) - (4)(9)(2) + (5)(9)(1) - (5)(3)(0) + (6)(3)(2) - (6)(8)(1)$$
$$\det(B) = 0 - 72 + 45 - 0 + 36 - 48$$
$$\det(B) = -39.$$

Therefore our solution, det(B), equals -39.

8. Given the following matrix, find the characteristic polynomial that results from:

$$Z = \begin{bmatrix} 2 & 1 & 1 \\ -3 & 6 & -3 \\ 1 & 4 & 0 \end{bmatrix}$$

Solution:

Using the definition of a characteristic polynomial, we can solve for x in order to find the resulting polynomial. We can do this by taking the determinant as follows,

$$\det(xI - Z).$$

First we compute the value inside of the determinant declaration:

$$xI - Z = \begin{bmatrix} x-2 & -1 & -1 \\ 3 & x-6 & 3 \\ -1 & 4 & 0 \end{bmatrix}.$$

Next we solve the determinant for a 3X3, this time calculating the expressions with x involved as an entry. Using the given equation for the determinant:

$$\det(xI - Z) = aei - afh + bfg - bdi + cdh - ceg$$
$$\det(xI - Z) = (x\text{-}2)(x\text{-}6)(0) - (x\text{-}2)(3)(4) + (\text{-}1)(3)(\text{-}1) - (\text{-}1)(3)(0) + (\text{-}1)(3)(4) - (\text{-}1)(x\text{-}6)(\text{-}1)$$
$$\det(xI - Z) = 0 - 12x - 24 + 3 - 0 - 12 - x - 6$$
$$\det(xI - Z) = -13x - 39.$$

Thus, the characteristic polynomial for the given matrix is -13x - 39.

9. What is the determinant of the following matrix?

$$A = \begin{bmatrix} 1 & k^2 & k \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$$

Solution:

By theorem 2.9:

$$\det A = k \det \begin{bmatrix} 1 & k^2 & k \\ 0 & 1 & 0 \\ 0 & 1 & \frac{1}{k} \end{bmatrix} = k \det \begin{bmatrix} 1 & k^2 & k \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{k} \end{bmatrix} = k\left(\frac{1}{k}\right) = 1$$

10. Solve the following system of equations by Gauss-Jordan Elimination, or show that it has no solution:

$$6x_1 + 4x_2 + x_3 = 20$$
$$x_1 + 4x_2 + x_3 = 10$$
$$x_1 + 2x_2 + x_3 = 8$$

Solution:

Put into matrix form,

$$\begin{bmatrix} 6 & 4 & 1 & 20 \\ 1 & 4 & 1 & 10 \\ 1 & 2 & 1 & 8 \end{bmatrix}$$

Use elementary row operations:

$$\begin{bmatrix} 5 & 2 & 0 & 12 \\ 0 & 2 & 0 & 2 \\ 1 & 2 & 1 & 8 \end{bmatrix}$$ *Subtract the third row from the first and second rows.*

$$\begin{bmatrix} 5 & 0 & 0 & 10 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 6 \end{bmatrix}$$ *Subtract the second row from the first and third rows.*

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 6 \end{bmatrix}$$ *Divide second row by two, divide the first row by 5..*

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$ *Subtract the first from third row.*

The solution is then, $x_1 = 2$, $x_2 = 1$, $x_3 = 4$.

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343 Midterm I

1. Solve the system of equations using Gaussian elimination (15 pts.)

$$3x+2y-z=-15$$
$$5x+3y+2z=0$$
$$-3x-2y+z=15$$
$$3x+y+3z=11$$

Solution: The augmented matrix of the system is

$$\begin{bmatrix} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ -3 & -2 & 1 & 15 \\ 3 & 1 & 3 & 11 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & -1 & -15 \\ 0 & -1/3 & 11/3 & 25 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 26 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & -1 & -15 \\ 0 & -1/3 & 11/3 & 25 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 3 & 2 & -1 & -15 \\ 0 & -1/3 & 11/3 & 25 \\ 0 & 0 & -7 & -56 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & -1 & -15 \\ 0 & -1/3 & 11/3 & 25 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 0 & -7 \\ 0 & -1/3 & 0 & -13/3 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 3 & 2 & 0 & -7 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 0 & -33 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is code for $\begin{matrix} x=-11 \\ y=13 \\ z=8 \end{matrix}$

2. Solve the system A**x**=**b** using A^{-1}

$$A=\begin{bmatrix} 1/5 & 1/5 & -2/5 \\ 2 & 2 & 1 \\ 1/5 & -4/5 & 1/10 \end{bmatrix} \quad \mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \mathbf{b}=\begin{bmatrix} -1/5 \\ 3 \\ -33/10 \end{bmatrix}$$

Solution:

Find A^{-1} by adjoining A with I and reducing A to I.

$$[A|I]=\left[\begin{array}{ccc|ccc} 1/5 & 1/5 & -2/5 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 1/5 & -4/5 & 1/10 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 1/5 & -4/5 & 1/10 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 0 & 5 & -10 & 1 & 0 \\ 0 & -1 & 1/2 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & -1/2 & 1 & 0 & -1 \\ 0 & 0 & 5 & -10 & 1 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & -1/2 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1/5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2/5 & 0 \\ 0 & 1 & 0 & 0 & 1/10 & -1 \\ 0 & 0 & 1 & -2 & 1/5 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3/10 & 1 \\ 0 & 1 & 0 & 0 & 1/10 & -1 \\ 0 & 0 & 1 & -2 & 1/5 & 0 \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} 1 & 3/10 & 1 \\ 0 & 1/10 & -1 \\ -2 & 1/5 & 0 \end{bmatrix}$$

$$A^{-1}\mathbf{b}=\begin{bmatrix} 1 & 3/10 & 1 \\ 0 & 1/10 & -1 \\ -2 & 1/5 & 0 \end{bmatrix}\begin{bmatrix} -1/5 \\ 3 \\ -33/10 \end{bmatrix}=\begin{bmatrix} -13/5 \\ 18/5 \\ 1 \end{bmatrix}$$

$x=-13/5$

$y=18/5$

$z=1$

3. Given the following matrices

$$A=\begin{bmatrix} 1 & 3 & 4 \\ 7 & 2 & 6 \\ 9 & 8 & 5 \end{bmatrix},\ B=\begin{bmatrix} -1 & 2 & 4 \\ -6 & -5 & 7 \\ -12 & -3 & -1 \end{bmatrix},\ C=\begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}$$

Find
(a) A+B
(b) B-A
(c) 2C (15pts.)

Solution: Matrices of the same size—such as A and B—can be added or subtracted from one another by adding or subtracting its corresponding entries.

$$A+B=\begin{bmatrix}1&3&4\\7&2&6\\9&8&5\end{bmatrix}+\begin{bmatrix}-1&2&4\\-6&-5&7\\-12&-3&-1\end{bmatrix}=\begin{bmatrix}0&5&8\\1&-3&13\\-3&5&4\end{bmatrix}$$

$$A-B=\begin{bmatrix}1&3&4\\7&2&6\\9&8&5\end{bmatrix}-\begin{bmatrix}-1&2&4\\-6&-5&7\\-12&-3&-1\end{bmatrix}=\begin{bmatrix}2&1&0\\13&7&-1\\21&11&6\end{bmatrix}$$

The product of a scalar and a matrix is found by multiplying the scalar to each of the entries of the matrix.

$$2C=2\begin{bmatrix}0&3\\2&4\end{bmatrix}=\begin{matrix}0&6\\4&8\end{matrix}$$

4. For the matrix $A=\begin{bmatrix}3&5\\4&7\end{bmatrix}$, compute A^2, A^{-2}, and A^2+3A (15 pts.)

Solution: A^2 is found by multiplying A to itself.

$$A^2=\begin{bmatrix}3&5\\4&7\end{bmatrix}\begin{bmatrix}3&5\\4&7\end{bmatrix}=\begin{bmatrix}29&50\\40&69\end{bmatrix}$$

A^{-2} is found by taking the inverse of A^2.

$$A^{-2}=\begin{bmatrix}29&50\\40&69\end{bmatrix}^{-1}=(1/(29\times 69-50\times 40))\begin{bmatrix}69&-50\\-40&29\end{bmatrix}=\begin{bmatrix}69&-50\\-40&29\end{bmatrix}$$

A^2+3A is also easily found.

$$A^2+3A=\begin{bmatrix}29&50\\40&69\end{bmatrix}+3\begin{bmatrix}3&5\\4&7\end{bmatrix}=\begin{bmatrix}29&50\\40&69\end{bmatrix}+\begin{bmatrix}9&15\\12&21\end{bmatrix}=\begin{bmatrix}38&65\\52&90\end{bmatrix}$$

5. Find A^{-1} using elementary row operations (15 pts.)

$$A=\begin{bmatrix}1/5&1/5&-2/5\\2&2&1\\1/5&-4/5&1/10\end{bmatrix}$$

Solution: Adjoin A with I and apply elementary row operations:

$$\left[\begin{array}{ccc|ccc} 1/5 & 1/5 & -2/5 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 1/5 & -4/5 & 1/10 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 1/5 & -4/5 & 1/10 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 0 & 5 & -10 & 1 & 0 \\ 0 & -1 & 1/2 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & -1/2 & 1 & 0 & -1 \\ 0 & 0 & 5 & -10 & 1 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & -1/2 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1/5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2/5 & 0 \\ 0 & 1 & 0 & 0 & 1/10 & -1 \\ 0 & 0 & 1 & -2 & 1/5 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3/10 & 1 \\ 0 & 1 & 0 & 0 & 1/10 & -1 \\ 0 & 0 & 1 & -2 & 1/5 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 3/10 & 1 \\ 0 & 1/10 & -1 \\ -2 & 1/5 & 0 \end{bmatrix}$$

6. Solve the system of equations simultaneously

$$\begin{aligned} x + 3y + 5z &= b_1 \\ -15x - 30y &= b_2 \\ 2x + 5y + 4z &= b_3 \end{aligned} \qquad \text{(15 pts.)}$$

$$\text{When } \begin{matrix} b_1 = 1 \\ b_2 = 0 \\ b_3 = -1 \end{matrix}\ ,\ \begin{matrix} b_1 = 0 \\ b_2 = 15 \\ b_3 = 1 \end{matrix}\ ,\ \begin{matrix} b_1 = -1 \\ b_2 = -15 \\ b_3 = 0 \end{matrix}$$

Solution: Adjoin the coefficient matrix of the system with the coefficient matrices of **b**:

$$\left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 1 & 0 & -1 \\ -15 & -30 & 0 & 0 & 15 & -15 \\ 2 & 5 & 4 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 1 & 0 & -1 \\ 0 & 15 & 75 & 15 & 15 & -30 \\ 0 & -1 & -6 & -3 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 1 & 0 & -1 \\ 0 & 1 & 5 & 1 & 1 & -2 \\ 0 & -1 & -6 & -3 & 1 & 2 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 1 & 0 & -1 \\ 0 & 1 & 5 & 1 & 1 & -2 \\ 0 & 0 & -1 & -2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 1 & 0 & -1 \\ 0 & 1 & 5 & 1 & 1 & -2 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c|c|c} 1 & 3 & 0 & -9 & 10 & -1 \\ 0 & 1 & 0 & -9 & 11 & -2 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 18 & -23 & 5 \\ 0 & 1 & 0 & -9 & 11 & -2 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

7. Prove that the product of 2 lower triangular matrices is another lower triangular matrix (15 pts.)

Solution: Let $A = \lfloor a_{ij} \rfloor$, $B = \lfloor b_{ij} \rfloor$, and $C = \lfloor c_{ij} \rfloor$ for $C = AB$ where C is the product of the lower triangular matrices A and B. By the nature of lower triangular matrices, C is lower triangular if $c_{ij} = 0$ for $i < j$. Therefore, by the definition of matrix multiplication

$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$

Assuming $i < j$, this can be regrouped into

$$c_{ij} = \underbrace{a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{i(j-1)}b_{(j-1)j}}_{\text{Row \# of b is less than column \# of b}} + \underbrace{a_{ij}b_{jj} + \ldots + a_{in}b_{nj}}_{\text{Row \# of a is less than column \# of a}}$$

So $c_{ij} = 0$.

8. Compute det(A) using the definition of a determinant.

A= $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (15 pts.)

Solution: The definition of a determinant is

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \ldots a_{n\sigma(n)}$$

Where $\operatorname{sgn}(\sigma)$ is -1 multiplied by the number of inversions in σ.
Permutations of n are (1,2) and (2,1).
Using $\sigma(1) = 1$, $\sigma(2) = 2$, the elementary product is $a_{11}a_{22}$ or $1 \times 4 = 4$.
Using $\sigma(1) = 2$, $\sigma(2) = 1$, the elementary product is $a_{12}a_{21}$ or $2 \times 3 = 6$

There are no inversions in (1,2) so its signed elementary product is +4
There is one inversion in (2,1) so its signed elementary product is -6

The sum of all elementary products is -2. So $\det(A) = -2$

9. Find det(A) using cofactor expansion on the 2nd row (15 pts.)

$$A=\begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

Solution: To find det(A) using cofactor expansion on the i^{th} row, use

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

First find the minors of the second row:

$$M_{21} = \det\begin{bmatrix} 4 & 3 \\ 2 & 0 \end{bmatrix} = -6$$
$$M_{22} = \det\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} = -6$$
$$M_{23} = \det\begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix} = -4$$

Then find the cofactors of the second row:

$$C_{21} = (-1)^{2+1} M_{21} = 6$$
$$C_{22} = (-1)^{2+2} M_{22} = -6$$
$$C_{23} = (-1)^{2+3} M_{23} = 6$$

Plug the numbers into the original formula to get

$$\det(A) = \sum_{j=1}^{3} a_{2j} C_{2j}$$
$$= 2(6) + 0(-6) + 1(6) = 18$$

10. (15 pts.) Compute det(A) by reducing A to an upper triangular matrix

$$A=\begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & -2 & 0 & -2 \\ -3 & 1 & -3 & 0 \\ 1 & 9 & 3 & 2 \end{bmatrix}$$

Soulution:

$$\begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & -2 & 0 & -2 \\ -3 & 1 & -3 & 0 \\ 1 & 9 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & 3 & 0 & 5 \\ 0 & -4 & 0 & -16/3 \\ 0 & 4 & -3 & 5 \\ 0 & 8 & 3 & 1/3 \end{bmatrix} \sim \begin{bmatrix} 3 & 3 & 0 & 5 \\ 0 & -4 & 0 & -16/3 \\ 0 & 0 & -3 & -1/3 \\ 0 & 8 & 3 & 1/3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 3 & 3 & 0 & 5 \\ 0 & -4 & 0 & -16/3 \\ 0 & 0 & -3 & -1/3 \\ 0 & 0 & 3 & -31/3 \end{bmatrix} \sim \begin{bmatrix} 3 & 3 & 0 & 5 \\ 0 & -4 & 0 & -16/3 \\ 0 & 0 & -3 & -1/3 \\ 0 & 0 & 0 & -32/3 \end{bmatrix}$$

The product of the main diagonal of A reduced to upper triangular form is

$$3 \times -4 \times -3 \times (-32/3) = -384$$

Because the only row operation we used to obtain a matrix in upper triangular form was adding a multiple of one row to another, we do not need to multiply the product of the main diagonal of that upper triangular matrix by any constant. So

$$\det(A) = -384$$

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Answers and solutions =

1. (15 pts.) Place the following set of linear equations into an augmented matrix A and solve using the Gaussian- Jordan elimination process.

3x + 5y – z = 13
x – 10y + 2z = -12
7x – 2y + 2z = 24

Solution:

$$A = \begin{pmatrix} 3 & 5 & -1 & 13 \\ 1 & 10 & 2 & -12 \\ 7 & -2 & 2 & 24 \end{pmatrix}$$ Interchange row 2 for row 1 then, new row 2 for row 3

$$A = \begin{pmatrix} 1 & -10 & 2 & -12 \\ 7 & -2 & 2 & 24 \\ 3 & 5 & -1 & 13 \end{pmatrix}$$ row2 – 7(row 1)

$$A = \begin{pmatrix} 1 & 10 & 2 & -12 \\ 0 & 68 & -12 & 108 \\ 0 & 35 & -7 & 49 \end{pmatrix}$$ row 2 x 1/4
row 3 x 1/7

$$A = \begin{pmatrix} 1 & -10 & 2 & -12 \\ 0 & 17 & -3 & 27 \\ 0 & 5 & -1 & 7 \end{pmatrix}$$ 17(row 3) – 5(row 2)

$$A = \begin{pmatrix} 1 & -10 & 2 & -12 \\ 0 & 17 & -3 & 27 \\ 0 & 0 & -2 & -16 \end{pmatrix}$$ row 3 x -1/2

$$A = \begin{pmatrix} 1 & -10 & 2 & -12 \\ 0 & 17 & -3 & 27 \\ 0 & 0 & 1 & 8 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -10 & 2 & -12 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 8 \end{pmatrix}$$ row 1 + 10(row 2)

$$A = \begin{pmatrix} 1 & 0 & 2 & 18 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 8 \end{pmatrix}$$ row 1 – 2(row 3)

$$A = \left(\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 8 \end{array}\right) \Rightarrow \begin{array}{l} x = 2 \\ y = 3 \\ z = 8 \end{array}$$

2. (15 pts.) Prove ~~why~~ that? if $4A^2$ is an invertible symmetric matrix, then $2I + 2A^{-1}A$ is also symmetric.

Solution:

but $2I + 2A^{-1}A = 2I + 2I = 4I = (4I)^T$

$4A^2$	$2I + 2A^{-1}A$
[4(A•A)	2I + 2I] x A^{-1}
$4(A^{-1}\bullet A\bullet A)$	$2I A^{-1} + 2I A^{-1}$
4IA	$2A^{-1} + 2A^{-1}$
4A	$4A^{-1}$

I don't get it.

Based on Theorem 1.7.3 "If A is invertible symmetric matrix, then A^{-1} is symmetric." $4A^{-1}$ is the inverse of the invertible symmetric matrix 4A; therefore, if 4A is a invertible symmetric matrix, then its inverse ($4A^{-1}$) must also be symmetric.

3. (20 pts.) Find the "inverse of the following set of linear equations" ? and solve the matrix for x, y, and z using A^{-1} .

w + x + 4y + 4z = 3
w + 3x +7y + 9z = 5
x – 2y – 3z = 4
w – 2x – 4y – 6z = 3

Answer: "|A|I| => |I| A^{-1} |"

Solution:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 1 & 0 & 0 & 0 \\ 1 & 3 & 7 & 9 & 0 & 1 & 0 & 0 \\ 2 & 1 & -2 & -3 & 0 & 0 & 0 & 1 \\ 1 & -2 & -4 & -6 & 0 & 0 & 0 & 1 \end{array}\right)$$
A I

add -1(R1) to R2
add -1(R1) to R3

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 5 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 0 & 0 & 1 & 0 \\ 0 & -3 & -8 & -10 & -1 & 0 & 0 & 1 \end{array}\right)$$

swap R2 with R3 then,
add -2(R2) to R3
add 3(R2) to R4

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 11 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 5 & -4 & 3 & 0 & 1 \end{array}\right)$$

swap R3 with R4 then,
add -7(R3) to R4

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 5 & -4 & 3 & 0 & 1 \\ 0 & 0 & 0 & -24 & 27 & -20 & -2 & -7 \end{array}\right)$$

multiply R4 by -1/24 then,
add -5(R4) to R3
add 3(R4) to R2
add -4(R4) to R1

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 4 & 0 & 11/2 & -10/3 & -1/3 & -7/6 \\ 0 & 1 & -2 & 0 & -27/3 & 5/2 & 5/4 & 7/8 \\ 0 & 0 & 1 & 0 & 13/8 & -7/6 & -5/12 & -11/24 \\ 0 & 0 & 0 & 1 & -9/8 & 5/6 & 1/12 & 7/24 \end{array}\right)$$

add 2(R3) to R2
add -4(R3) to R1

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & -1 & 4/3 & 4/3 & 2/3 \\ 0 & 1 & 0 & 0 & -1/8 & 1/6 & 5/12 & 1/24 \\ 0 & 0 & 1 & 0 & 13/8 & -7/6 & 5/4 & -11/24 \\ 0 & 0 & 0 & 1 & -9/8 & 5/6 & 1/12 & 7/24 \end{array}\right)$$

add -1(R2) to R1

$$\underbrace{\left(\begin{array}{cccc}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{array}\right.}_{I}\left|\underbrace{\begin{array}{cccc} \mathbf{-7/8} & \mathbf{7/6} & \mathbf{11/12} & \mathbf{17/24} \\ \mathbf{-1/8} & \mathbf{1/6} & \mathbf{5/12} & \mathbf{1/24} \\ \mathbf{13/8} & \mathbf{-7/6} & \mathbf{5/4} & \mathbf{-11/24} \\ \mathbf{-9/8} & \mathbf{5/6} & \mathbf{1/12} & \mathbf{7/24} \end{array}}_{A^{-1}}\right)$$

4. (15 pts.) Solve the previous matrix for x, y, and z using A^{-1} .

Theorem 1.6.2 "If A is an invertible n x n matrix, then for each n x n matrix b, the system of equations Ax = b has exactly one solution, namely, $x = A^{-1}$ b."

Solution:

$$\underbrace{\begin{pmatrix} -7/8 & 7/6 & 11/12 & 17/24 \\ -1/8 & 1/6 & 5/12 & 1/24 \\ 13/8 & -7/6 & 5/4 & -11/24 \\ -9/8 & 5/6 & 1/12 & 7/24 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 3 \\ 5 \\ 4 \\ 3 \end{pmatrix}}_{b} = \underbrace{\begin{pmatrix} 9 \\ 2 \\ -4 \\ 2 \end{pmatrix}}_{x}$$

w = 9
x = 3
y = -4
z = 2

5. (15 pts.) If A and B are both invertible, symmetric matrices, and k is some nonzero constant, is C an invertible, symmetric matrix? Explain your reasoning. $C = \{[(kA + B)^2]^{-1}\}^T$

Answer : Yes
Solution:

a) A nonzero constant k times an invertible, symmetric matrix A results in an invertible, symmetric matrix. Theorem 1.7.2
b) Two invertible, symmetric matrices added together result in a invertible, symmetric matrix. Theorem 1.7.2 – not true
c) If a matrix A is invertible and symmetric then it's inverse is also invertible and symmetric. Theorem 1.4.10 and 1.7.4
d) If A is an invertible, symmetric matrix, then it's transpose is also invertible and symmetric. Theorem 1.4.10 and 1.7.2

$I + (-I) = 0$

6. (15 pts.) Given that A is an n x n matrix and $B = A + A^T$ and $C = A - A^T$, prove that B is symmetric and C is skew symmetric. Hint: that means that $B = B^T$ and $C = -C^T$.

Solution: $B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = B$

$C^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -C$

7. (10 pts.) Is it possible to have AB = I if $B \neq A^{-1}$? Justify your answer.

Answer: Yes
Solution:

If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 97 & 67 \end{pmatrix}$

Then AB = I and $B \neq A^{-1}$

8. (20 pts.) Find the inverse of matrix A using its adjoint.

$$A = \begin{pmatrix} 5 & 3 & -1 \\ -2 & 1 & -4 \\ 1 & 2 & -6 \end{pmatrix}$$

Solution:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

det(A) : 5(-6 - -8) – 3(12 - -4) + -1(-4 - 1) = -33
Cofactors for adj(A)

$C_{11} = (-1)^{1+1}M_{11} = (1)[6 - -8] = 2$
$C_{12} = (-1)^{1+2}M_{12} = (-1)[-12 - -4] = 8$
$C_{13} = (-1)^{1+3}M_{13} = (1)[-4 - 1] = -5$
$C_{21} = (-1)^{2+1}M_{21} = (-1)[18 - -2] = -22$
$C_{22} = (-1)^{2+2}M_{22} = (1)[30 - -1] = 31$
$C_{23} = (-1)^{2+3}M_{23} = (-1)[10 - 3] = -7$
$C_{31} = (-1)^{3+1}M_{31} = (1)[-12 - -1] = -11$
$C_{32} = (-1)^{3+2}M_{32} = (-1)[-20 - 2] = 22$
$C_{33} = (-1)^{3+3}M_{32} = (1)[5 - -6] = 11$

Cofactor Matrix

$$\begin{pmatrix} 2 & 8 & -5 \\ -22 & 31 & -7 \\ -11 & 22 & 11 \end{pmatrix} \qquad \text{adj(A)} = [\text{Cofactor Matrix}]^T = \begin{pmatrix} 2 & -22 & -11 \\ 8 & 31 & 22 \\ -5 & -7 & 11 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -2/33 & 22/33 & 11/33 \\ -8/33 & -31/33 & -22/33 \\ 5/33 & 7/33 & -11/33 \end{pmatrix}$$

9. (5 pts.) What is the determinant of the following lower triangular matrix A.

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 5 & 9 & 0 & 0 \\ 7 & 4 & 1 & 0 \\ 3 & 5 & 6 & 2 \end{pmatrix}$$

Solution:

The determinant of a triangular matrix is the product of the diagonal.

det(A) = (2 x 9 x 1 x 2) = 36 ?

10. (10 pts.) a) Find the determinant for matrices A and C by inspection.

what is C?

Solutions:

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

det(A) = 2
Elementary matrix – row 1 x k

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

det(B) = 1

Elementary matrix – row 2 + row 3

b) (10 pts.) Find the determinant of a matrix C, which is the result of the addition of matrix A and matrix B, remembering the properties of the determinant function.

$A = \begin{pmatrix} 2 & 3 \\ -5 & 5 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$

Not true!

Solution:

C = A + B det(C) = det(A) + ~~det(B)~~ = (10 - -15) + (4 - -3) = 32 **det(C) = 32**

According to Theorem 2.3.1, this only works because matrix A and B differ only by one row.

$\det C = \det \begin{bmatrix} 4 & 6 \\ -6 & 7 \end{bmatrix} = 28 + 36 = 64 \neq 32.$

1. Write the augmented matrix of the following systems of linear equations. If none exists, specify why.

a.
$$\begin{aligned} 2x1 - 3x2 + 4x3 &= 2 \\ 10x2 - 6x3 &= 4 \\ 5x1 - 5x3 &= 0 \end{aligned}$$

b.
$$\begin{aligned} 2x1 &= 3x2 + 4 \\ 4 - 2x2 &= 5x3 - 3x2 \\ 7 + x1 + x2 + x3 &= 0 \end{aligned}$$

c.
$$\begin{aligned} 2x1 + 3x2 &= 18 \\ 17x1 - 3x2 &= 5 \\ x1x2 &= 1 \end{aligned}$$

d.
$$\begin{aligned} x1 &= 2 \\ 3/x3 + 4 &= 0 \end{aligned}$$

Solution:

a.
$$\begin{bmatrix} 2 & -3 & 4 & 2 \\ 0 & 10 & -6 & 4 \\ 5 & 0 & -5 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 2 & -3 & 0 & 4 \\ 0 & 1 & -5 & -4 \\ 1 & 2 & 3 & -7 \end{bmatrix}$$

c. Not possible, x1x2 is not linear.

d.
$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 4 & -3 \end{bmatrix}$$

2) Solve the following system of equations using Gaussian Elimination and back-substitution.

$$2x_1 + 3x_2 + 1x_3 = 10$$

$$0x_1 + 3x_2 + 1x_4 = 6$$

$$0x_1 + 3x_2 + 2x_3 = 9$$

$$4x_1 + 6x_2 + 2x_3 = 20$$

Solution:

$$x_z = 2$$

$$x_2 = 1$$

$$x_3 = 3$$

Check:

Since the fourth equation is simply double the first, it can be discarded and we'll still have enough information to complete the problem (3 eqs. and 3 unknowns)

We begin by finding the augmented matrix for the equation. Then we reduce it to row-echelon form.

$$A = \left[\begin{array}{ccc|c} 2 & 3 & 1 & 10 \\ 0 & 3 & 1 & 6 \\ 0 & 3 & 2 & 9 \end{array}\right] \approx \left[\begin{array}{ccc|c} 2 & 3 & 1 & 10 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right] \approx \left[\begin{array}{ccc|c} 1 & 3/2 & 1/2 & 5 \\ 0 & 1 & 1/3 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

From this we learn that $x_3 = 3$, allowing us to use back-substitution so that

$$x_2 + \frac{1}{3}x_3 = 2 \quad \rightarrow \quad x_2 + \frac{1}{3}(3) = 2 \quad \rightarrow \quad x_2 = 2 - 1 \quad = \quad 1$$

and,

$$x_1 + \frac{3}{2}x_2 + x_3 = 5 \quad \rightarrow \quad x_1 + \frac{3}{2}(1) + \frac{1}{2}(3) = 5 \quad \rightarrow \quad x_1 + 3 = 5 \quad \rightarrow \quad x_1 = 2$$

3) Solve this augmented matrix, A, using Gauss-Jordan elimination.

$$A = \begin{bmatrix} 1 & 5 & -3 & 0 & 19 \\ 3 & 16 & -7 & 2 & 69 \\ 2 & 12 & 3 & 16 & 63 \\ 1 & 3 & -11 & -10 & -25 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 5 & -3 & 0 & 19 \\ 3 & 16 & -7 & 2 & 69 \\ 2 & 12 & 3 & 16 & 63 \\ 1 & 3 & -11 & -10 & -25 \end{bmatrix} => \begin{bmatrix} 1 & 5 & -3 & 0 & 19 \\ 0 & 1 & 2 & 2 & 12 \\ 0 & 2 & 9 & 16 & 63 \\ 0 & -2 & -8 & -10 & -44 \end{bmatrix} => \begin{bmatrix} 1 & 5 & -3 & 0 & 19 \\ 0 & 1 & 2 & 2 & 12 \\ 0 & 2 & 9 & 16 & 63 \\ 0 & 0 & 1 & 6 & 19 \end{bmatrix}$$

$$=> \begin{bmatrix} 1 & 5 & -3 & 0 & 19 \\ 0 & 1 & 2 & 2 & 12 \\ 0 & 0 & 3 & 12 & 39 \\ 0 & 0 & 1 & 6 & 19 \end{bmatrix} => \begin{bmatrix} 1 & 5 & -3 & 0 & 19 \\ 0 & 1 & 2 & 2 & 12 \\ 0 & 0 & 1 & 4 & 13 \\ 0 & 0 & 1 & 6 & 19 \end{bmatrix} => \begin{bmatrix} 1 & 5 & -3 & 0 & 19 \\ 0 & 1 & 2 & 2 & 12 \\ 0 & 0 & 1 & 4 & 13 \\ 0 & 0 & 0 & 2 & 6 \end{bmatrix} =>$$

$$\begin{bmatrix} 1 & 5 & -3 & 0 & 19 \\ 0 & 1 & 2 & 2 & 12 \\ 0 & 0 & 1 & 4 & 13 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} => \begin{bmatrix} 1 & 5 & -3 & 0 & 19 \\ 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} => \begin{bmatrix} 1 & 5 & 0 & 0 & 22 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} =>$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

W=2
X=4
Y=1
Z=3

4) Determine whether each of the following statements is True or False.

(A) $(A+B)C = CA + CB$

(B) $A^T = -A$

(C) $(A^T)^{-1} = (A^{-1})^T$

(D) $(ABCDE)^{-1} = A^{-1}B^{-1}C^{-1}D^{-1}E^{-1}$

(E) $A^{-1}A = I$ and $AA^{-1} = I$

(F) $(ABC)^T = C^T B^T A^T$

(G) $(A+B)^T = B^T + A$

(H) If (AB) is invertible, then A and B are both invertible

(I) A is symmetric if and only if $A^T = -A$

(J) If $BA = I$ then $B = A^{-1}$

Solutions:

(A)False.
(B)False
(C)True
(D)False
(E)True
(F)True
(G)~~True~~ FALSE
(H)~~True~~ FALSE, unless square
(I)False
(J)~~True~~ FALSE, " "

5) Let X = $\begin{bmatrix} 5 & 2 \\ -3 & 1 \end{bmatrix}$ and Y = $\begin{bmatrix} 1 & 7 \\ 0 & 2 \end{bmatrix}$

Find:

3X² + 3(XT) transpose?

X-1 + 4X

(2X – 3Y)T

Solution:

a) 3X² + 3(XT)

$=> 3*\begin{bmatrix} 5 & 2 \\ -3 & 1 \end{bmatrix}^2 + 3*\begin{bmatrix} 5 & 2 \\ -3 & 1 \end{bmatrix}^T$

$=> 3*\begin{bmatrix} 19 & 12 \\ -18 & -5 \end{bmatrix} + 3*\begin{bmatrix} 5 & -3 \\ 2 & 1 \end{bmatrix} => \begin{bmatrix} 57 & 36 \\ -54 & -15 \end{bmatrix} + \begin{bmatrix} 15 & -9 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 72 & 27 \\ -48 & -12 \end{bmatrix}$

b) X-1 + 4X

$=> \begin{bmatrix} 5 & 2 \\ -3 & 1 \end{bmatrix}^{-1} + 4*\begin{bmatrix} 5 & 2 \\ -3 & 1 \end{bmatrix} => \begin{bmatrix} 1/11 & -2/11 \\ 3/11 & 5/11 \end{bmatrix} + \begin{bmatrix} 20 & 8 \\ -12 & 4 \end{bmatrix} => \begin{bmatrix} 221/11 & 86/11 \\ -129/11 & 49/11 \end{bmatrix}$

c) (2X – 3Y)T

$=> (2*\begin{bmatrix} 5 & 2 \\ -3 & 1 \end{bmatrix} - 3*\begin{bmatrix} 1 & 7 \\ 0 & 2 \end{bmatrix})^T => (\begin{bmatrix} 10 & 4 \\ -6 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 21 \\ 0 & 6 \end{bmatrix})^T => (\begin{bmatrix} 7 & -17 \\ -6 & -4 \end{bmatrix})^T$

$=> \begin{bmatrix} 7 & -6 \\ -17 & -4 \end{bmatrix}$

6) Solve the following systems of linear equations simultaneously.

a. $x1 - 2x2 + 3x3 = 10$
$5x1 + 3x3 = 30$
$x1 + 2x2 - 2x3 = 1$

b. $x1 - 2x2 + 3x3 = \frac{29}{2}$
$5x1 + 3x3 = \frac{29}{2}$
$x1 + 2x2 - 2x3 = \frac{-19}{2}$

c. $x1 - 2x2 + 3x3 = -16$
$5x1 + 3x3 = 34$
$x1 + 2x2 - 2x3 = 30$

Solution:
Since the coefficients are identical, we can use the given identity (x = A-1b) as follows: invert the coefficient matrix and multiply by the 'b' matrices.

$$A=\left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 5 & 0 & 3 & 0 & 1 & 0 \\ 1 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 10 & -12 & -5 & 1 & 0 \\ 0 & -4 & 5 & 1 & 0 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -12/10 & -1/2 & 1/10 & 0 \\ 0 & -4 & 5 & 1 & 0 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -12/10 & -1/2 & 1/10 & 0 \\ 0 & 0 & 1/5 & -1 & 2/5 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -12/10 & -1/2 & 1/10 & 0 \\ 0 & 0 & 1 & -5 & 2 & -5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -13/2 & 5/2 & -6 \\ 0 & 0 & 1 & -5 & 2 & -5 \end{array}\right]$$

=> $\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -12 & 5 & -12 \\ 0 & 1 & 0 & -13/2 & 5/2 & -6 \\ 0 & 0 & 1 & -5 & 2 & -5 \end{array}\right]$ => $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 3 \\ 0 & 1 & 0 & -13/2 & 5/2 & -6 \\ 0 & 0 & 1 & -5 & 2 & -5 \end{array}\right]$

Thus A-1 = $\begin{bmatrix} 3 & -1 & 3 \\ -13/2 & 5/2 & -6 \\ -5 & 2 & -5 \end{bmatrix}$ and our solutions can be determined as the product of A-1 and b as follows:

$\begin{bmatrix} 3 & -1 & 3 \\ -13/2 & 5/2 & -6 \\ -5 & 2 & -5 \end{bmatrix}\begin{bmatrix} 10 \\ 30 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ thus for part a. x1=3 x2=4 x3=5

Similarly,

$\begin{bmatrix} 3 & -1 & 3 \\ -13/2 & 5/2 & -6 \\ -5 & 2 & -5 \end{bmatrix}\begin{bmatrix} 29/2 \\ 29/2 \\ -19/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1 \\ 4 \end{bmatrix}$ thus for part b. x1=1/2 x2=-1 x3=4

$\begin{bmatrix} 3 & -1 & 3 \\ -13/2 & 5/2 & -6 \\ -5 & 2 & -5 \end{bmatrix}\begin{bmatrix} -16 \\ 34 \\ 28 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ -2 \end{bmatrix}$ thus for part c. x1=8 x2=9 x3=-2

(note: we could also simply put the original matrix into reduced-row echelon form to arrive at the same solutions.

$\left[\begin{array}{ccc|c|c|c} 1 & -2 & 3 & 10 & 29/2 & 8 \\ 5 & 0 & 3 & 30 & 29/2 & 9 \\ 1 & 2 & -2 & 1 & -19/2 & -2 \end{array}\right]$ => $\left[\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 3 & 1/2 & 8 \\ 0 & 1 & 0 & 4 & -1 & 9 \\ 0 & 0 & 1 & 5 & 4 & -2 \end{array}\right]\right]$

7) Given the following matrices:

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 10 & 5 & 0 \\ 13 & 10 & 1 \\ 3 & 4 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 6 & 4 & 2 \end{bmatrix}$

Identify whether the following are row equivalent:

Matrices A and C
Matrices A and B
Matrices B and C

Solution:
Matrices A and C ARE NOT row equal.
Matrices A and B ARE row equal.
Matrices B and C ARE NOT row equal.

Check:
If A and C are row equal, then C must be able to reduce to equal A. So, we have:

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 6 & 4 & 2 \end{bmatrix} \approx \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{which cannot be reduced to} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Therefore, A and C are not row equivalent.}$$

To show that A and B are row equal, we perform operations on A to make it equal B.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \approx \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \approx \begin{bmatrix} 10 & 5 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \approx \begin{bmatrix} 10 & 5 & 0 \\ 10 & 6 & 0 \\ 3 & 4 & 1 \end{bmatrix} \approx \begin{bmatrix} 10 & 5 & 0 \\ 13 & 10 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

Therefore, A and B are row equal.

A and B are row equal, but A and C aren't. So, it follows that B and C aren't row equal.

8) Find the determinate for matrix A. What does this tell us about A? A =
$$\begin{bmatrix} 2 & 1 & 4 \\ 5 & 3 & 2 \\ -2 & 2 & 6 \end{bmatrix}$$

Solution:
The determinant of A is the sum of all signed elementary products of A, as σ runs through all possible permutations.

det(A) = (2*3*6)+ (4*5*2)+(2*-2*1) –(4*3*-2) – (2*2*2) – (6*1*5)

det(A) = 36+40-4+24-8-30
det(A)= 58

This tells us, because det(A) is not equal to 0, matrix A has an inverse.

9) Using signed elementary products, compute the determinant of the following matrix

$$\begin{bmatrix} -7 & -4 & 3 \\ 8 & 1 & 6 \\ 4 & 3 & -5 \end{bmatrix}$$

Solution:

a. $A = A_{3x3}$ there are thus 3!=6 permutations.

=> sign(1,2,3)(-7*1*-5) +
sign(1,3,2)(-7*6*3) +
sign(2,1,3)(-4*8*-5) +
sign(2,3,1)(-4*6*4) +
sign(3,1,2)(3*8*3) +
sign(3,2,1)(3*1*4)

= (35) - (-126) - (160) + (-96) + (72) - (12) = -35

Thus det A = -35

10) Given the matrix $A = \begin{bmatrix} 3 & 6 & 0 \\ 1 & 3 & 2 \\ 5 & 0 & 1 \end{bmatrix}$, find A^{-1} using the adjunct of A.

Solution:

$$\begin{bmatrix} 1/21 & -2/21 & 4/21 \\ 1/7 & 1/21 & -2/21 \\ -5/21 & -10/21 & 1/21 \end{bmatrix}$$

Check:

Since $A^{-1} = \frac{1}{\det(A)} adj(A)$, we begin by finding det(A).

Using Cofactors we find that $\det(A) = 3\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 5 & 1 \end{vmatrix} = 3(3) - 6(-9) = 63$

To find adj(A) we begin by finding Cofactors for each entry of A

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = 3 \qquad C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 2 \\ 5 & 1 \end{vmatrix} = 9 \qquad C_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} = -15$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 6 & 0 \\ 0 & 1 \end{vmatrix} = -6 \qquad C_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 0 \\ 5 & 1 \end{vmatrix} = 3 \qquad C_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 6 \\ 5 & 0 \end{vmatrix} = -30$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 6 & 0 \\ 3 & 2 \end{vmatrix} = 12 \qquad C_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} = -6 \qquad C_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 6 \\ 1 & 3 \end{vmatrix} = 3$$

This gives the matrix: $\begin{bmatrix} 3 & 9 & -15 \\ -6 & 3 & -30 \\ 12 & -6 & 3 \end{bmatrix} \rightarrow adj(A) = \begin{bmatrix} 3 & -6 & 12 \\ 9 & 3 & -6 \\ -15 & -30 & 3 \end{bmatrix}$

So, $A^{-1} = \dfrac{1}{63}\begin{bmatrix} 3 & -6 & 12 \\ 9 & 3 & -6 \\ -15 & -30 & 3 \end{bmatrix} = \begin{bmatrix} 1/21 & -2/21 & 4/21 \\ 1/7 & 1/21 & -2/21 \\ -5/21 & -10/21 & 1/21 \end{bmatrix}$

Question #1 (10pts.)

a) Which of the following are systems of linear equations? Include your reasoning on why examples may not qualify.

b) For each set of equations that are systems of linear equations, write the corresponding augmented matrix.

a.
$$\begin{aligned} (2)^{1/2} x_1 + 4x_2 - 3x_3 &= 10 \\ 7x_1 \qquad - 9x_3 &= 5^2 \\ 3x_2 - 5x_3 &= 3/4 \end{aligned}$$

b.
$$\begin{aligned} (\pi)x + 12y - \cos(\pi/3)w &= 1 \\ 10y + 7z &= 2^{-3/4} \\ 4x - 7z + 9w &= 5 \\ y - z + w - 3 &= 10 \end{aligned}$$

c.
$$\begin{aligned} (c^{1/2}/b)x_1 + x_2 - mx_3 &= n \\ a^2x_1 + (b/2)x_2 &= d \\ jx_1 + kx_3 &= e \end{aligned}$$

d.
$$\begin{aligned} (\pi x)^{1/2} + 2y - 3z &= 12^2 \\ 6x + (3)^{1/2} - 7z^2 &= 4 \\ 9x - 2y + 10z &= 5 \end{aligned}$$

Solution:

a) Options a, b, and c all qualify as systems of linear equations. Option d is not linear because of the terms "$(\pi x)^{1/2}$" in the first equation and "$7z^2$" in the second equation.

b) Corresponding augmented matrices:

a. $\begin{bmatrix} \sqrt{2} & 4 & -3 & 10 \\ 7 & 0 & -9 & 25 \\ 0 & 3 & -5 & 3/4 \end{bmatrix}$

b. $\begin{bmatrix} \pi & 12 & 0 & -\cos(\pi/3) & 1 \\ 0 & 10 & 7 & 0 & 2^{-3/4} \\ 4 & 0 & -7 & 9 & 5 \\ 0 & 1 & -1 & 1 & 13 \end{bmatrix}$

c. $\begin{bmatrix} \sqrt{c}/b & 1 & -m & n \\ a^2 & b/2 & 0 & d \\ j & 0 & k & e \end{bmatrix}$

d. Not a system of linear equations

Question#2(10pts)

Find the coefficients a,b,c,d so that the curve with the points (-2,-1), (0,7), (2,-1), (3,4) will follow the equation: $ax^3 + bx^2 + cx + d = y$

Solution:

b. Consider the data as a system of linear equations:

$$(-2)^3a + (-2)^2b + (-2)c + d = -1$$
$$(0)^3a + (0)^2b + (0)c + d = 7$$

$(2)^3a + (2)^2b + (2)c + d = -1$
$(3)^3a + (3)^2b + (3)c + d = 4$

c. Create and solve the augmented matrix for the system:

$$\begin{bmatrix} -2^3 & -2^2 & -2 & 1 & -1 \\ 0^3 & 0^2 & 0 & 1 & 7 \\ 2^3 & 2^2 & 2 & 1 & -1 \\ 3^3 & 3^2 & 3 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} -8 & 4 & -2 & 1 & -1 \\ 0 & 0 & 0 & 1 & 7 \\ 8 & 4 & 2 & 1 & -1 \\ 27 & 9 & 3 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} -8 & 4 & -2 & 1 & -1 \\ 8 & 4 & 2 & 1 & -1 \\ 27 & 9 & 3 & 1 & 4 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} -8 & 4 & -2 & 1 & -1 \\ 0 & 8 & 0 & 2 & -2 \\ 27 & 9 & 3 & 1 & 4 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 1/4 & -1/8 & 1/8 \\ 0 & 8 & 0 & 2 & -2 \\ 0 & 45/2 & -15/4 & 35/8 & 5/8 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1/2 & 1/4 & -1/8 & 1/8 \\ 0 & 1 & 0 & 1/4 & -1/4 \\ 0 & 9 & -3/2 & 7/4 & 1/4 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 1/4 & -1/8 & 1/8 \\ 0 & 1 & 0 & 1/4 & -1/4 \\ 0 & 0 & -3/2 & -1/2 & 5/2 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1/2 & 1/4 & -1/8 & 1/8 \\ 0 & 1 & 0 & 1/4 & -1/4 \\ 0 & 0 & 1 & 1/3 & -5/3 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 1/4 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1/2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim$$

a = 1
b = -2
c = -4
d = 7

Therefore: $x^3 - 2x^2 - 4x + 7 = y$

Question #3(10pts)

$$A=\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} \quad B=\begin{bmatrix} 5 & 0 & -2 \\ 2 & 3 & 9 \end{bmatrix} \quad C=\begin{bmatrix} 7 & -1 & 0 \\ 4 & -5 & 8 \\ 1 & 2 & 6 \end{bmatrix} \quad D=\begin{bmatrix} 1 & -7 & 4 \\ 6 & 5 & 3 \\ -2 & 0 & 1 \end{bmatrix} \quad E=\begin{bmatrix} 2 & -3 \\ -4 & -1 \\ 0 & 5 \end{bmatrix}$$

Compute the following (where possible):

a. C+D b. AC c. E(2B)-4D d. $(3A^T)D$

Solution:

a.

$$C+D=\begin{bmatrix}7 & -1 & 0\\ 4 & -5 & 8\\ 1 & 2 & 6\end{bmatrix}+\begin{bmatrix}1 & -7 & 4\\ 6 & 5 & 3\\ -2 & 0 & 1\end{bmatrix}=\begin{bmatrix}8 & -8 & 4\\ 10 & 0 & 11\\ -1 & 2 & 7\end{bmatrix}$$

b. AC is not defined

d. E(2B) – 4D

$$2B= 2\begin{bmatrix}5 & 0 & -2\\ 2 & 3 & 9\end{bmatrix}=\begin{bmatrix}10 & 4 & -4\\ 4 & 3 & 18\end{bmatrix} \qquad -4D= -4\begin{bmatrix}1 & -7 & 4\\ 6 & 5 & 3\\ -2 & 0 & 1\end{bmatrix}=\begin{bmatrix}-4 & 28 & -16\\ -24 & -20 & -12\\ 8 & 0 & -4\end{bmatrix}$$

$$E(2B)=\begin{bmatrix}2 & -3\\ -4 & -1\\ 0 & 5\end{bmatrix}\begin{bmatrix}10 & 4 & -4\\ 4 & 3 & 18\end{bmatrix}=\begin{bmatrix}-2 & -18 & 46\\ -36 & 6 & 2\\ 20 & 30 & 90\end{bmatrix}$$

$$E(2B) + -4D=\begin{bmatrix}-2 & -18 & 46\\ -36 & 6 & 2\\ 20 & 30 & 90\end{bmatrix}+\begin{bmatrix}-4 & 28 & -16\\ -24 & -20 & -12\\ 8 & 0 & -4\end{bmatrix}=\begin{bmatrix}-6 & 10 & 30\\ -60 & -14 & -10\\ 28 & 30 & 86\end{bmatrix}$$

a = 1
b = -2
c = -4
d = 7

Therefore: $x^3 -2x^2 -4x + 7 = y$

e. $(3A^T)D$

$$3A= 3\begin{bmatrix}-1\\ 2\\ 4\end{bmatrix}=\begin{bmatrix}-3\\ 6\\ 12\end{bmatrix} \qquad 3A^T=\begin{bmatrix}-1\\ 2\\ 4\end{bmatrix}^T = [\,-3\ \ 6\ \ 12\,]$$

$$(3A^T)D= [\,-3\ \ 6\ \ 12\,]\begin{bmatrix}1 & -7 & 4\\ 6 & 5 & 3\\ -2 & 0 & 1\end{bmatrix} = [\,9\ \ 51\ \ 18\,]$$

2. Let A and B be square matrices of the same size. Find $(A+B)^3$.

Solution:

a. $(A+B)^3 = (A+B)(A+B)(A+B)$ by definition

b. $(A+B)(A+B)(A+B) = (A^2 + AB + BA + B^2)(A+B)$ by left distributive law and associative law for multiplication of matrices

c. $(A^2 + AB + BA + B^2)(A+B) = A^3 + ABA + BA^2 + B^2A + A^2B + AB^2 + BAB + B^3$ by left distributive law

Section 1.5

Calculate the inverse of the following matrices if possible using the method [A|I]

a)

$$A=\begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

b)

$$A=\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$$

c)

$$A=\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$$

Question #4 (10pts)

a)Paul is twice as old as Andrea. Andrea's age added to Paul's age is the same as Melissa's age. The sum of their ages is 48. How old is each person?

b)Eric has two times as many apples as Sam. If Eric and Sam combined their apples they would have as many as John. When they all combined their apples they have 66. How Many apples do they each have?

c)90 people are in a huge group that is divided into three separate groups. Group one and two combined have as many as group three. Group one has half as many as group two. How many are in the individual groups?

Solve all of this questions using one matrix.

Key for Section 1.5

a) $\left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$ Begin in the form [A|I]

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$ Switch rows 1 and 2

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$ Subtract 3 times row 1 from row 3

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$ Subtract 2 times row 1 from row 2

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \\ 0 & 4 & -10 & 1 & -3 & 0 \end{array}\right]$ Switch rows 2 and 3

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & -2/5 & 1/5 \\ 0 & 4 & -10 & 1 & -3 & 0 \end{array}\right]$ Divide row 2 by 5

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & -2/5 & 1/5 \\ 0 & 0 & -2 & 1 & -7/5 & -4/5 \end{array}\right]$ Subtract 4 times row 2 from row 3

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -2 & 1 & -7/5 & -4/5 \end{array}\right]$ Add row 3 from row 2

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array}\right]$ Divide row 3 by -2

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -11/10 & -6/5 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array}\right]$ Subtract 3 times row 3 from row 1

$$A^{-1} = \begin{bmatrix} 3/2 & -11/10 & -6/5 \\ -1 & 1 & 1 \\ -1/2 & 7/10 & 2/5 \end{bmatrix}$$

b) $\left[\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right]$ Start in the form [A|I]

$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right]$ Multiply row 1 by -1

$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right]$ Subtract 2 times row 1 from row 2

$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right]$ Add 4 times row 1 to row 3

$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 1 & -7/10 & 1/5 & 1/10 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right]$ Divide row 2 by 10

$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 1 & -7/10 & 1/5 & 1/10 & 0 \\ 0 & 0 & 0 & -4 & 0 & 1 \end{array}\right]$ Add 10 times row 2 to row 3

No inverse exists due to a row of zeroes

c) $\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$ Start in the form [A|I]

$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$ Subtract row 1 from row 2 and 3

$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$ Subtract row 2 from row 3

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \text{Divide row 1 by 2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1/2 & 3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \text{Subtract 3 times row 3 from row 1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \text{Subtract 3 times row 2 from row}$$

$$A^{-1} = \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Question #5 (10pts)

The system of linear equations that correlate for all three questions are as follows

$x_1 - x_2 - 0 = b_1$
$x_1 + x_2 - x_3 = b_2$
$x_1 + x_2 + x_3 = b_3$

Part a) has a vector **b** = $\begin{bmatrix} 0 \\ 0 \\ 48 \end{bmatrix}$

Part b) has a vector **b** = $\begin{bmatrix} 0 \\ 0 \\ 66 \end{bmatrix}$

Part c) has a vector **b** = $\begin{bmatrix} 0 \\ 0 \\ 90 \end{bmatrix}$

This can be represented in the matrix A and vector **b**

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \quad \mathbf{b}_a = \begin{bmatrix} 0 \\ 0 \\ 48 \end{bmatrix} \quad \mathbf{b}_b = \begin{bmatrix} 0 \\ 0 \\ 66 \end{bmatrix} \quad \mathbf{b}_c = \begin{bmatrix} 0 \\ 0 \\ 90 \end{bmatrix}$$

All three of these equations can be solved at once

$$\left[\begin{array}{ccc|c|c|c} 1 & -2 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 48 & 66 & 90 \end{array}\right]$$

Using elementary row reduction and reducing the first section of the matrix into the Identity form the respectful vectors will be the answers because $\mathbf{x}=A^{-1}\mathbf{b}$

$$\left[\begin{array}{ccc|c|c|c} 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 48 & 66 & 90 \end{array}\right] \text{Subtract row 1 from row 2 and row 3}$$

$$\left[\begin{array}{ccc|c|c|c} 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1/3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 48 & 66 & 90 \end{array}\right] \text{Divide row 2 by 3}$$

$$\left[\begin{array}{ccc|c|c|c} 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1/3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 48 & 66 & 90 \end{array}\right] \text{Subtract 3 times row 2 from row 3}$$

$$\left[\begin{array}{ccc|c|c|c} 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1/3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 24 & 33 & 45 \end{array}\right] \text{Divide row 3 by 2}$$

$$\left[\begin{array}{ccc|c|c|c} 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 8 & 11 & 15 \\ 0 & 0 & 1 & 24 & 33 & 45 \end{array}\right] \text{Add 1/3 times row 3 to row 2}$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 16 & 22 & 30 \\ 0 & 1 & 0 & 8 & 11 & 15 \\ 0 & 0 & 1 & 24 & 33 & 45 \end{array}\right] \text{Add 2 times row 2 to row 1}$$

Questions #6-7 from 1.7(worth 10pts each)

Section 1.7

Question

Show that A and B commute is a - d = 7b ?

$$A := \begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix} \qquad B := \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

Solution

The remark on page 72 of the text says that if, AB = BA, then the product AB will be symmetric. In other words, the product of two symmetric matrices is symmetric if and only if the matrices commute. Thus, it is necessary to show that that AB = BA and AB is symmetric.

$$AB = \begin{pmatrix} 2a + b & 2b + d \\ a - 5b & b - 5d \end{pmatrix}$$

$$BA = \begin{pmatrix} 2a + b & b - 5d \\ 2b + d & b - 5d \end{pmatrix}$$

Now it is necessary to show that 2b + d = a - 5b.

a - d = 7b

d = a - 7b, therefore subsitution gives

2b + a - 7b = a - 5b

so 2b + d = a - 5b

therefore, AB = BA and both are symmetric. According to the aforestated theorem,
A and B must commute because the product of the two symmetric matrices is symmetric.

Section 2.1 (handout)

Question

Compute the determinant of matrix A showing the sum of the signed elementary products.

$$A := \begin{pmatrix} 1 & 2 & 4 \\ 3 & 2 & 0 \\ 2 & 1 & 2 \end{pmatrix}$$

Solution

Defintion 6 in the packet states that det(A) = $\Sigma \ \mathrm{sgn}\ (\sigma) \cdot a_{1\sigma(1)} \ldots\ldots a_{n\sigma(n)}$

The different σ permutations are (1,2,3); (2,1,3); (3,2,1); (2,3,1); (3,1,2); (1,3,2). Therefore,

det (A) = 1*(1)(2)(2) + (-1)(2)(3)(2) + (-1)(4)(2)(2) + (1)(2)(0)(2) + (1)(4)(3)(1) + (-1)(1)(0)(1)

det(A) = 4 - 12 - 16 + 0 + 12 + 0; therefore

det(A) = -14

Question #8 (10pts)

For the given matrix $A = \begin{bmatrix} f & 0 & r \\ k & l & m \\ a & 0 & b \end{bmatrix}$ find the det(A) by cofactor expansion along any row or column.

Solution:

By cofactor expansion along column two the det(A)

$$= -0 \bullet \begin{vmatrix} k & m \\ a & b \end{vmatrix} + l \bullet \begin{vmatrix} f & r \\ a & b \end{vmatrix} - 0 \bullet \begin{vmatrix} f & r \\ k & m \end{vmatrix}$$

$$= l \begin{vmatrix} f & r \\ a & b \end{vmatrix}$$

$$= flb - rla.$$

Other solutions for cofactor expansion on other rows or columns include:

flb - ~~fm~~-rla, by expansion along row 1

fb - ra, by expansion along row2

-arl + bfl, by expansion along row 3

flb - rla, by expansion along column 1

and

-rla + bfl, by expansion along column 3

have to be same.

Question #9 (10pts)

Find the *characteristic polynomial*, also known as the *characteristic equation*, of the matrix $Q = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$.

Solution:

The ~~*characteristic polynomial*, or~~ *characteristic equation*, ~~satisfies~~ is the equation $\det(XI - Q) = 0$. It follows that the

$$\det(XI\text{-}Q) = \left| \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} (x-3) & -1 \\ -2 & (x-2) \end{vmatrix} = 0$$

$$\Rightarrow (x-3)(x-2) - 2 = 0$$

$$\Rightarrow x^2 - 5x + 4 = 0.$$

Question #10 (10pts)

Find the *Eigen Value(s)* for the given matrix Q in the previous problem.

Solution:
The Eigen Values of this matrix are the corresponding roots for x in the characteristic polynomial. It follows that with the quadratic formula, a=1, b=-5, c=4, the Eigen Values are 1 and 4.

1.) Solve the following system of equations by using Gause-Jordan Elimination:

$$3x + 2z = 3$$
$$4x + 2y + 5z = 16$$
$$3x + y + 3z = 10$$

Solution:

First, we need to write this system of equations in matrix form and then reduce it into reduced row echelon form by using the elementary row operations, which gives us:

$$\begin{bmatrix} 3 & 0 & 2 & 3 \\ 4 & 2 & 5 & 16 \\ 3 & 1 & 3 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 & 2 & 3 \\ 0 & 6 & 7 & 36 \\ 0 & 1 & 1 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 & 2 & 3 \\ 0 & 1 & 7/6 & 6 \\ 0 & 0 & 1/6 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2/3 & 1 \\ 0 & -1 & 0 & -13 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$ which gives us our answer that x = 5, y = 13, and z = -6.

2.) Given that A and B are square matrices of the same size and AB = BA show that $(A + B)(A - B) = A^2 - B^2$ or that $(A + B)(A - B) \neq A^2 - B^2$.

...

It is easy to see that such a series of operations is defined for square matrices of the same size therefore we need only determine what the algebraic expansion of $(A + B)(A - B)$ is. We know that the left and right distributive laws for scalar numbers holds true for matrices, therefore

$$(A + B)(A - B) = A^2 - AB + BA - B^2 .$$

Since we are given that AB = BA we can simplify by canceling out the two middle terms leaving us with

$$(A + B)(A - B) = A^2 - B^2 .$$

3.) If matrix $A = \begin{bmatrix} 2 & 11 & 5 & 10 \\ 0 & 4 & 0 & 0 \\ 0 & 8 & 0 & 3 \\ 1 & 6 & 7 & 9 \end{bmatrix}$ find the det(A).

...

Many different methods can be used to find the determinant but only the most efficient (as a computer runs) will be shown here.

not cofactor expansion!

We know

$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + a_{4j}C_{4j}$ and $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + a_{i4}C_{i4}$

where a_{ij} is the element in the *i*th row *j*th column and C_{ij} is the cofactor in the *i*th row *j*th column, so choosing to expand row 2 we have

$\det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + a_{24}C_{24}$.

Since a_{21}, a_{23} and a_{24} are all 0 we are left with

$$\det(A) = a_{22}C_{22} \Rightarrow \det(A) = 4(-1)^4 \begin{vmatrix} 2 & 5 & 10 \\ 0 & 0 & 3 \\ 1 & 7 & 9 \end{vmatrix}$$

choosing again to use row 2 we obtain

$$\det(A) = 3(-1)^5 \begin{vmatrix} 2 & 5 \\ 1 & 7 \end{vmatrix} \Rightarrow \det(A) = (-3)(9) = -27$$

4.) Find the inverses of the following matrices:

a. $\begin{bmatrix} 5 & 8 \\ 7 & 10 \end{bmatrix}$

b. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Solutions: Each of these problems comes from the section 1.4. They can be easily determined by the theorem 1.4.5.

a. For this matrix you have to find the inverse of the determinant and multiply it by $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. It should look like this:

$$1/(5*10 - 8*7) \begin{bmatrix} 10 & -7 \\ -8 & 5 \end{bmatrix} = \begin{bmatrix} -5/3 & 7/6 \\ 4/3 & -5/6 \end{bmatrix}$$

b. This problem is exactly similar to the previous one only with variables. It requires the 1.4.5 Thereom. The solution should look more like a proof and look like this:

$$1/(ad\text{-}bc) \begin{bmatrix} d & -c \\ -b & d \end{bmatrix} = \begin{bmatrix} d/(ad\text{-}bc) & -c/(ad\text{-}bc) \\ -b/(ad\text{-}bc) & a/(ad\text{-}bc) \end{bmatrix}$$

5.) Find the inverse of matrix [A]:

$$\mathbf{A} = \left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 2 & 4 & 0 & 1 & 0 \\ 2 & 6 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 1 & 0 \\ -1 & 2 & 4 & 0 & 1 & 0 \\ 2 & 6 & 3 & 0 & 0 & 1 \end{array}\right]$$

we *added 2nd row to 1st row*

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 1 & 0 \\ 0 & 5 & 11 & 1 & 2 & 0 \\ 2 & 6 & 3 & 0 & 0 & 1 \end{array}\right]$$

we *added 1st row to 2nd row*

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 1 & 0 \\ 0 & 5 & 11 & 1 & 2 & 0 \\ 0 & 0 & -11 & -2 & -2 & 1 \end{array}\right]$$

we *added –2 times row 1 to row 3*

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 1 & 0 \\ 0 & 1 & \frac{11}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 1 & \frac{2}{11} & \frac{2}{11} & -\frac{1}{11} \end{array}\right]$$

we *divided row 2 by 5 and row 3 by* -11

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{5} & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{11} & \frac{2}{11} & -\frac{1}{11} \end{array}\right]$$

we *added –11/5 times row 3 to row 2*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 7 & \frac{8}{5} & 1 & \frac{3}{5} \\ 0 & 1 & 0 & -\frac{1}{5} & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{11} & \frac{2}{11} & -\frac{1}{11} \end{array}\right]$$

we *added –3 times row 2 to row 1*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{18}{55} & -\frac{3}{11} & -\frac{2}{55} \\ 0 & 1 & 0 & -\frac{1}{5} & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{11} & \frac{2}{11} & -\frac{1}{11} \end{array}\right] = \mathbf{A}^{-1}$$

we *added –7 times row 3 to* row 1

6.) Suppose you have matrix A and matrix B. If a = -1, and c = 4 what must b equal for BA to be a symmetric matrix? What does this tell you about matrix A and matrix B?

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Solution:
Because A and B are symmetric matrices, if BA is a symmetric matrix that means that matrix A and matrix B must commute, meaning that BA = AB. This gives us a series of equations. If a = -1 and our answer matrix is

$$AB = BA = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$ then we know that:

<u>AB</u>		<u>BA</u>
$(3 \times -1) + b$	$= w =$	$b + (3 \times -1)$
$3b + 4$	$= x =$	$-1 + 5b$
$-1 + 5b$	$= y =$	$3b +$
$b + 20$	$= z =$	$b + 20$

Using the equation 3b + 4 = -1 + 5b, we find that b is equal to 5/2, which gives us a symmetric matrix when AB and BA.

7.) Compute the determinant of the following matrix. What values of a would make computing the determinant undefined?

– never undefined!

$$\begin{bmatrix} 5 & 1 & 3 \\ a & 0 & 0 \\ 0 & 1 & 8 \end{bmatrix}$$

Solution:

There are many ways to solve the determinant of this matrix. One way is to use the equation for a 3x3 matrix which is:

aei + bfg + cdh – ceg – afh – bdi, from the matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

This translates as:

5(0)(8) + 1(0)(0) + 3(a)(1) – 3(0)(0) – 5(0)(1) – 1(a)(8) =
0 + 0 + 3a – 0 – 0 – 8a = -5a
The determinant, then, is equal to 5a. If a = 0, the determinant is undefined because there would be a row of all zeros.

8.) Find all **x** such that A**x** = **b** where A = $\begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}$ and **b** = $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

...

We know that if A^{-1} exists it is unique A**x** = **b** has only one solution for every n x 1 matrix **b**. Since **x** = A^{-1}**b** it is also unique, therefore we need only find A^{-1} to find the solution.

Because A is a 2 x 2 matrix we can use the formula $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Therefore $A^{-1} = \frac{1}{(5)(4) - (3)(7)}\begin{bmatrix} 4 & -7 \\ -3 & 5 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} -4 & 7 \\ 3 & -5 \end{bmatrix}$

and

$$A^{-1}\mathbf{b} = \begin{bmatrix} -4 & 7 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} (-4)(2) + (7)(-3) \\ (3)(2) + (-5)(-3) \end{bmatrix} \Rightarrow A^{-1}\mathbf{b} = \begin{bmatrix} -29 \\ 15 \end{bmatrix}.$$

Therefore $\mathbf{x} = \begin{bmatrix} -29 \\ 15 \end{bmatrix}$.

9.) Find λ I – A for the linear system:

$$x_1 + 2x_2 = \lambda x_1$$
$$4x_1 + 3x_2 = \lambda x_2$$

Solution: $$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} \lambda-1 & -2 \\ -4 & \lambda-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so:

$$\lambda I - A = \begin{bmatrix} \lambda-1 & -2 \\ -4 & \lambda-3 \end{bmatrix}$$

10.) List how many inversions there are in the following permutations of (1, 2, ,3, 4, 5).

a. (5, 4, 3, 2, 1)

b. (2, 4, 3, 1, 5)

Solutions: These problems require that the numbers of integers that follow j_1, j_2, j_3, etc. are added together to determine the total amount of inversions there are. The work looks something similar to this:

(5,4), (5,3), (5,2) (5,1), (4,3), (4,2), (4,1), (3,2), (3,1), (2,1)

a. There are 4 less than five, 3 less than four, 2 less than three and 1 less than 2. Once these are added there are a total of 9 inversions.
10 $4+3+2+1 = 7+3 = 10$

b. There are 1 less than two, 2 less than four, 1 less than three and 0 less than one or then five. Once the inversions are added together there becomes 4 inversions total.

1. Given the system of equations, find the solution to x_1, x_2, x_3, and x_4 using Gauss Jordan elimination.

$$\begin{aligned} 3x_1 + 7x_2 - x_3 - 2x_4 &= 2 \\ -6x_2 + 4x_3 + x_4 &= 1 \\ x_1 + 2x_2 \quad - 4x_4 &= 4 \\ 3x_2 - 9x_3 \quad &= 6 \end{aligned}$$

ANSWER

$$\begin{pmatrix} 3 & 7 & -1 & -2 & 2 \\ 0 & -6 & 4 & 1 & 1 \\ 1 & 2 & 0 & -4 & 4 \\ 0 & 3 & -9 & 0 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{7}{3} & \frac{-1}{3} & \frac{-2}{3} & \frac{2}{3} \\ 0 & 1 & \frac{-2}{3} & \frac{-1}{6} & \frac{-1}{6} \\ 0 & 3 & -9 & 0 & 6 \\ 1 & 2 & 0 & -4 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{7}{3} & \frac{-1}{3} & \frac{-2}{3} & \frac{2}{3} \\ 0 & 1 & \frac{-2}{3} & \frac{-1}{6} & \frac{-1}{6} \\ 0 & 0 & -7 & \frac{1}{2} & \frac{13}{2} \\ 0 & \frac{-1}{3} & \frac{1}{3} & \frac{-10}{3} & \frac{10}{3} \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & \frac{7}{3} & \frac{-1}{3} & \frac{-2}{3} & \frac{2}{3} \\ 0 & 1 & \frac{-2}{3} & \frac{-1}{6} & \frac{-1}{6} \\ 0 & 0 & 1 & \frac{-1}{14} & \frac{-13}{14} \\ 0 & 0 & \frac{1}{9} & \frac{-61}{18} & \frac{59}{18} \end{pmatrix} \to \begin{pmatrix} 1 & \frac{7}{3} & \frac{-1}{3} & \frac{-2}{3} & \frac{2}{3} \\ 0 & 1 & \frac{-2}{3} & \frac{-1}{6} & \frac{-1}{6} \\ 0 & 0 & 1 & \frac{-1}{14} & \frac{-13}{14} \\ 0 & 0 & 0 & \frac{-71}{21} & \frac{71}{21} \end{pmatrix} \to \begin{pmatrix} 1 & \frac{7}{3} & \frac{-1}{3} & \frac{-2}{3} & \frac{2}{3} \\ 0 & 1 & \frac{-2}{3} & \frac{-1}{6} & \frac{-1}{6} \\ 0 & 0 & 1 & \frac{-1}{14} & \frac{-13}{14} \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & \frac{7}{3} & \frac{-1}{3} & \frac{-2}{3} & \frac{2}{3} \\ 0 & 1 & \frac{-2}{3} & \frac{-1}{6} & \frac{-1}{6} \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{7}{3} & \frac{-1}{3} & \frac{-2}{3} & \frac{2}{3} \\ 0 & 1 & 0 & \frac{-1}{6} & \frac{-5}{6} \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{7}{3} & \frac{-1}{3} & \frac{-2}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & \frac{-1}{3} & \frac{-2}{3} & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \frac{-2}{3} & \frac{8}{3} \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \to$$

x_1= 2
x_2= --1
x_3= --1
x_4= --1

2. Solve for the equation using gaussian elimination.

$5x_1 + 4x_2 + 3x_3 = -3$
$x_1 + 2x_2 + 3x_3 = 3$
$3x_1 + 2x_2 + x_3 = -3$

ANSWER

$$\begin{pmatrix} 5 & 4 & 3 & -3 \\ 1 & 2 & 3 & 3 \\ 3 & 2 & 1 & -3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 3 & 3 \\ 3 & 2 & 1 & -3 \\ 5 & 4 & 3 & -3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & -4 & -8 & -12 \\ 0 & -6 & -12 & -18 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \mapsto$$

$x_1 + 2x_2 + 3x_3 = 3$
$x_2 + 2x_3 = 3$
$x_3 = t$
$x_2 = 3 - 2t$
$x_1 = 3\text{-}3t - 2(3 - 2t)$
$= t - 3$

3. Solve the systems in all parts simultaneously

a) $x_1 + 3x_2 - x_3 = -1$ | b) $x_1 + 3x_2 - x_3 = 6$
$5x_2 + 6x_3 = 1$ | $5x_2 + 6x_3 = 2$
$-2x_1 + 4x_3 = 3$ | $-2x_1 + 4x_3 = -3$

ANSWER

$$\begin{pmatrix} 1 & 3 & -1 & -1 & 6 \\ 0 & 5 & 6 & 1 & 2 \\ -2 & 0 & 4 & 3 & -3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 3 & -1 & -1 & 6 \\ 0 & 1 & \frac{6}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & 6 & 2 & 1 & 9 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 3 & -1 & -1 & 6 \\ 0 & 1 & \frac{6}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{-26}{5} & \frac{-1}{5} & \frac{33}{5} \end{pmatrix} \mapsto$$

$$\begin{pmatrix} 1 & 3 & -1 & -1 & 6 \\ 0 & 1 & \frac{6}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{26} & \frac{-33}{26} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 3 & -1 & -1 & 6 \\ 0 & 1 & 0 & \frac{2}{13} & \frac{25}{13} \\ 0 & 0 & 1 & \frac{1}{26} & \frac{-33}{26} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & -1 & \frac{-19}{13} & \frac{3}{13} \\ 0 & 1 & 0 & \frac{2}{13} & \frac{25}{13} \\ 0 & 0 & 1 & \frac{1}{26} & \frac{-33}{26} \end{pmatrix} \mapsto$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{-491}{78} & \frac{-27}{26} \\ 0 & 1 & 0 & \frac{2}{13} & \frac{25}{13} \\ 0 & 0 & 1 & \frac{1}{26} & \frac{-33}{26} \end{pmatrix}$$

a) $x_1 = -491/78$ | b) $x_1 = -27/26$
$x_2 = 2/13$ | $x_2 = 25/13$
$x_3 = 1/26$ | $x_3 = -33/26$

4. Given the matrices:

$$A := \begin{pmatrix} 3 & 6 & 4 \\ 1 & 2 & 7 \end{pmatrix} \qquad B := \begin{pmatrix} 1 & 4 & -6 & 2 \\ 7 & 3 & 4 & 5 \\ -1 & 6 & 9 & -8 \end{pmatrix} \qquad C := \begin{pmatrix} 0 & 1 & 3 & 5 \\ -6 & -1 & 0 & 2 \end{pmatrix}$$

Find $(AB)C^T$

ANSWER

$$A := \begin{pmatrix} 3 & 6 & 4 \\ 1 & 2 & 7 \end{pmatrix} \cdot B := \begin{pmatrix} 1 & 4 & -6 & 2 \\ 7 & 3 & 4 & 5 \\ -1 & 6 & 9 & -8 \end{pmatrix} \rightarrow \qquad AB := \begin{pmatrix} 41 & 54 & 42 & 4 \\ 8 & 52 & 65 & -44 \end{pmatrix}$$

$$AB := \begin{pmatrix} 41 & 54 & 42 & 4 \\ 8 & 52 & 65 & -44 \end{pmatrix} \cdot \qquad C^T = \begin{pmatrix} 0 & -6 \\ 1 & -1 \\ 3 & 0 \\ 5 & 2 \end{pmatrix} \rightarrow \qquad (AB)C^T = \begin{pmatrix} 200 & -292 \\ 27 & -188 \end{pmatrix}$$

5. Given this 3x3 matrix, find the determinant

$$\begin{pmatrix} 3 & 6 & 8 \\ 2 & 7 & 4 \\ 9 & 1 & 5 \end{pmatrix}$$

ANSWER

$$(3\cdot 7\cdot 5) + (6\cdot 4\cdot 9) + (8\cdot 2\cdot 1) - (8\cdot 7\cdot 9) - (3\cdot 4\cdot 1) - (6\cdot 2\cdot 5) = -239$$

6. Find the Determinants

$$D := \begin{pmatrix} 1 & \frac{17}{4} & 2 & 0 & 1 \\ 2 & \frac{8}{9} & 1 & 0 & 3 \\ 3 & \frac{7}{5} & 3 & 0 & 2 \\ 4 & \frac{1}{3} & 4 & 2 & 4 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad E := \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 3 \\ 3 & 3 & 0 & 2 \\ 4 & 4 & 2 & 4 \end{pmatrix} \qquad F := \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 3 & 2 \end{pmatrix}$$

ANSWER := 12

det D = $D_{51}C_{51} + D_{52}C_{52} + D_{53}C_{53} + D_{54}C_{54} + D_{55}C_{55} = D_{52}C_{52} = C_{52}$=det (E)

det E = $E_{13}C_{13} + E_{23}C_{23} + E_{33}C_{33} + E_{43}C_{43} = E_{43}C_{43} = 2C_{43}$ =2det (F)

det F = $(1\cdot1\cdot2) + (2\cdot3\cdot3) + (1\cdot2\cdot3) - (2\cdot2\cdot2) - (1\cdot3\cdot3) - (1\cdot1\cdot3) = 6$

Since det D = det E = 2det F = 2 x 6 = 12

7. Find $\left(\left(G^{-4}\cdot G^{2}\right)^{T}\right)^{-1}$ When $G := \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

ANSWER

$$\left(\left(G^{-4}\cdot G^{2}\right)^{T}\right)^{-1} = \left[\left(G^{-4}\cdot G^{2}\right)^{-1}\right]^{T} = \left[\left(G^{-2}\right)^{-1}\right]^{T} = \left(G^{2}\right)^{T}$$

$$G = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \quad x \quad G = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \quad = \quad G^{2} = \begin{pmatrix} 7 & 18 \\ 6 & 19 \end{pmatrix}$$

$$\left(G^{2}\right)^{T} = \begin{pmatrix} 7 & 6 \\ 18 & 19 \end{pmatrix}$$

8. Find the Cofactor of the entry m_{32}

$$m := \begin{pmatrix} 1 & 3 & 6 & 2 \\ 4 & 8 & 7 & 6 \\ 3 & 2 & 2 & 2 \\ 1 & 6 & 5 & 3 \end{pmatrix}$$

ANSWER

Remove row 3 and column 2 and find the determinant for the resulting 3 x 3 matrix.

$(A_{13} \times A_{22} \times A_{31}) + (A_{11} \times A_{23} \times A_{32}) + (A_{22} \times A_{21} \times A_{33}) - (A_{11} \times A_{22} \times A_{33}) - (A_{12} \times A_{23} \times A_{31}) - (A_{31} \times A_{21} \times A_{32})$ = minor and since 3+2 (entry) is odd the cofactor is (-1) minor so (-1)19 = -19

9. Find the trace of the identity matrix I_n.

ANSWER

The trace of a square matrix is the sum of the entries on the main diagonal. The identity matrix I_n has n entries on the main diagonal, each of which is equal to 1. Thus, the trace $Tr(I_n) = 1 \times n = n$

10. Find $\left(kA^T\right)^{-1}$ Where k is a nonzero scalar and A is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

ANSWER

From Theorem 1.4.8c we know that $\left(kA^T\right)^{-1} = \frac{1}{k}\left(A^T\right)^{-1}$ From theorem 1.4.10 we know that $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$. Thus $\left(kA^T\right)^{-1} = \frac{1}{k}\left(A^{-1}\right)^T$

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \rightarrow \left(A^{-1}\right)^T = \frac{1}{ad - bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} \frac{d}{kad - kbc} & \frac{-c}{kad - kbc} \\ \frac{-b}{kad - kbc} & \frac{a}{kad - kbc} \end{pmatrix}$$

1. (10 points) Solve the following system of linear equations:

$$2x_1+x_2-x_3=2$$
$$3x_2+5x_3=2$$
$$3x_1-2x_3=0$$

Solution: The augmented matrix for the system is:

$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & -2 & 0 \end{bmatrix}$$

Reducing this matrix to reduced row-echelon form, we obtain:

$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & -1/2 & 1 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & -1/2 & 1 \\ 0 & 3 & 5 & 2 \\ 0 & -3/2 & -1/2 & -3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1/2 & -1/2 & 1 \\ 0 & 1 & 5/3 & 2/3 \\ 0 & 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & -1/2 & 1 \\ 0 & 1 & 5/3 & 2/3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 7/3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 7/3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The corresponding system of equations is:

$$x_1 \qquad = -2/3$$
$$x_2 \quad = 7/3$$
$$x_1 = -1$$

By inspection, $x_1 = -2/3$, $x_2 = 7/3$, $x_3 = -1$

2. (15 points) Find AA^T, when $A = \begin{bmatrix} 5 & 7 & 1 & 2 \\ 0 & 3 & 6 & 1 \\ 2 & 1 & 0 & 4 \\ 3 & 2 & 2 & 1 \end{bmatrix}$

Solution:

$$A^T = \begin{bmatrix} 5 & 0 & 2 & 3 \\ 7 & 3 & 1 & 2 \\ 1 & 6 & 0 & 2 \\ 2 & 1 & 4 & 1 \end{bmatrix}$$

$$\text{Then, } AA^T = \begin{bmatrix} 5 & 7 & 1 & 2 \\ 0 & 3 & 6 & 1 \\ 2 & 1 & 0 & 4 \\ 3 & 2 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 5 & 0 & 2 & 3 \\ 7 & 3 & 1 & 2 \\ 1 & 6 & 0 & 2 \\ 2 & 1 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5(5)+7(7)+1+2(2) & 7(3)+5+2 & 5(2)+7+2(4) & 5(3)+7(2)+2+2 \\ 3(7)+6+2 & 3(3)+6(6)+1 & 3+4 & 3(2)+6(2)+1 \\ 2(5)+7+4(2) & 3+4 & 2(2)+1+4(4) & 2(3)+2+4 \\ 3(5)+2(7)+2+2 & 2(3)+2(6)+1 & 3(2)+2+4 & 3(3)+2(2)+2(2)+1 \end{bmatrix}$$

$$= \begin{bmatrix} 79 & 29 & 25 & 33 \\ 29 & 46 & 7 & 19 \\ 25 & 7 & 21 & 12 \\ 33 & 19 & 12 & 18 \end{bmatrix}$$

3. (15 points) Find A(B+C)-AC, when

$$A = \begin{bmatrix} 0 & 1 & 3 \\ 5 & 2 & 4 \\ 1 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 4 & 1 \\ 3 & 0 & 2 \end{bmatrix}$$

Solution:

By the left distributive law, we know that A(B+C)=AB+AC, so the answer will be AB

$$AB = \begin{bmatrix} 2+3 & 3+3(3) & 1 \\ 5(2)+2(2)+4 & 2(3)+4(3) & 5+2 \\ 2+2(2)+1 & 2(3)+3 & 1+2 \end{bmatrix} = \begin{bmatrix} 5 & 12 & 1 \\ 18 & 18 & 7 \\ 7 & 9 & 3 \end{bmatrix}$$

Check:

$$B+C=\begin{bmatrix}3 & -1 & 1\\ 4 & 7 & 2\\ 4 & 3 & 2\end{bmatrix}$$

$$A(B+C)=\begin{bmatrix}4+3(4) & 7+3(3) & 2+3(2)\\ 5(3)+2(4)+4(4) & -5+2(7)+4(3) & 5+2(2)+4(2)\\ 3+2(4)+4 & -1+2(7)+3 & 1+2(2)+2\end{bmatrix}=\begin{bmatrix}16 & 16 & 8\\ 39 & 21 & 17\\ 15 & 16 & 7\end{bmatrix}$$

$$AC=\begin{bmatrix}2+3(3) & 4 & 1+3(2)\\ 5+2(2)+4(3) & -5+2(4) & 2+4(2)\\ 1+2(2)+3 & -1+2(4) & 2+2\end{bmatrix}=\begin{bmatrix}11 & 4 & 7\\ 21 & 3 & 10\\ 8 & 7 & 4\end{bmatrix}$$

$$A(B+C)-AC=\begin{bmatrix}5 & 12 & 1\\ 18 & 18 & 7\\ 7 & 9 & 3\end{bmatrix}=AB.$$

4. (15 points) Solve the systems:

a) $2x+\ y+4z=10$ b) $2x+\ y+4z=6$
$x+3y+2z=7$ $x+3y+2z=9$
$2y+\ z=5$ $2y+\ z=11$

Solution:

$$\left[\begin{array}{ccc|c|c}2 & 1 & 4 & 10 & 6\\ 1 & 3 & 2 & 7 & 9\\ 0 & 2 & 1 & 5 & 11\end{array}\right]\text{Switch with second row.}$$

$$\left[\begin{array}{ccc|c|c}1 & 3 & 2 & 7 & 9\\ 2 & 1 & 4 & 10 & 6\\ 0 & 2 & 1 & 5 & 11\end{array}\right]\text{Add -2 times first row.}$$

$$\left[\begin{array}{ccc|c|c}1 & 3 & 2 & 7 & 9\\ 0 & -5 & 0 & -4 & -12\\ 0 & 2 & 1 & 5 & 11\end{array}\right]\text{Multiply by -(1/5).}$$

$$\begin{bmatrix} 1 & 3 & 2| & 7| & -12 \\ 0 & 1 & 0| & 4/5| & 12/5 \\ 0 & 2 & 1| & 5| & 11 \end{bmatrix}$$ Add -3 times second row.

Add -2 times second row.

$$\begin{bmatrix} 1 & 0 & 2| & 23/5| & 9/5 \\ 0 & 1 & 0| & 4/5| & 12/5 \\ 0 & 0 & 1| & 17/5| & 31/5 \end{bmatrix}$$ Add -2 times third row.

$$\begin{bmatrix} 1 & 0 & 0| & -11/5| & -51/5 \\ 0 & 1 & 0| & 4/5| & 12/5 \\ 0 & 0 & 1| & 17/5| & 31/5 \end{bmatrix}$$ Add -2 times third row.

$$\begin{bmatrix} 1 & 0 & 0| & -11/5| & -51/5 \\ 0 & 1 & 0| & 4/5| & 12/5 \\ 0 & 0 & 1| & 17/5| & 31/5 \end{bmatrix}$$

a) x=-11/5 b) x=-53/5
y=4/5 y=12/5
z=17/5 z=31/5

Check:

a) 2x+y+4z=10
2(-11/5)+4/5+4(17/5)=10
-22/5+4/5+68/5=7
50/5=10

x+3y+2z=7
-11/5+3(4/5)+2(17/5)=7
-11/5+12/5+34/5=7
35/5=7

2y+z=5
2(4/5)+17/5=5
8/5+17/5=5
25/5=5

b) 2x+y+4z=6
2(-53/5)+12/5+4(31/5)=6
-106/5+12/5+124/5=6
30/5=6

x+3y+2z=9
-53/5+3(12/5)+2(31/5)=9
-53/5+36/5+62/5=9
45/5=9

2y+z=11
2(12/5)+31/5=11
24/5+31/5=11
55/5=11

5. (15 points) Compute the following determinant of A.

Let $A=\begin{bmatrix} 7 & -5 & 3 \\ 0 & 8 & 16 \\ -10 & 6 & 4 \end{bmatrix}$

Solution:

Using the shortcut for a 3x3 determinant, rewrite A as:

$$\begin{matrix} 7 & -5 & 3 & 7 & -5 \\ 0 & 8 & 16 & 0 & 8 \\ -10 & 6 & 4 & -10 & 6 \end{matrix}$$

For diagonals which go to the right, multiply the terms and add. For diagonals that go to the left, multiply and subtract.

The sum is the determinant.

det(A)= (7)(8)(4)+(-5)(16)(-10)+(3)(0)(6)-(3)(8)(-10)-(7)(16)(6)-(-5)(0)(4)
=592

6. (15 points) Compute the following: AB where $A=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 5 & 9 & 0 & 0 & 0 \\ 10 & 3 & 8 & 0 & 0 \\ 2 & 6 & 4 & 1 & 0 \\ 4 & 7 & -3 & -5 & 7 \end{bmatrix}$ and $B=A^T$

Solution:

$AB=AA^T$

$$AA^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 5 & 9 & 0 & 0 & 0 \\ 10 & 3 & 8 & 0 & 0 \\ 2 & 6 & 4 & 1 & 0 \\ 4 & 7 & -3 & -5 & 7 \end{bmatrix} \begin{bmatrix} 1 & 5 & 10 & 2 & 4 \\ 0 & 9 & 3 & 6 & 7 \\ 0 & 0 & 8 & 4 & -3 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 10 & 2 & 4 \\ 5 & 106 & 77 & 64 & 83 \\ 10 & 77 & 173 & 62 & 37 \\ 2 & 64 & 62 & 57 & 33 \\ 4 & 83 & 37 & 33 & 158 \end{bmatrix}$$

7. (15 points) Find the inverse of A where $A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$$-2R_1+R_2 \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -6 & -2 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$-3R_1+R_3 \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -6 & -2 & 1 & 0 \\ 0 & -6 & -8 & -3 & 0 & 1 \end{array}\right]$$

$$-3R_2+R3 \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -6 & -2 & 1 & 0 \\ 0 & 0 & 10 & 3 & -3 & 1 \end{array}\right]$$

$$-1/2R_2,\ 1/10R_3 \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 & -1/2 & 0 \\ 0 & 0 & 10 & 3/10 & -3/10 & 1/10 \end{array}\right]$$

$$R_1-3R_3,\ R_2-3R_3 \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1/10 & 9/10 & -3/10 \\ 0 & 1 & 0 & 1/10 & 4/10 & -3/10 \\ 0 & 0 & 1 & 3/10 & -3/10 & 1/10 \end{array}\right]$$

$$R_1-2R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/10 & 1/10 & 3/10 \\ 0 & 1 & 0 & 1/10 & 4/10 & -3/10 \\ 0 & 0 & 1 & 3/10 & -3/10 & 1/10 \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} -1/10 & 1/10 & 3/10 \\ 1/10 & 4/10 & -3/10 \\ 3/10 & -3/10 & 1/10 \end{bmatrix}$$

8. (15 points) Find the determinate of matrix *A*.

$$A = \begin{bmatrix} 0 & 1 & 0 & 5 & 0 \\ 6 & 2 & 1 & 3 & 4 \\ 0 & 7 & 0 & 2 & 0 \\ 5 & 1 & 1 & 3 & 1 \\ 0 & 2 & 0 & 6 & 0 \end{bmatrix}$$

Solution

1. Cofactor expansion along row 2.

$$\det A = -6\det\begin{bmatrix} 1 & 0 & 5 & 0 \\ 7 & 0 & 2 & 0 \\ 1 & 1 & 3 & 1 \\ 2 & 0 & 6 & 0 \end{bmatrix} + 2\det\begin{bmatrix} 0 & 0 & 5 & 0 \\ 0 & 0 & 3 & 0 \\ 5 & 1 & 2 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} - \det\begin{bmatrix} 0 & 1 & 5 & 0 \\ 0 & 7 & 2 & 0 \\ 5 & 1 & 3 & 1 \\ 0 & 2 & 6 & 0 \end{bmatrix} + 3\det\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 5 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \end{bmatrix} - 4\det\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 7 & 0 & 2 \\ 5 & 1 & 1 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix}$$

Next using the idea that $\det A = \det(A^T)$, transpose several sub-matrices to give easier determinates, specifically the 1st, 3rd, and 5th matrices. Such that

$$\det A = -6\det\begin{bmatrix} 1 & 7 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 5 & 2 & 3 & 6 \\ 0 & 0 & 1 & 0 \end{bmatrix} + 2\det\begin{bmatrix} 0 & 0 & 5 & 0 \\ 0 & 0 & 3 & 0 \\ 5 & 1 & 2 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} - \det\begin{bmatrix} 0 & 0 & 5 & 0 \\ 1 & 7 & 1 & 2 \\ 5 & 2 & 3 & 6 \\ 0 & 0 & 1 & 0 \end{bmatrix} + 3\det\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 5 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \end{bmatrix} - 4\det\begin{bmatrix} 0 & 0 & 5 & 0 \\ 1 & 7 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 5 & 2 & 3 & 6 \end{bmatrix}$$

Now evaluate each new determinate with cofactor expansions.

1st matrix along row 2: $(-6)(1)\det\begin{bmatrix} 1 & 7 & 2 \\ 5 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} = 0$, due to the row of zeros in the matrix.

2nd matrix along row 1: $(2)(-5)\det\begin{bmatrix} 0 & 0 & 0 \\ 5 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = 0$, due to the rows of zeros in the matrix.

3rd matrix along row 1: $(-1)(-5)\det\begin{bmatrix} 1 & 7 & 2 \\ 5 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} = 0$, due to the row of zeros in the matrix.

4th matrix along row 1: $(3)(-1)\det\begin{bmatrix} 0 & 0 & 0 \\ 5 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 0$, due to the rows of zeros in the matrix.

5th matrix along row 1: $(-4)(-5)\det\begin{bmatrix} 1 & 7 & 2 \\ 0 & 0 & 0 \\ 5 & 2 & 6 \end{bmatrix} = 0$, due to the row of zeros in the matrix.

Finally add all the final values of cofactor expansions to get the $\det A$. $0+0+0+0+0 = 0$

9 & 10. (35 points) Given:

$$B = \begin{bmatrix} 2 & 4 & -1 & 0 \\ 0 & 0 & -2 & 1 \\ 3 & 1 & -4 & 1 \\ 1 & 2 & 0 & -3 \end{bmatrix} \quad \text{and } B\text{'s matrix of cofactors} = \begin{bmatrix} * & * & * & * \\ -43 & -16 & -30 & -5 \\ 22 & -11 & 0 & 0 \\ -7 & 1 & -10 & -20 \end{bmatrix}$$

a) Find the adj$B$. b) Find det$B$ using the cofactors from row one. c) Find B^{-1}.

Solution

a) The column missing needed for adj$B$ is the cofactors of the first row of B.

$$C_{11} = \det\begin{bmatrix} 0 & -2 & 1 \\ 1 & -4 & 1 \\ 2 & 0 & -3 \end{bmatrix} = -2 \qquad C_{12} = \det\begin{bmatrix} 0 & -2 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & -3 \end{bmatrix} = 16$$

$$C_{13} = \det\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 1 \\ 1 & 2 & -3 \end{bmatrix} = 5 \qquad C_{14} = \det\begin{bmatrix} 0 & 0 & -2 \\ 3 & 1 & -4 \\ 1 & 2 & 0 \end{bmatrix} = 10$$

Using these values for the first row then transposing, then

$$adjB = \begin{bmatrix} -2 & -43 & 22 & -7 \\ 16 & -16 & -11 & 1 \\ 5 & -30 & 0 & -10 \\ 10 & -5 & 0 & -20 \end{bmatrix}$$ (see last page for rest of the cofactor determinates)

b) Using the cofactors solved for in part a,

det$B$ = $b_{11}C_{11} - b_{12}C_{12} + b_{13}C_{13} - b_{14}C_{14}$= 2(-2)+4(16)+(-1)(5)+(0)(10) = 55

c) Using the det$B$ and the adj$B$ solving for B^{-1}.

$$B^{-1} = (1/\det B)*\text{adj}(B) = 1/55\begin{bmatrix} -2 & 43 & 22 & -7 \\ 16 & -16 & -11 & 1 \\ 5 & -30 & 0 & -10 \\ 10 & -5 & 0 & -20 \end{bmatrix}$$

$$= \begin{bmatrix} -2/55 & -43/55 & 2/5 & -7/55 \\ 16/55 & 14/55 & -1/5 & 1/55 \\ 1/11 & -6/11 & 0 & -2/11 \\ 2/11 & -1/11 & 0 & -4/11 \end{bmatrix}$$

$$C_{11}=\det\begin{bmatrix}0 & -2 & 1\\ 1 & -4 & 1\\ 2 & 0 & -3\end{bmatrix}=-2$$

$$C_{12}=\det\begin{bmatrix}0 & -2 & 1\\ 3 & -4 & 1\\ 1 & 0 & -3\end{bmatrix}=16$$

$$C_{13}=\det\begin{bmatrix}0 & 0 & 1\\ 3 & 1 & 1\\ 1 & 2 & -3\end{bmatrix}=5$$

$$C_{14}=\det\begin{bmatrix}0 & 0 & -2\\ 3 & 1 & -4\\ 1 & 2 & 0\end{bmatrix}=10$$

$$C_{21}=\det\begin{bmatrix}4 & -1 & 0\\ 1 & -4 & 1\\ 2 & 0 & -3\end{bmatrix}=-43$$

$$C_{22}=\det\begin{bmatrix}2 & -1 & 0\\ 3 & -4 & 1\\ 1 & 0 & -3\end{bmatrix}=-16$$

$$C_{23}=\det\begin{bmatrix}2 & 4 & 0\\ 3 & 1 & 1\\ 1 & 2 & -3\end{bmatrix}=-30$$

$$C_{24}=\det\begin{bmatrix}2 & 4 & -1\\ 3 & 1 & 4\\ 1 & 2 & 0\end{bmatrix}=-5$$

$$C_{31}=\det\begin{bmatrix}4 & -1 & 0\\ 0 & -2 & 1\\ 2 & 0 & -3\end{bmatrix}=22$$

$$C_{32}=\det\begin{bmatrix}2 & -1 & 0\\ 0 & -2 & 1\\ 1 & 0 & -3\end{bmatrix}=-11$$

$$C_{33}=\det\begin{bmatrix}2 & 4 & 0\\ 0 & 0 & 1\\ 1 & 2 & -3\end{bmatrix}=0$$

$$C_{34}=\det\begin{bmatrix}2 & 4 & -1\\ 0 & 0 & -2\\ 1 & 2 & 0\end{bmatrix}=0$$

$$C_{41}=\det\begin{bmatrix}4 & -1 & 0\\ 0 & -2 & 1\\ 1 & -4 & 1\end{bmatrix}=-7$$

$$C_{42}=\det\begin{bmatrix}2 & -1 & 0\\ 0 & -2 & 1\\ 3 & -4 & 1\end{bmatrix}=1$$

$$C_{43}=\det\begin{bmatrix}2 & 4 & 0\\ 0 & 0 & 1\\ 3 & 1 & 1\end{bmatrix}=-10$$

$$C_{44}=\det\begin{bmatrix}2 & 4 & -1\\ 0 & 0 & -2\\ 3 & 1 & -4\end{bmatrix}=-20$$

For problems 1-4, given that:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 1 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 4 & 4 \\ 4 & 8 & -2 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 8 \\ -3 & -9 \\ 6 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 3 & 8 \\ 4 & 6 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix}$$

Compute the following:

1. $A^{-1}B$ (15 pts.)

a) First we find A^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 6 & 8 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -4 & -24 & -6 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 6 & 3/2 & -1/4 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -9/2 & -25/4 & 6 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 25/2 & -8 \\ 0 & 1 & 0 & -9/2 & -25/4 & 6 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

b) Now we can find the product

$$A^{-1}B = \begin{bmatrix} -6 & 25/2 & -8 \\ -9/2 & -25/4 & 6 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 4 & 4 \\ 4 & 8 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 14 & 34 \\ -7 & -25 & -37 \\ -3 & -8 & 3 \end{bmatrix}$$

2. $B^TC - D^T$ (15 pts.)

a) First we find that $B^T = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 8 \\ 1 & 4 & -2 \end{bmatrix}$ and $D^T = \begin{bmatrix} 1 & 4 \\ 3 & 6 \\ 8 & 1 \end{bmatrix}$

b) Next we find the product $B^TC = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 8 \\ 1 & 4 & -2 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ -3 & -9 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 22 & 10 \\ 36 & 4 \\ -20 & -42 \end{bmatrix}$

c) Finally we find the difference $B^TC - D^T = \begin{bmatrix} 22 & 10 \\ 36 & 4 \\ -20 & -42 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 3 & 6 \\ 8 & 1 \end{bmatrix} = \begin{bmatrix} 21 & 6 \\ 33 & -2 \\ -28 & -43 \end{bmatrix}$

3. $C^TD^T + 5E$ (15 pts.)

a) First we find that $C^T = \begin{bmatrix} 4 & -3 & 6 \\ 8 & -9 & 5 \end{bmatrix}$ and $D^T = \begin{bmatrix} 1 & 4 \\ 3 & 6 \\ 8 & 1 \end{bmatrix}$

b) Next we find the products of C^TD^T and 5E

$$C^TD^T = \begin{bmatrix} 4 & -3 & 6 \\ 8 & -9 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & 6 \\ 8 & 1 \end{bmatrix} = \begin{bmatrix} 43 & 4 \\ 21 & -17 \end{bmatrix}$$

$$5E = 5\begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 25 & 5 \end{bmatrix}$$

c) Finally we do the summation of $C^TD^T + 5E$

$$C^TD^T + 5E = \begin{bmatrix} 43 & 4 \\ 21 & -17 \end{bmatrix} + \begin{bmatrix} 10 & 20 \\ 25 & 5 \end{bmatrix} = \begin{bmatrix} 53 & 24 \\ 46 & -12 \end{bmatrix}$$

4. $(E^2\text{-}10I)(\text{-}D)$ (15 pts.)

a) First we find that $E^2 = \begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 24 & 12 \\ 15 & 21 \end{bmatrix}$ $10I = 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

and $-D = \begin{bmatrix} -1 & -3 & -8 \\ -4 & -6 & -1 \end{bmatrix}$

b) From part a, we subtract $E^2 - 10I = \begin{bmatrix} 14 & 12 \\ 15 & 11 \end{bmatrix}$

c) Last of all we find the final product $(E^2\text{-}10I)(\text{-}D)$

$$(E^2\text{-}10I)(\text{-}D) = \begin{bmatrix} 14 & 12 \\ 15 & 11 \end{bmatrix}\begin{bmatrix} -1 & -3 & -8 \\ -4 & -6 & -1 \end{bmatrix} = \begin{bmatrix} -62 & -114 & -136 \\ -59 & -111 & -142 \end{bmatrix}$$

5. Solve the system of equations (15 pts.)

$$\begin{aligned} x + 2y + z + 3w &= 4 \\ x \quad\quad + 3z - w &= -4 \\ y - 2z + 5w &= 3 \\ 3x + 4y + 6z \quad &= 7 \end{aligned}$$

a) To solve the system of equations, we set up the equations in a matrix and solve for the variables

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 1 & 0 & 3 & -1 & -4 \\ 0 & 1 & -2 & 5 & 3 \\ 3 & 4 & 6 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 1 & -2 & 5 & 3 \\ 1 & 0 & 3 & -1 & -4 \\ 3 & 4 & 6 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 1 & -2 & 5 & 3 \\ 0 & -2 & 2 & -4 & -8 \\ 0 & -2 & 3 & -9 & -5 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 1 & -2 & 5 & 3 \\ 0 & 0 & -2 & 6 & -2 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 1 & -2 & 5 & 3 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 1 & -2 & 5 & 3 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 1 & -2 & 5 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 & 7 \\ 0 & 1 & -2 & 0 & 8 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

From this matrix, we find that: x = 1, y = 4, z = -2, and w = -1

6. Compute the inverse of the following matrix: (15 pts.)

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ 1 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 1 & 0 & 1 & d \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} a & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & b & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & d & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/a & 0 & 0 & 0 \\ 0 & b & 0 & 0 & -1/a & 1 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & d & -1/a & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/ab & 1/b & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & d & -1/a & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/ab & 1/b & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/c & 0 \\ 0 & 0 & 1 & d & -1/a & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/ab & 1/b & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/c & 0 \\ 0 & 0 & 0 & d & -1/a & 0 & -1/c & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/ab & 1/b & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/c & 0 \\ 0 & 0 & 0 & 1 & -1/ad & 0 & -1/cd & 1/d \end{array}\right]$$

7. Solve the systems in all parts simultaneously (15 pts)

$2x_1 + x_2 + 4x_3 = 0$

$4x_1 + 3x_2 + 8x_3 = 1$

$6x_1 + 3x_2 + 13x_3 = 0$

Solution:

Use the identity that Ax=b has one solution that is x= A^{-1} b

A= $\begin{bmatrix} 2 & 1 & 4 \\ 4 & 3 & 8 \\ 6 & 3 & 13 \end{bmatrix}$ x= $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ b= $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Take the inverse of A

$$A^{-1} = \begin{bmatrix} \frac{15}{2} & \frac{-1}{2} & -2 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

By the identity the solution of the system is

$$x = A^{-1} b \begin{bmatrix} \frac{15}{2} & \frac{-1}{2} & -2 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ 1 \\ 0 \end{bmatrix}$$

8. Factor A into the form A= BD, where D is a diagonal matrix. (15 pts)

$$\begin{bmatrix} 2a & 5b & 3c & -4d \\ 2e & 5f & 3g & -4h \\ 2i & 5j & 3k & -4l \\ 2m & 5n & 3o & -4p \end{bmatrix}$$

Solution:

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 2a & 5b & 3c & -4d \\ 2e & 5f & 3g & -4h \\ 2i & 5j & 3k & -4l \\ 2m & 5n & 3o & -4p \end{bmatrix}$$

9. Find det of A^{-1}: (20 pts.)

1/3 ?

$$A= \begin{bmatrix} 3 & 6 & 7 & 2 \\ 2 & 4 & 9 & 7 \\ 3 & 2 & 5 & 6 \\ 1 & 3 & 8 & 4 \end{bmatrix} \cdot 1/3$$

1	2	7/3	2/3		
2	4	9	7	+ (R1 · -2)	1
3	2	5	6	+ (R1 · -3)	1
1	3	8	4	+ (R1 · -1)	1

1	2	7/3	2/3		
0	0	13/3	17/3	switch R2 with R4	-1
0	-4	-2	4		
0	1	17/3	10/3		

1	2	7/3	2/3		
0	1	17/3	10/3		
0	-4	-2	4	+ (R2 · 4)	1
0	0	13/3	17/3		

1	2	7/3	2/3
0	1	17/3	10/3
0	0	62/3	52/3

$$\begin{matrix} 0 & 0 & 13/3 & 17/3 \end{matrix} \quad + R3 \cdot (-13/62) \qquad 1 \qquad C = 1\cdot 1\cdot -1\cdot 1\cdot 1\cdot 1\cdot (1/3) = -1/3$$

$$B = \begin{bmatrix} 1 & 2 & 7/3 & 2/3 \\ 0 & 1 & 17/3 & 10/3 \\ 0 & 0 & 62/3 & 52/3 \\ 0 & 0 & 0 & 17/3 \end{bmatrix}$$

det(B)= $1 \cdot 1 \cdot 62/3 \cdot 17/3 = 1054/9$

det(A)= 1054/9) /C = -1054/3

det(A^{-1})= 1/ det(A) = -3/1054

10. Compute the determinant of the matrix: (10 pts)

$$A = \begin{bmatrix} 1 & 3 & 6 & 7 & 2 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 7 & 0 & 3 & 2 & 6 & 4 & 0 \\ 5 & 0 & 1 & 6 & 4 & 0 & 0 \\ 2 & 4 & 0 & 3 & 1 & 3 & 4 \\ 5 & 0 & 0 & 4 & 0 & 0 & 0 \\ 9 & 0 & 2 & 5 & 7 & 0 & 0 \end{bmatrix}$$

$$\det(A) = \sum (-1)^{i+j} \cdot a_{ij} \cdot C_{ij}$$

$$\det(A) = (-1)^{2+4} \cdot 8 \cdot \det \begin{bmatrix} 1 & 3 & 6 & 2 & 0 & 0 \\ 7 & 0 & 3 & 6 & 4 & 0 \\ 5 & 0 & 1 & 4 & 0 & 0 \\ 2 & 4 & 0 & 1 & 3 & 4 \\ 5 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & 2 & 7 & 0 & 0 \end{bmatrix} = 1 \cdot 8 \cdot ((-1)^{4+6} \cdot 4 \cdot \det \begin{bmatrix} 1 & 3 & 6 & 2 & 0 \\ 7 & 0 & 3 & 6 & 4 \\ 5 & 0 & 1 & 4 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 9 & 0 & 2 & 7 & 0 \end{bmatrix})$$

$$= 1 \cdot 8 \cdot 1 \cdot 4 \cdot ((-1)^{1+2} \cdot 3 \cdot \det \begin{bmatrix} 7 & 3 & 6 & 4 \\ 5 & 1 & 4 & 0 \\ 5 & 0 & 0 & 0 \\ 9 & 2 & 7 & 0 \end{bmatrix})$$

$$= 1 \cdot 8 \cdot 1 \cdot 4 \cdot -1 \cdot 3 \cdot ((-1)^{3+1} \cdot 5 \cdot \det \begin{bmatrix} 3 & 6 & 4 \\ 1 & 4 & 0 \\ 2 & 7 & 0 \end{bmatrix}$$

$$=1 \cdot 8 \cdot 1 \cdot 4 \cdot -1 \cdot 3 \cdot 1 \cdot 5 \cdot (-1)^{1+3} \cdot 4 \cdot \det \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$$

$$= 1 \cdot 8 \cdot 1 \cdot 4 \cdot -1 \cdot 3 \cdot 1 \cdot 5 \cdot 4 \cdot (7-8)$$

$$= 1920 = \det(A)$$

1. Consider the system of equations

$$\begin{array}{llll} x & -y & +2z & =0 \\ 2x & & +z & =1 \\ x & -y & -z & =3 \\ -x & +2y & +2z & =0 \end{array}.$$

(a) Write the augmented matrix for this system.

(b) Use elementary row operations to reduce the augmented matrix to row echelon form.

(c) Classify this system as inconsistent or consistent, and give a solution if consistent.

Solution

(a) The augmented matrix is

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & -1 & -1 & 3 \\ -1 & 2 & 2 & 0 \end{bmatrix}$$

(b) We can reduce the augmented matrix as follows:

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & -1 & -1 & 3 \\ -1 & 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -3 & 3 \\ 0 & 1 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & -11 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & -30 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -30 \end{bmatrix}.$$

This last matrix is in row echelon form, so we are finished. Note, however, that there are many other possible solutions.

(c) As the last row of the row-echelon form matrix we gave above represents $0 = -30$, which is patently false, it is clear that this system of equations is inconsistent.

2. (a) What is the determinant of the matrix I_3,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}?$$

(b) Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

by reducing it to I_3. List each necessary elementary row operation and its effect on det$A$; then solve the relation between det$A$ and I_3.

Solution

(a) As the determinant of a diagonal matrix is the product of the diagonal entries, $\det I_3 = 1$.

(b) The following elementary row operations, with their corresponding effects on the determinant, will transform A into I_3:

- add -2 times the first row to the second: determinant unchanged
- add $-\frac{1}{3}$ times the second row to the third: determinant unchanged
- add $\frac{30}{13}$ times the third row to the second: determinant unchanged
- add $\frac{15}{13}$ times the third row to the first: determinant unchanged
- add $\frac{2}{3}$ times the second row to the first: determinant unchanged
- multiply second row by $-\frac{1}{3}$: determinant multiplied by $-\frac{1}{3}$
- multiply third row by $\frac{3}{13}$: determinant multiplied by $\frac{3}{13}$.

We have now reduced A to I_3; the relation between det$A$ and $\det I_3 = 1$ is

$$-\frac{1}{3} * \frac{3}{13} * \det A = 1$$

$$\det A = 1 * (-3) * \frac{13}{3} = -13$$

3. Consider the matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Find

(a) A^T

(b) The trace of A.

Solution

(a) The transpose of a matrix is defined as $A^T_{ij} = A_{ji}$, so

$$A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}.$$

(b) The trace of $A = A_{n \times n}$ is given by the sum of the diagonal entries ($\sum_{i=1}^{n} a_{ii}$), so the trace of A is given by $a + e + i$.

4. Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix}; \; B = \begin{bmatrix} -1 & 1 & 3 \\ 0 & -2 & 3 \\ 2 & 1 & 0 \end{bmatrix}; \; C = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -2 & 1 \\ -1 & -2 & 3 \end{bmatrix}$$

and their inverses,

$$A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -1 & \frac{8}{3} & \frac{2}{3} \\ 1 & -\frac{5}{3} & -\frac{2}{3} \end{bmatrix}; \; B^{-1} = \begin{bmatrix} -\frac{1}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & -\frac{2}{7} & \frac{1}{7} \\ \frac{4}{21} & \frac{1}{7} & \frac{2}{21} \end{bmatrix}; \; C^{-1} = \begin{bmatrix} \frac{2}{9} & \frac{1}{9} & -\frac{1}{9} \\ \frac{7}{18} & -\frac{5}{9} & \frac{1}{18} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

(a) Express $(ABC)^{-1}$ as a product of 3 matrices (do not actually multiply).

(b) Express $(A^{-1}B^{-1}C^{-1})^T$ as a product of 3 matrices.

(c) Find the inverse of $3A$.

(d) Find any matrix D such that $B^{-1}D = C$.

Solution

(a) The inverse of the product of several invertible matrices in the product of their inverses in reverse order, so $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$, which is expressed as

$$\begin{bmatrix} \frac{2}{9} & \frac{1}{9} & -\frac{1}{9} \\ \frac{7}{18} & -\frac{5}{9} & \frac{1}{18} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -\frac{1}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & -\frac{2}{7} & \frac{1}{7} \\ \frac{4}{21} & \frac{1}{7} & \frac{2}{21} \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ -1 & \frac{8}{3} & \frac{2}{3} \\ 1 & -\frac{5}{3} & -\frac{2}{3} \end{bmatrix}.$$

(b) Similarly, the transpose of a product of two or more matrices is equal to the product of their transposes in reverse order. Thus, $(A^{-1}B^{-1}C^{-1})^T = (C^{-1})^T(B^{-1})^T(A^{-1})^T$, which is

$$\begin{bmatrix} \frac{2}{9} & \frac{1}{9} & -\frac{1}{9} \\ \frac{7}{18} & -\frac{5}{9} & \frac{1}{18} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}^T \begin{bmatrix} -\frac{1}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & -\frac{2}{7} & \frac{1}{7} \\ \frac{4}{21} & \frac{1}{7} & \frac{2}{21} \end{bmatrix}^T \begin{bmatrix} -1 & 2 & 1 \\ -1 & \frac{8}{3} & \frac{2}{3} \\ 1 & -\frac{5}{3} & -\frac{2}{3} \end{bmatrix}^T,$$

or

$$\begin{bmatrix} \frac{2}{9} & \frac{7}{18} & \frac{1}{3} \\ \frac{1}{9} & -\frac{5}{9} & -\frac{1}{3} \\ -\frac{1}{9} & \frac{1}{18} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -\frac{1}{7} & \frac{2}{7} & \frac{4}{21} \\ \frac{1}{7} & -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{1}{7} & \frac{2}{21} \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 2 & \frac{8}{3} & -\frac{5}{3} \\ 1 & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$

(c) We know that for any non-zero scalar and any invertible matrix A, $(kA)^{-1} = \frac{1}{k}A^{-1}$. Since 3 is a non-zero scalar and A is invertible, $(3A)^{-1} = \frac{1}{3}A^{-1}$, or

$$\frac{1}{3}\begin{bmatrix} -1 & 2 & 1 \\ -1 & \frac{8}{3} & \frac{2}{3} \\ 1 & -\frac{5}{3} & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{8}{9} & \frac{2}{9} \\ \frac{1}{3} & -\frac{5}{9} & -\frac{2}{9} \end{bmatrix}.$$

(d) If $B^{-1}D = C$, then $BB^{-1}D = BC \Rightarrow ID = BC \Rightarrow D = BC$. Given B and C, we find that

$$D = \begin{bmatrix} -1 & 1 & 3 \\ 0 & -2 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 2 & -2 & 1 \\ -1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -4 & -8 & 9 \\ -7 & -2 & 7 \\ 8 & -2 & 3 \end{bmatrix}$$

5. Which of the following are elementary matrices?

(a)

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(e)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(f)

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Solution

(a) YES corresponds to adding two times the second row to the first row.

(b) NO requires two elementary row operations.

(c) YES multiply first row by three.

(d) NO not square.

(e) YES switch first and third rows.

(f) NO requires two elementary row operations.

6. (a) Consider the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & 4 & 2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}.$$

Solve for x_1, x_2, and x_3.

(b) Later it is found that

$$\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Solve for x_1, x_2, and x_3 with the new $\mathbf{b}$.

Solution

(a) If we combine A and $\mathbf{b}$ and reduce A to reduced row echelon form, we will obtain $\mathbf{x}$ where $\mathbf{b}$ was:

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 2 & 2 & 2 & 2 \\ 3 & 4 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 2 & 2 & -4 \\ 0 & 4 & 2 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 1 & -3 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0- & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow x_1 = 3; x_2 = -1; x_3 = -1.$$

(b) We follow exactly the same procedure as in (a):

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 2 & 2 & 2 & 1 \\ 3 & 4 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 2 & 2 & -3 \\ 0 & 4 & 2 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -\frac{3}{2} \\ 0 & 2 & 1 & -2 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -\frac{3}{2} \\ 0 & 0- & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow x_1 = 2; x_2 = -\frac{1}{2}; x_3 = -1.$$

7. Consider the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

(a) Is B invertible? If so, calculate B^{-1}. If not, which entry should be changed to make B invertible?

(b) Calculate B^3

Solution

(a) Because B is a diagonal matrix with a 0 entry on the main diagonal, it is not invertible. Noting that B has a row and/or column of 0 entries is also an acceptable explanation for why it is not invertible.

To make B invertible, the entry b_{22} should be changed to any non-zero real number. Changing any other entry would leave an empty row or column, so that B would still not be invertible.

(b) Because B is a diagonal matrix, B^n is calculated by raising each diagonal entry to the nth power. The resulting matrix would be:

$$B^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 216 \end{bmatrix}.$$

8. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}.$$

(a) List all possible permutations σ in the set S_3, along with sign(σ) for each one.

(b) Use these to find the determinant of A.

Solution

(a) $S_3 = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$
$\text{sign}(1,2,3) = 1$; $\text{sign}(1,3,2) = -1$; $\text{sign}(2,1,3) = -1$; $\text{sign}(2,3,1) = 1$;
$\text{sign}(3,1,2) = 1$; $\text{sign}(3,2,1) = -1$.

(b) The definition of determinant states that

$$\det A = \sum_{\sigma \in S} \text{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)}.$$

Applying this to A, we have

$$\begin{aligned} \det A &= (a_{11} * a_{22} * a_{33}) - (a_{11} * a_{23} * a_{32}) - (a_{12} * a_{21} * a_{33}) \\ &\quad + (a_{12} * a_{23} * a_{31}) + (a_{13} * a_{21} * a_{32}) - (a_{13} * a_{22} * a_{31}) \\ &= (1 * 2 * 1) - (1 * 3 * 1) - (3 * 2 * 1) + (3 * 3 * 3) + (2 * 2 * 1) - (2 * 2 * 3) \\ &= 2 - 3 - 6 + 27 + 4 - 12 = 12 \end{aligned}$$

9. Consider the matrix

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & -3 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) What is M_{11}?

(b) What is C_{22}?

(c) What row or column should one expand on to calculate $\det A$ as efficiently as possible?

(d) Calculate $\det A$ by cofactor expansion, expanding on the row or column in (c).

Solution

(a)

$$M_{11} = \det \begin{bmatrix} -3 & 5 \\ 0 & 2 \end{bmatrix} = -6.$$

(b)

$$C_{22} = (-1)^{2+2} \det \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} = 1 * 2 = 2.$$

(c) To calculate $\det A$ most efficiently, one should expand on the third row, as it has 2 zeros.

(d)

$$\det A = 0 + 0 + 2 * \det \begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix} = 2 * (-3 - 8) = 2 * -11 = -22.$$

10. Consider the matrix A and its inverse, A^{-1}:

$$A = \begin{bmatrix} 2 & 0 & 1 & 4 \\ 0 & 2 & 3 & 1 \\ 1 & -2 & 0 & 2 \\ -1 & -1 & 3 & 1 \end{bmatrix}; \; A^{-1} = \begin{bmatrix} -\frac{6}{5} & \frac{8}{5} & \frac{11}{5} & -\frac{-6}{5} \\ \frac{2}{5} & -\frac{1}{5} & -\frac{11}{15} & \frac{1}{15} \\ -\frac{3}{5} & \frac{4}{5} & \frac{14}{15} & -\frac{4}{15} \\ 1 & -1 & -\frac{4}{3} & \frac{2}{3} \end{bmatrix}.$$

(a) Find $\det A$ using elementary row operations

(b) Find the adjoint of A

(c) Find the determinant of $2A$

Solution

(a) Each elementary row operation is associated with some operation on the value of the determinant, so we can reduce A to upper triangular form, keeping track of each elementary row operation, find the determinant of the upper triangular matrix, and then apply the appropriate operations to obtain $\det A$.

$$\begin{bmatrix} 2 & 0 & 1 & 4 \\ 0 & 2 & 3 & 1 \\ 1 & -2 & 0 & 2 \\ -1 & -1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 2 & 3 & 1 \\ 2 & -0 & 1 & 4 \\ -1 & -1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 4 & 1 & 0 \\ 0 & -3 & 3 & 3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 4 & 1 & 0 \\ 0 & -3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & -5 & -2 \\ 0 & 0 & \frac{15}{2} & \frac{9}{2} \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{2}{5} \\ 0 & 0 & \frac{15}{2} & \frac{9}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 & \frac{3}{2} \end{bmatrix}.$$

As the determinant of the upper triangular matrix is $1 * 1 * 1 * \frac{3}{2} = \frac{3}{2}$, and we multiplied $\det A$ by -1 and divided by 2 and -5 to obtain that $\det A = \frac{3}{2} * (-1) * 2 * (-5) = 15$.

(b) Since $A\text{adj}A = \det A * I$, $\text{adj}A = A^{-1}\det A * I = \det A^{-1}$. We know $\det A$ and A^{-1}, so

$$\text{adj}A = 15 \begin{bmatrix} -\frac{6}{5} & \frac{8}{5} & \frac{11}{5} & -\frac{-6}{5} \\ \frac{2}{5} & -\frac{1}{5} & -\frac{11}{15} & \frac{1}{15} \\ -\frac{3}{5} & \frac{4}{5} & \frac{14}{15} & -\frac{4}{15} \\ 1 & -1 & -\frac{4}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} -18 & 24 & 33 & -18 \\ 6 & -3 & -11 & 1 \\ -9 & 12 & 14 & -4 \\ 15 & -15 & -20 & 10 \end{bmatrix}.$$

(c) ?

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Math 343 Exam 1

1. (30pts) For the following problems, let matrices A through F be equal to the following:

$$A=\begin{bmatrix}2 & 3\\ 4 & 7\end{bmatrix}\quad B=\begin{bmatrix}1 & 2\\ 0 & 3\\ 2 & 5\end{bmatrix}\quad C=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\quad E=\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 3\\ 1 & 2 & 2\end{bmatrix}\quad F=\begin{bmatrix}3 & 2 & 1 & 0\\ 0 & 5 & 6 & 7\\ 8 & 3 & 0 & 5\end{bmatrix}$$

Based on these values, compute the following:

a) $2A + C^2$
b) A^{-1}
c) C^{-1}
d) E^{-1}
e) BA
f) $(BC)^T$
g) Assuming that F represents a system of linear equations, solve the system using Gauss-Jordan Elimination.

Solution:

a) $$2A=\begin{bmatrix}4 & 6\\ 8 & 14\end{bmatrix}\quad C^2=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}=\begin{bmatrix}5 & 10\\ 10 & 20\end{bmatrix}\Rightarrow 2A+C^2=\begin{bmatrix}4+5 & 6+10\\ 8+10 & 14+20\end{bmatrix}=\begin{bmatrix}9 & 16\\ 18 & 34\end{bmatrix}$$

b) $$A^{-1}=\frac{1}{2\times 7-4\times 3}\begin{bmatrix}7 & -3\\ -4 & 2\end{bmatrix}=\begin{bmatrix}7/2 & -3/2\\ -2 & 1\end{bmatrix}$$

c) $$C^{-1}=\frac{1}{1\times 4-2\times 2}\begin{bmatrix}4 & -2\\ -2 & 1\end{bmatrix}\Rightarrow 1\times 4-2\times 2=0 \therefore \text{C is singualar}$$

d) $$E^{-1}\Rightarrow\left[\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 2 & 1 & 3 & 0 & 1 & 0\\ 1 & 2 & 2 & 0 & 0 & 1\end{array}\right]$$

$$E^{-1}\Rightarrow\left[\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 0 & -3 & -3 & -2 & 1 & 0\\ 0 & 0 & 1 & 1 & 0 & -1\end{array}\right]$$

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$$E^{-1} = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 0 & 3 \\ 0 & 3 & 0 & -1 & -1 & 3 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$E^{-1} = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$E^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4/3 & 2/3 & 1 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

e) $BA = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 10 & 17 \\ 12 & 21 \\ 24 & 41 \end{bmatrix}$

f) $(BC)^T = \left(\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \right)^T = \left(\begin{bmatrix} 5 & 10 \\ 6 & 12 \\ 12 & 24 \end{bmatrix} \right)^T = \begin{bmatrix} 5 & 6 & 12 \\ 10 & 12 & 24 \end{bmatrix}$

First row - reduce matrix F

g)

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 5 & 6 & 7 \\ 8 & 3 & 0 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 5 & 6 & 7 \\ 0 & -7/3 & -8/3 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 5 & 6 & 7 \\ 0 & -7 & -8 & 15 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 2/5 & 124/5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 2 & 0 & -62 \\ 0 & 5 & 0 & -365 \\ 0 & 0 & 1 & 62 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 2 & 0 & -62 \\ 0 & 1 & 0 & -73 \\ 0 & 0 & 1 & 62 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 & 0 & 84 \\ 0 & 1 & 0 & -73 \\ 0 & 0 & 1 & 62 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 28 \\ 0 & 1 & 0 & -73 \\ 0 & 0 & 1 & 62 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 28 \\ x_2 = -73 \\ x_3 = 62 \end{matrix}$$

2. (15pts) For the following problem let matrix B equal the following:

$$B = \begin{bmatrix} 2 & 6 & 3 \\ 4 & 7 & 5 \\ 1 & 0 & 2 \end{bmatrix}$$

Show how you would find the determinant of B using the following theorem:

$$\det(B) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) B_1\sigma_{(1)} \cdots B_n\sigma_n$$

Solution:

$$\det(B) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) B_1\sigma_{(1)} \cdots B_n\sigma_n$$

$$= sign(1,2,3) * 2 * 7 * 2 + sign(1,3,2) * 2 * 5 * 0 + sign(2,1,3) * 6 * 4 * 2 + sign(2,3,1) * 6 * 5 * 1 +$$
$$sign(3,1,2)(3 * 4 * 0) + sign(3,2,1) * 3 * 7 * 1$$
$$= 28 - 0 - 48 + 30 - 0 - 21$$
$$= -11$$

3. (15pts) Using the method of cofactor expansion, compute the determinant of the following matrix:

$$\begin{bmatrix} 2 & 2 & 6 & 5 & 3 & 2 \\ 0 & 7 & 0 & 8 & 0 & 1 \\ 4 & 6 & 7 & 7 & 5 & 4 \\ 0 & 9 & 0 & 2 & 0 & 0 \\ 1 & 5 & 0 & 8 & 2 & 3 \\ 0 & 3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

By expanding on the sixth row :

$$\det\begin{bmatrix} 2 & 2 & 6 & 5 & 3 & 2 \\ 0 & 7 & 0 & 8 & 0 & 1 \\ 4 & 6 & 7 & 7 & 5 & 4 \\ 0 & 9 & 0 & 2 & 0 & 0 \\ 1 & 5 & 0 & 8 & 2 & 3 \\ 0 & 3 & 0 & 0 & 0 & 0 \end{bmatrix} = (+3)\det\left(\begin{bmatrix} 2 & 6 & 5 & 3 & 2 \\ 0 & 0 & 8 & 0 & 1 \\ 4 & 7 & 7 & 5 & 4 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 8 & 2 & 3 \end{bmatrix}\right)$$

$$= (+3)(-2)\det\left(\begin{bmatrix} 2 & 6 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 4 & 7 & 5 & 4 \\ 1 & 0 & 2 & 3 \end{bmatrix}\right)$$

$$= (+3)(-2)(+1)\det\left(\begin{bmatrix} 2 & 6 & 3 \\ 4 & 7 & 5 \\ 1 & 0 & 2 \end{bmatrix}\right)$$

$$= (+3)(-2)(+1)(-11), \text{(according to problem 2)}$$

$$= 66$$

4. (10pts) Prove that the matrix $A = \begin{bmatrix} a\,b \\ c\,d \end{bmatrix}$ is invertible if, and only if, $ad - bc = 0$.

Solution:

By definition, if $A = \begin{bmatrix} a\,b \\ c\,d \end{bmatrix}$, then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}.$$

If $ad - bc = 0$, then we will be dividing by zero, causing the solution to fail.

5. (10pts) For the following problem, let A be a square matrix that is equal to the following:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Determine whether A is invertible, and if so, find its inverse.

Solution:

If A is invertible, then A^{-1} can be found by using row operations and the identity matrix I_n.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \text{Interchange rows 2 and 3.}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \text{Add -1 times row 2 to rows 1 and 3.}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix} \rightarrow \text{Add -2 times row 1 to row 3.}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 2/3 & -1/3 & -1/3 \end{bmatrix} \rightarrow \text{Multiply the 3}^{\text{rd}}\text{ row by } -1/3.$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1/3 & 2/3 & \frac{-1}{3} \\ 0 & 0 & 1 \\ 2/3 & -1/3 & -1/3 \end{bmatrix} \rightarrow \text{Add - 2 times the } 3^{rd} \text{ row to row 1.}$$

Thus, $A^{-1} = \begin{bmatrix} -1/3 & 2/3 & \frac{-1}{3} \\ 0 & 0 & 1 \\ 2/3 & -1/3 & -1/3 \end{bmatrix}$

6. (10pts) Show that for matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}$, if A is an elementary matrix (E_3), then at least 1 entry in the last row of A must be zero.

Solution: By definition, an elementary matrix (E_n) is a matrix that can be obtained from the nxn identity matrix I_n by performing a single elementary row operation.

For the possibilities:

(a) If $c = 1$ then either a or b may be any value, while the other is zero, and the single elementary row operation will be performed on the nonzero value to make it zero by adding a multiple of another row to the last row.

(b) If $c = 1; a, b = 0$ then the single elementary row operation would be to multiply the last row by the constant 1.

(c) If $c \neq 1,0$ then both a and b must be zero and the single elementary row operation will be performed on c to make it 1.

(d) If $c = 0$ then the matrix A is no longer able to take on the form of an elementary matrix (E_n), regardless of the values of a and b .

Thus we see that in order for A to be considered an elementary matrix (E_n) a or b must be zero.

7. (15pts) For the following system of equations, what conditions must ~~x, y,~~ a, b and ~~z~~ c satisfy for the system of equations to be consistent?

$$\begin{aligned} x + y - z &= a \\ z &= b \\ 2x + y + 2z &= c \end{aligned}$$

Solution:

The augmented matrix would appear in the form:

$$\begin{bmatrix} 1 & 1 & -1 & a \\ 0 & 0 & 1 & b \\ 2 & 1 & 2 & c \end{bmatrix}$$

Reducing this to row-echelon form yields:

$$\begin{bmatrix} 1 & 0 & 0 & -a-3b+c \\ 0 & 1 & 0 & 2a+4b-c \\ 0 & 0 & 1 & b \end{bmatrix}$$

As we can see, there are no restrictions on a, b, and c; hence the ~~conditions~~ solution regarding x, ~~y, and z are:~~ is

$$x = -a - 3b + c$$
$$y = 2a + 4b - c$$
$$z = b$$

8. (10pts) If A is a symmetric n x n matrix and B is n x m, prove that B^TAB is a symmetric m x m matrix.

Solution: A is given as a symmetric matrix; therefore, $A^T = A$. Assuming that AB is defined then:

$$(B^TAB)^T = B^TA^T(B^T)^T = B^TAB,$$

From this we see that the transpose of B^TAB is simply itself, which illustrates its symmetry.

9. (15pts) Show that the following ~~expression~~ equation is true.

$$\det(A) = \begin{vmatrix} 2a & 2b & b-c \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix} = -2(a-b)^2(a+b)$$

Solution:

$$\begin{vmatrix} 2a & 2b & b-c \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix} \sim \begin{vmatrix} 2a+2b & 2b+2a & b+a \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix} \sim (a+b) \begin{vmatrix} 2 & 2 & 1 \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix}$$

*Continued on the following page.

$$\sim(a+b)\begin{vmatrix} 0 & 2 & 1 \\ 2(b-a) & 2a & a+c \\ 0 & a+b & b \end{vmatrix} \sim 2(a+b)(a-b)\begin{vmatrix} 2 & 1 \\ a+b & b \end{vmatrix} = -2(a+b)(a-b)^2$$

First add row 2 to row 1. Then take a common factor of (a+b) through the determinant sign. Next add subtract column 2 from column 1. Then take the determinant through column 1.

10. (20pts) Given the formula $\det(B) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) B_1\sigma_{(1)} \cdots B_n\sigma_n$, derive a formula for the determinant of the following matrix:

$$B = \begin{bmatrix} a & b & c & d \\ b & e & f & c \\ c & f & e & b \\ d & c & b & a \end{bmatrix}$$

Solution: You can find the signed elementary products by setting up the following table.

Associated Permutation	sgn(σ)	sign(σ)	Elementary Product	Signed Elementary P.
(1, 2, 3, 4)	0	+	a*e*e*a	+a*a*e*e
(1, 2, 4, 3)	1	-	a*e*c*c	-a*c*c*e
(1, 3, 2, 4)	1	-	a*f*f*a	-a*a*f*f
(1, 3, 4, 2)	2	+	a*f*c*b	+a*b*c*f
(1, 4, 2, 3)	2	+	a*b*f*c	+a*b*c*f
(1, 4, 3, 2)	3	-	a*b*e*b	-a*b*b*e
(2, 1, 3, 4)	1	-	b*b*e*a	-a*b*b*e
(2, 1, 4, 3)	2	+	b*b*c*c	+b*b*c*c
(2, 3, 1, 4)	2	+	b*f*c*a	+a*b*c*f
(2, 3, 4, 1)	3	-	b*f*c*d	-b*c*d*f
(2, 4, 1, 3)	3	-	b*b*c*c	-b*b*c*c
(2, 4, 3, 1)	4	+	b*b*e*d	+b*b*e*d
(3, 1, 2, 4)	2	+	c*b*f*a	+a*b*c*f
(3, 1, 4, 2)	3	-	c*b*c*b	-b*b*c*c
(3, 2, 1, 4)	3	-	c*e*c*a	-a*c*c*e
(3, 2, 4, 1)	4	+	c*e*c*d	+c*c*d*e
(3, 4, 1, 2)	4	+	c*b*c*b	+b*b*c*c
(3, 4, 2, 1)	5	-	c*b*f*d	-b*c*d*f
(4, 1, 2, 3)	3	-	d*b*f*c	-b*c*d*f
(4, 1, 3, 2)	4	+	d*b*e*b	+b*b*d*e
(4, 2, 1, 3)	4	+	d*e*c*c	+c*c*d*e
(4, 2, 3, 1)	5	-	d*e*e*d	-d*d*e*e
(4, 3, 1, 2)	5	-	d*f*c*b	-b*c*d*f
(4, 3, 2, 1)	6	+	d*f*f*d	+d*d*f*f

From here you simply combine the signed elementary products and simplify. The final answer should be:

$$\det(B) = a^2e^2 - 2ac^2e - a^2f^2 + 4abcf - 2ab^2e - 4bcdf + 2c^2de + 2b^2de - d^2e^2 + d^2f^2$$

1.

Theorem 1.4.6

If A and B are invertible matrices of the same size, then AB is invertible and **$(AB)^{-1} = B^{-1}A^{-1}$**

Use matrices A and B listed below to verify this theorem.

$$A=\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix} \quad B=\begin{bmatrix}3 & 2\\ 2 & 3\end{bmatrix}$$

__**Solution**__: $AB = \begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\ 2 & 3\end{bmatrix} = \begin{bmatrix}3+6 & 2+9\\ 6+2 & 4+3\end{bmatrix} = \begin{bmatrix}9 & 11\\ 8 & 7\end{bmatrix}$

$$(AB)^{-1}=\begin{bmatrix}-7/25 & 11/25\\ 8/25 & -9/25\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}-1/5 & 3/5\\ 2/5 & -1/5\end{bmatrix} \quad B^{-1}=\begin{bmatrix}3/5 & -2/5\\ -2/5 & 3/5\end{bmatrix}$$

$$B^{-1}A^{-1}=\begin{bmatrix}3/5 & -2/5\\ -2/5 & 3/5\end{bmatrix}\begin{bmatrix}-1/5 & 3/5\\ 2/5 & -1/5\end{bmatrix} = \begin{bmatrix}-7/25 & 11/25\\ 8/25 & -9/25\end{bmatrix}$$

$$(AB)^{-1}=\begin{bmatrix}-7/25 & 11/25\\ 8/25 & -9/25\end{bmatrix} = B^{-1}A^{-1}= \begin{bmatrix}-7/25 & 11/25\\ 8/25 & -9/25\end{bmatrix}$$

2.

Solve this system of equations using A^{-1}.

$2x_1+x_2+4x_3 = -1$
$x_1+5x_2-2x_3 = 12$
$x_1+x_2+x_3 = 2$

__**Solution**__: $A\underline{x} = \underline{b}$

$\underline{x} = A^{-1}\underline{b}$

$$A^{-1} = \begin{bmatrix}-7/5 & -3/5 & 22/5\\ 3/5 & 2/5 & -8/5\\ 4/5 & 1/5 & -9/5\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}-7/5 & -3/5 & 22/5\\ 3/5 & 2/5 & -8/5\\ 4/5 & 1/5 & -9/5\end{bmatrix}\begin{bmatrix}-1\\ 12\\ 2\end{bmatrix} = \begin{bmatrix}7/5-36/5+44/5\\ -3/5+24/3-16/5\\ -4/5+12/5-18/5\end{bmatrix} = \begin{bmatrix}3\\ 1\\ -2\end{bmatrix} = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

So $x_1=3$, $x_2=1$, and $x_3= -2$

3.

What conditions must b_1, b_2, and b_3 satisfy in order for this system of equations to be consistent?

$x_1+x_2+x_3 = b_1$
$x_1+2x_3 = b_2$
$-x_2+2x_3 = b_3$

Solution: $\begin{bmatrix} 1 & 1 & 1 & b_1 \\ 1 & 0 & 2 & b_2 \\ 0 & -1 & 2 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & b_1 \\ 0 & -1 & 2 & b_3 \\ 1 & 0 & 2 & b_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & b_1 \\ 0 & -1 & 2 & b_3 \\ 0 & -1 & 1 & b_2 - b_1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & b_1 \\ 0 & -1 & 2 & b_3 \\ 0 & 0 & -1 & b_2 - b_1 - b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & b_1 + b_3 \\ 0 & -1 & 2 & b_3 \\ 0 & 0 & -1 & b_2 - b_1 - b_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2b_1 + 3b_2 - 2b_3 \\ 0 & -1 & 0 & -2b_1 + 2b_2 - b_3 \\ 0 & 0 & -1 & b_2 - b_1 - b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2b_1 + 3b_2 - 2b_3 \\ 0 & 1 & 0 & 2b_1 - 2b_2 + b_3 \\ 0 & 0 & 1 & -b_2 + b_1 + b_3 \end{bmatrix}$$

Therefore: $x_1 = -2b_1 + 3b_2 - 2b_3$
$x_2 = 2b_1 - 2b_2 + b_3$
$x_3 = -b_2 + b_1 + b_3$

none !

4.

Solve by Gauss-Jordan Elimination

$$\begin{bmatrix} 6 & 12 & -2 & -30 \\ 10 & 6 & 4 & 0 \\ 6 & 2 & 6 & 22 \\ -12 & -8 & 4 & 60 \end{bmatrix}$$

?

Solution:

$x_1 = -4$
$x_2 = 2$
$x_3 = 7$

5.

Find A^{-1} using its Adjoint

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$$

Solution:

det(A) = 4

$$\text{adj}(A) = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & -4 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}\text{adj(A)} = (4)\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1/2 \end{bmatrix}$$

6.

Solve the system of equations using matricies:

$$\begin{aligned}
2x_0 + 2x_1 + 4x_2 + 2x_4 &= 42 \\
-2x_0 - 1x_1 - 2x_2 - 6x_4 &= -143 \\
-1x_0 - 1x_1 - 5x_2 + 5x_4 &= 123 \\
4x_0 + 4x_1 + 8x_2 + 6x_3 + 4x_4 &= 138 \\
-2x_0 - 2x_1 - 4x_2 + 2x_4 &= 58 \\
\tfrac{1}{2}x_1 + 1x_2 - 2x_4 &= -\tfrac{101}{2}
\end{aligned}$$

Solution:

Take the system of equations

$$
\begin{aligned}
2x_0 + 2x_1 + 4x_2 + 2x_4 &= 42 \\
-2x_0 - 1x_1 - 2x_2 - 6x_4 &= -143 \\
-1x_0 - 1x_1 - 5x_2 + 5x_4 &= 123 \\
4x_0 + 4x_1 + 8x_2 + 6x_3 + 4x_4 &= 138 \\
-2x_0 - 2x_1 - 4x_2 + 2x_4 &= 58 \\
\tfrac{1}{2}x_1 + 1x_2 - 2x_4 &= -\tfrac{101}{2}
\end{aligned}
$$

and put it in Matrix Form $\Rightarrow \begin{bmatrix} 2 & 2 & 4 & 0 & 2 & 42 \\ -2 & -1 & -2 & 0 & -6 & -143 \\ -1 & -1 & -5 & 0 & 5 & 123 \\ 4 & 4 & 8 & 6 & 4 & 138 \\ -2 & -2 & -4 & 0 & 2 & 58 \\ 0 & \tfrac{1}{2} & 1 & 0 & -2 & -\tfrac{101}{2} \end{bmatrix}$

Times the first row by 1 and add to the 2nd row $\Rightarrow \begin{bmatrix} 2 & 2 & 4 & 0 & 2 & 42 \\ 0 & 1 & 2 & 0 & -4 & -101 \\ -1 & -1 & -5 & 0 & 5 & 123 \\ 4 & 4 & 8 & 6 & 4 & 138 \\ -2 & -2 & -4 & 0 & 2 & 58 \\ 0 & \tfrac{1}{2} & 1 & 0 & -2 & -\tfrac{101}{2} \end{bmatrix}$

Times the first row by $\frac{1}{2}$ add it to the 2nd row $\Rightarrow$ $$\begin{bmatrix} 2 & 2 & 4 & 0 & 2 & 42 \\ 0 & 1 & 2 & 0 & -4 & -101 \\ 0 & 0 & -3 & 0 & 6 & 144 \\ 4 & 4 & 8 & 6 & 4 & 138 \\ -2 & -2 & -4 & 0 & 2 & 58 \\ 0 & \frac{1}{2} & 1 & 0 & -2 & -\frac{101}{2} \end{bmatrix}$$

Times the 1st row by -2 add it to the 3rd row $\Rightarrow$ $$\begin{bmatrix} 2 & 2 & 4 & 0 & 2 & 42 \\ 0 & 1 & 2 & 0 & -4 & -101 \\ 0 & 0 & -3 & 0 & 6 & 144 \\ 0 & 0 & 0 & 6 & 0 & 138 \\ -2 & -2 & -4 & 0 & 2 & 58 \\ 0 & \frac{1}{2} & 1 & 0 & -2 & -\frac{101}{2} \end{bmatrix}$$

Add the first row to the second 5th row $\Rightarrow$ $$\begin{bmatrix} 2 & 2 & 4 & 0 & 2 & 42 \\ 0 & 1 & 2 & 0 & -4 & -101 \\ 0 & 0 & -3 & 0 & 6 & 144 \\ 0 & 0 & 0 & 6 & 0 & 54 \\ 0 & 0 & 0 & 0 & 4 & 100 \\ 0 & \frac{1}{2} & 1 & 0 & -2 & -\frac{101}{2} \end{bmatrix}$$

Times the 1st row by - $\frac{1}{4}$ add it to the 6th row $\Rightarrow$ $$\begin{bmatrix} 2 & 2 & 4 & 0 & 2 & 42 \\ 0 & 1 & 2 & 0 & -4 & -101 \\ 0 & 0 & -3 & 0 & 6 & 144 \\ 0 & 0 & 0 & 6 & 0 & 54 \\ 0 & 0 & 0 & 0 & 4 & 100 \\ -\frac{1}{2} & 0 & 0 & 0 & -\frac{5}{2} & -61 \end{bmatrix}$$

Divide the 1st row by 2 $\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 21 \\ 0 & 1 & 2 & 0 & -4 & -101 \\ 0 & 0 & -3 & 0 & 6 & 144 \\ 0 & 0 & 0 & 6 & 0 & 54 \\ 0 & 0 & 0 & 0 & 4 & 100 \\ -\frac{1}{2} & 0 & 0 & 0 & -\frac{5}{2} & -61 \end{bmatrix}$

Divide the 3rd row by -3 $\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 21 \\ 0 & 1 & 2 & 0 & -4 & -101 \\ 0 & 0 & 1 & 0 & -2 & -48 \\ 0 & 0 & 0 & 6 & 0 & 54 \\ 0 & 0 & 0 & 0 & 4 & 100 \\ -\frac{1}{2} & 0 & 0 & 0 & -\frac{5}{2} & -61 \end{bmatrix}$

Divide the 4th row by 6 $\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 21 \\ 0 & 1 & 2 & 0 & -4 & -101 \\ 0 & 0 & 1 & 0 & -2 & -48 \\ 0 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 & 4 & 100 \\ -\frac{1}{2} & 0 & 0 & 0 & -\frac{5}{2} & -61 \end{bmatrix}$

Divide the 5th row by 4 $\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 21 \\ 0 & 1 & 2 & 0 & -4 & -101 \\ 0 & 0 & 1 & 0 & -2 & -48 \\ 0 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 25 \\ -\frac{1}{2} & 0 & 0 & 0 & -\frac{5}{2} & -61 \end{bmatrix}$

Multiply the 6th row by 2 $\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 21 \\ 0 & 1 & 2 & 0 & -4 & -101 \\ 0 & 0 & 1 & 0 & -2 & -48 \\ 0 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 25 \\ -1 & 0 & 0 & 0 & -5 & -122 \end{bmatrix}$

Times the 2nd row by -1 add it to the first row $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 5 & 122 \\ 0 & 1 & 2 & 0 & -4 & -101 \\ 0 & 0 & 1 & 0 & -2 & -48 \\ 0 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 25 \\ -1 & 0 & 0 & 0 & -5 & -122 \end{bmatrix}$

Times the 3rd row by -2 add it to the 2nd row $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 5 & 122 \\ 0 & 1 & 0 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & -2 & -48 \\ 0 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 25 \\ -1 & 0 & 0 & 0 & -5 & -122 \end{bmatrix}$

We times the 5th row by 2 and add it to the 2nd row $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 5 & 122 \\ 0 & 1 & 0 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 25 \\ -1 & 0 & 0 & 0 & -5 & -122 \end{bmatrix}$

We add the 1st row to the 6th row $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 5 & 122 \\ 0 & 1 & 0 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 25 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

We times the 5 row by -5 and add it to row 1 $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 25 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

And the answer is $x_1 = -3$ $x_2 = -5$ $x_3 = 2$ $x_4 = 9$ $x_5 = 25$

7.

Compute the determinate by using cofactors of the matrix A.

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 4 \\ 0 & 2 & 0 & 0 & 0 \\ -3 & 1 & -5 & 2 & 1 \\ 3 & 9 & 5 & 0 & 0 \\ 5 & -3 & 1 & 0 & 0 \end{bmatrix}$$

Solution:

We can take out the second row because all are zero except one

$$\det(A) = 2 \times \det\left(\begin{bmatrix} 3 & 1 & 0 & 4 \\ -3 & -5 & 2 & 1 \\ 3 & 5 & 0 & 0 \\ 5 & 1 & 0 & 0 \end{bmatrix}\right)$$

$$\det(A) = 2 \times (-2) \times \det\left(\begin{bmatrix} 3 & 1 & 4 \\ 3 & 5 & 0 \\ 5 & 1 & 0 \end{bmatrix}\right)$$

$$\det(A) = 2 \times (-2) \times 4 \times \det\left(\begin{bmatrix} 3 & 5 \\ 5 & 1 \end{bmatrix}\right)$$

$$\det(A) = 2 \times (-2) \times 4 \times (25 - 3)$$

$$\det(A) = 352$$

8.

Consider the following matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 8 & 6 & 2 \\ 2 & 5 & 1 \end{bmatrix}$$

Compute the following:

$(A + B)C$

Solution:

$$\left(\begin{bmatrix}1 & 4\\3 & 2\end{bmatrix}+\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}\right)\left(\begin{bmatrix}8 & 6 & 2\\2 & 5 & 1\end{bmatrix}\right)=\begin{bmatrix}3 & 9\\4 & 5\end{bmatrix}\begin{bmatrix}8 & 6 & 2\\2 & 5 & 1\end{bmatrix}$$

$$=\begin{bmatrix}3\cdot 8+9\cdot 2 & 3\cdot 6+9\cdot 5 & 3\cdot 2+9\cdot 1\\4\cdot 8+5\cdot 2 & 4\cdot 6+5\cdot 5 & 4\cdot 2+5\cdot 1\end{bmatrix}=\begin{bmatrix}42 & 63 & 15\\42 & 49 & 13\end{bmatrix}$$

9.

Find the inverse of the following matrix A

$$A=\begin{bmatrix}k & 0 & 0\\0 & 1 & 0\\0 & 0 & k\end{bmatrix}$$

Solution:

Reduce A to the identity matrix while simultaneously applying the same row operations to I to produce A^{-1} using an augmented matrix . Start with $[A \mid I]$.

$$\left[\begin{array}{ccc|ccc}0 & 0 & k & 1 & 0 & 0\\0 & 1 & 0 & 0 & 1 & 0\\k & 0 & 0 & 0 & 0 & 1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}0 & 0 & 1 & \frac{1}{k} & 0 & 0\\0 & 1 & 0 & 0 & 1 & 0\\k & 0 & 0 & 0 & 0 & 1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}0 & 0 & 1 & \frac{1}{k} & 0 & 0\\0 & 1 & 0 & 0 & 1 & 0\\1 & 0 & 0 & 0 & 0 & \frac{1}{k}\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 0 & 0 & \frac{1}{k}\\0 & 1 & 0 & 0 & 1 & 0\\0 & 0 & 1 & \frac{1}{k} & 0 & 0\end{array}\right]$$

Now that the left half is equal to I, the right side must equal A^{-1}, so the matrix is in the form $[I \mid A^{-1}]$ and $A^{-1}=\begin{bmatrix}0 & 0 & \frac{1}{k}\\0 & 1 & 0\\\frac{1}{k} & 0 & 0\end{bmatrix}$

10.

Determine whether the following matrix is invertible, and if possible, find it's inverse:

$$A=\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\5 & 0 & 3\end{bmatrix}$$

Solution:

A is an invertible matrix because it is triangular and all the entries on the main diagonal are non-zero.

Find it's inverse using an augmented matrix in the form $[A \mid I]$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\0 & 2 & 0 & 0 & 1 & 0\\5 & 0 & 3 & 0 & 0 & 1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 0 & \frac{1}{2} & 0\\5 & 0 & 3 & 0 & 0 & 1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 0 & \frac{1}{2} & 0\\0 & 0 & 3 & -5 & 0 & 1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 0 & \frac{1}{2} & 0\\0 & 0 & 1 & \frac{-5}{3} & 0 & \frac{1}{3}\end{array}\right]$$

Now, the matrix is in the form $[I \mid A^{-1}]$, so $A^{-1}=\begin{bmatrix}1 & 0 & 0\\0 & \frac{1}{2} & 0\\\frac{-5}{3} & 0 & \frac{1}{3}\end{bmatrix}$

Math 343 Midterm 1 Exam Key September 2005

Problem 1:

Find the Transpose of A = $\begin{bmatrix} 8 & 6 & 7 & 2 \\ 4 & 3 & 8 & 9 \\ 1 & 0 & 5 & 7 \\ 4 & 3 & 2 & 9 \end{bmatrix}$.

Solution 1:

If A= $\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$ then the transpose of A, denoted as A^T, is found by interchanging the rows of A with the columns of A.

Thus $A^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 8 & 4 & 1 & 4 \\ 6 & 3 & 0 & 3 \\ 7 & 8 & 5 & 2 \\ 2 & 9 & 7 & 9 \end{bmatrix}$.

Problem 2:

Consider matrices C = $\begin{bmatrix} 5 & 1 & 4 & 9 \\ 1 & 0 & 3 & 2 \\ 4 & 2 & 1 & 1 \end{bmatrix}$ and Q = $\begin{bmatrix} 5 & 1 & 4 & 9 \\ 1 & 0 & 3 & 2 \\ 8 & 2 & 13 & 8 \end{bmatrix}$ Find the elementary matrix E such that EC=Q.

Solution 2: Good.

The elementary matrix E will perform one elementary row operation on C. This elementary row operation performed on the identity matrix, I_{3x3}, is the same elementary row operation that is performed on C. The elementary row operation performed on I_{3x3} is 4 times the second row added to the third row.

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$. Check: EC=Q $\Rightarrow$ $EC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 4 & 9 \\ 1 & 0 & 3 & 2 \\ 4 & 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 & 9 \\ 1 & 0 & 3 & 2 \\ 8 & 2 & 13 & 8 \end{bmatrix} = Q$.

Problem 3:

Consider matrix A= $\begin{bmatrix} 2 & 3 & 6 & 7 & 1 \\ 4 & 6 & 5 & 4 & 4 \\ 1 & 5 & 3 & 2 & 1 \end{bmatrix}$. Find the reduced row-echelon form of matrix A

Solution 3:

$\begin{bmatrix} 1 & 5 & 3 & 2 & 1 \\ 4 & 6 & 5 & 4 & 4 \\ 2 & 3 & 6 & 7 & 1 \end{bmatrix}$ is obtained by swapping row 1 and 3 of matrix A.

$\sim \begin{bmatrix} 1 & 5 & 3 & 2 & 1 \\ 0 & -14 & -7 & -4 & 0 \\ 2 & 3 & 6 & 7 & 1 \end{bmatrix}$ Add the second row to -4 times the first row.

$\sim \begin{bmatrix} 1 & 5 & 3 & 2 & 1 \\ 0 & 1 & 1/2 & 2/7 & 0 \\ 0 & -7 & 0 & 3 & -1 \end{bmatrix}$ Add the third row to -2 times the first row.

$\sim \begin{bmatrix} 1 & 5 & 3 & 2 & 1 \\ 0 & 1 & 1/2 & 2/7 & 0 \\ 0 & 0 & 7/2 & 5 & -1 \end{bmatrix}$ Add the third row to 7 times the second row.

$\sim \begin{bmatrix} 1 & 5 & 0 & -16/7 & 13/7 \\ 0 & 1 & 0 & -3/7 & 2/35 \\ 0 & 0 & 1 & 10/7 & -2/7 \end{bmatrix}$ Add -1/2 times the third row to the second row and -3 times the third row to the first row.

$\sim \begin{bmatrix} 1 & 0 & 0 & -1/7 & 11/7 \\ 0 & 1 & 0 & -3/7 & 2/35 \\ 0 & 0 & 1 & 10/7 & -2/7 \end{bmatrix}$ Add 5 times the second row to the first row.

$\sim \begin{bmatrix} 1 & 0 & 0 & -1/7 & 11/7 \\ 0 & 1 & 0 & -3/7 & 2/35 \\ 0 & 0 & 1 & 10/7 & -2/7 \end{bmatrix}$ Is in reduced row-echelon form because all the leading ones are in columns where ever other entry is equal to zero, and starting with the leading one of row one, every other leading one below it is to the right.

Problem 4: Solve by Gauss-Jordan elimination:

$$\begin{array}{llllllll} x_1 & +3x_2 & -2x_3 & & +2x_5 & & = 0 \\ 2x_1 & +6x_2 & -5x_3 & -2x_4 & +4x_5 & -3x_6 & =-1 \\ & & 4x_3 & +8x_4 & & +12x_6 & = 4 \\ 2x_1 & +6x_2 & & +8x_4 & +4x_5 & +18x_6 & = 6 \end{array}$$

<u>Solution 4</u>:

The augmented matrix for the system of linear equations is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 4 & 8 & 0 & 12 & 4 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}.$$

Adding -2 times row 1 to rows 2 and 4 gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 4 & 8 & 0 & 12 & 4 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix};$$

Multiplying row 2 by -1 and adding -4 times the new row to rows 3 and 4

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix};$$

Interchanging rows 3 and 4 and multiplying the new row 3 by 1/6 gives row-echelon

form: $\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$

Adding -3 times row 3 to row 2 and adding 2 times row 2 to row 1 gives the reduced row-

echelon form: $\begin{bmatrix} 1 & 3 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

The corresponding system of linear equations is:

$$\begin{aligned} x_1 \quad +3x_2 \qquad\quad +4x_4 \quad +2x_5 \qquad &= 0 \\ x_3 \quad +2x_4 \qquad\qquad &= 0 \\ x_6 \quad &= 1/3 \end{aligned}$$

Solving for the leading variables yields

$$\begin{aligned} x_1 &= -3x_2 \quad -4x_4 \quad -2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \;\; 1/3 \end{aligned}$$

By assigning x_2, x_4, and x_5 arbitrary values r, s, and t, respectively, the general solution is given by

x_1= -3s-4r-2t, x_2= r, x_3= -2s, x_4= s, x_5 = t, x_6 = 1/3.

Problem 5:

Realizing $[A|I] \sim [I|A^{-1}]$, find the inverse of $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 4 \\ 3 & 5 & 2 \end{bmatrix}$:

<u>Solution 5</u>:

Setting matrix A side by side with the identity matrix I_{3x3} and applying elementary row operations to both matrices simultaneously will yield A^{-1}. (ie. $[A|I] \sim [I|A^{-1}]$.)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1/4 & 0 \\ 0 & -1 & -10 & -3 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1/4 & 0 \\ 0 & 0 & -9 & -3 & 1/4 & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 1/3 & -1/36 & -1/9 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1/2 & 0 \\ 0 & 1 & 1 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 1/3 & -1/36 & -1/9 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1/2 & 0 \\ 0 & 1 & 0 & -1/3 & 10/36 & 1/9 \\ 0 & 0 & 1 & 1/3 & -1/36 & -1/9 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -16/36 & 2/9 \\ 0 & 1 & 0 & -1/3 & 10/36 & 1/9 \\ 0 & 0 & 1 & 1/3 & -1/36 & -1/9 \end{array}\right] \Rightarrow \quad A^{-1} = \begin{bmatrix} 1/3 & -16/36 & 2/9 \\ -1/3 & 10/36 & 1/9 \\ 1/3 & -1/36 & -1/9 \end{bmatrix}$$

23
13
69
23 9
29 9
$\det A^3 = 13 \cdot 23 - 30 \cdot 10$
$= 299 - 300 = -1$

Problem 6:

Let A = $\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$. Find A^{-3}.

Solution 6:

There are two ways to solve A^{-3} because $A^{-3} = (A^{-1})^3 = (A^3)^{-1}$.

$$A^3 = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 30 \\ 10 & 23 \end{bmatrix} = A^3 \Rightarrow (A^3)^{-1} = \begin{bmatrix} 23 & -30 \\ -10 & 13 \end{bmatrix}.$$

The second possible way to solve for A^{-3} is similar, except that, first, find the inverse of A and then, multiply that matrix by itself three times. It yields the same result.

Problem 7:

Find the determinant of A = $\begin{bmatrix} 3 & 9 & 2 \\ 0 & 7 & 1 \\ 5 & 4 & 2 \end{bmatrix}$.

Solution: det (A) can be found be by copying the first two rows of A to the right of A in this manner:

$$\begin{matrix} A_{11} & A_{12} & A_{13} & A_{11} & A_{12} \\ A_{21} & A_{22} & A_{23} & A_{21} & A_{22} \\ A_{31} & A_{32} & A_{33} & A_{31} & A_{32} \end{matrix}$$

. Then multiply each diagonal of three entries from left to right and from right to left. Finally sum these terms adding the left to right terms and subtracting the right to left terms (this is the short-cut discussed in class) i.e.: $A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33}$. Thus for A the arraignment is

$$\begin{matrix} 3 & 9 & 2 & 3 & 2 \\ 0 & 7 & 1 & 0 & 7 \\ 5 & 4 & 2 & 5 & 4 \end{matrix}$$

and the det(A)

$= 3*7*2 + 9*1*5 + 2*0*4 - 2*7*5 - 3*1*4 - 2*0*5 = 5 =$ det(A).

Problem 8:

Consider the matrices A= $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ and B= $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$. Do A and B commute?

Solution 8: A and B will commute if AB=BA, by definition. Therefore, yes, A and B do commute because AB = $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ = BA.

Problem 9:

Solve the systems of (a) and (b) simultaneously/

$$\begin{array}{llcl} x_1 & -5x_2 & = & d_1 \\ 3x_1 & -13x_2 & = & d_2 \end{array} \qquad \begin{array}{llll} (a) & d_1 = 1, & d_2 = & 5 \\ (b) & d_1 = 2 & d_2 = & 12 \end{array}$$

Solution 9:

The two systems have the same coefficient matrix. If we augment this coefficient matrix with the columns of constants on the right sides of these systems, we obtain

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & 2 \\ 3 & -13 & 5 & 12 \end{array}\right]$$

Reducing to reduced row-echelon form yields

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & 2 \\ 3 & -13 & 5 & 12 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & -5 & 1 & 2 \\ 0 & 2 & 2 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & -5 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & 0 & 6 & 17 \\ 0 & 1 & 1 & 3 \end{array}\right].$$

It follows from the last two columns that the solution of (a) is $x_1 = 6$, $x_2 = 1$, and the solution of (b) is $x_1 = 17$, $x_2 = 3$.

Problem 10:

For the matrix A= $\begin{bmatrix} 2 & 4 & 5 \\ 1 & 3 & 1 \\ 2 & 6 & 8 \end{bmatrix}$ find the cofactors.

Solution 10:

The cofactor C_{ij} of A = $(-1)^{i+j} M_{ij}$ thus,

C_{11}= det $\begin{bmatrix} 3 & 1 \\ 1 & 8 \end{bmatrix}$ = 3(8) – 1(6) = 18

C_{12}= det $\begin{bmatrix} 1 & 1 \\ 2 & 8 \end{bmatrix}$ = (-1((1(8)-1(2))) = -6

C_{13}= det $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ = 1(6) – 3(2) = 0

C_{21} = det $\begin{bmatrix} 4 & 5 \\ 6 & 8 \end{bmatrix}$ = -1((4(8) – 6(5)) = -2

C_{22} = det $\begin{bmatrix} 2 & 5 \\ 2 & 8 \end{bmatrix}$ = 2(8) – 2(5) = 6

C_{23} = det $\begin{bmatrix} 2 & 4 \\ 2 & 6 \end{bmatrix}$ = -1((2(6) – 2(4)) = -4

$$C_{31} = \det \begin{bmatrix} 4 & 5 \\ 3 & 1 \end{bmatrix} = 4(1) - 3(5) = -11$$

$$C_{32} = \det \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix} = -1((2(1)-1(5)) = -3$$

$$C_{33} = \det \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} = 2(3) - 1(4) = 2$$

Problem 11:

Given $A = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 3 & 1 \\ 2 & 6 & 8 \end{bmatrix}$ Find det(A).

<u>Solution 11</u>:

Using the cofactors found in problem 10 and theorem 2.4 which states: $\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij}$

we choose to expand the determinate on row 2. Then we have that

$$\begin{aligned}\det(A) &= (1) \det \begin{bmatrix} 4 & 5 \\ 6 & 8 \end{bmatrix} + (3) \det \begin{bmatrix} 2 & 5 \\ 2 & 8 \end{bmatrix} + (1) \det \begin{bmatrix} 2 & 4 \\ 2 & 6 \end{bmatrix}. \\ &= 1(-2) + 3(6) + 1(-4) \\ &= -2 + 18 - 4 \\ &= 12\end{aligned}$$

1) Calculate the cofactor, adjoint, determinate and inverse of matrix A. Check your answer.

$$A=\begin{bmatrix}3 & 2 & -1\\ 1 & 6 & 3\\ 2 & -4 & 0\end{bmatrix}$$

Solution: (20)

The cofactor if A is found for each row/column combination by crossing out the row and the column and taking the determinate of the remaining matrix times the sign change of the checkerboard matrix.

Checkerboard Matrix = $\begin{bmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{bmatrix}$ $\qquad C_{11} = \det\begin{bmatrix}6 & 3\\ -4 & 0\end{bmatrix} = (6\times 0)+(-4\times 3)=12$

$C_{11} = (6\times 0)-(-4\times 3)=12$ $\quad C_{12} = (1\times 0)-(-2\times 3)=-(-6)$ $\quad C_{13} = (1\times 4)-(-6\times 2)=-16$

$C_{21} = (2\times 0)-(-1\times -4) = -(-14)$ $\quad C_{22} = (3\times 0)-(-1\times 2)=2$ $\quad C_{23} = (3\times -4)-(2\times 2) = -(16)$

$C_{31} = (2\times 3)-(-1\times 6)=12$ $\quad C_{32} = (3\times 3)-(-1\times 1) = -(10)$ $\quad C_{33} = (3\times 6)-(1\times 2)=16$

Cofactor Matrix = $\begin{bmatrix}12 & 6 & -16\\ 4 & 2 & 16\\ 12 & 10 & 16\end{bmatrix}$. To get the adjoint, transpose the cofactor matrix.

Adj(A)= $\begin{bmatrix}12 & 4 & 12\\ 6 & 2 & -10\\ -16 & 16 & 16\end{bmatrix}$. To calculate to determinant, select a row and add the minor of each column in that row.

$$\det(A) = 2\det\begin{bmatrix}2 & -1\\ 6 & 3\end{bmatrix}-(-4)\det\begin{bmatrix}3 & -1\\ 1 & 3\end{bmatrix}+0\det\begin{bmatrix}3 & 2\\ 1 & 6\end{bmatrix} = (2\times 12)-(-4\times 10)+0=64.$$

To find the inverse, use the equation $A^{-1} = \frac{1}{\det(A)}adj(A)$.

$$A^{-1} = \frac{1}{64}\begin{bmatrix}12 & 4 & 12\\ 6 & 2 & -10\\ -16 & 16 & 16\end{bmatrix} = \begin{bmatrix}\frac{3}{16} & \frac{1}{16} & \frac{3}{16}\\ \frac{3}{32} & \frac{1}{32} & \frac{-5}{32}\\ \frac{1}{4} & \frac{-1}{4} & \frac{1}{4}\end{bmatrix}$$

Check: $A\,A^{-1} = I$

$$\begin{bmatrix}3 & 2 & -1\\ 1 & 6 & 3\\ 2 & -4 & 0\end{bmatrix}\begin{bmatrix}\frac{3}{16} & \frac{1}{16} & \frac{3}{16}\\ \frac{3}{32} & \frac{1}{32} & \frac{-5}{32}\\ \frac{1}{4} & \frac{-1}{4} & \frac{1}{4}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

2) Solve the following equation by Gûass elimination and back substitution.

$$\begin{aligned} 10y - 4z + w &= 1 \\ x + 4y - z + w &= 2 \\ 3x + 2y + z + w &= 5 \\ -2x - 8y + 2z + 2w &= -4 \\ x - 6y + 3z \quad &= 1 \end{aligned}$$

Solution: (15)

First, transcribe this into an augmented matrix, then row reduce but trading rows, and adding multiples of one row to another.

$$\begin{bmatrix} 0 & 10 & -4 & 1 & 1 \\ 1 & 4 & -1 & 4 & 2 \\ 3 & 2 & 1 & 2 & 5 \\ -2 & -8 & 2 & -2 & -4 \\ 1 & -6 & 3 & 0 & 1 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 4 & -1 & 4 & 2 \\ 0 & 10 & -4 & 1 & 1 \\ 3 & 2 & 1 & 2 & 5 \\ -2 & -8 & 2 & -2 & -4 \\ 1 & -6 & 3 & 0 & 1 \end{bmatrix} \begin{array}{l} R_3 = R_3 + (-3)R_1 \\ R_4 = R_4 + (2)R_1 \\ R_5 = R_5 + (-1)R_1 \end{array}$$

$$\begin{bmatrix} 1 & 4 & -1 & 1 & 2 \\ 0 & 10 & -4 & 1 & 1 \\ 0 & -10 & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -10 & -4 & -1 & -1 \end{bmatrix} \begin{array}{l} R_2 = \frac{R_2}{10} \\ R_3 = R_3 + R_2 \\ R_5 = R_5 + R_2 \end{array} \begin{bmatrix} 1 & 4 & -1 & 1 & 2 \\ 0 & 1 & \frac{-2}{5} & \frac{1}{10} & \frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_1 = R_1 + (-4)R_2$$

$$\begin{bmatrix} 1 & 0 & 2/5 & 3/5 & -8/5 \\ 0 & 1 & -2/5 & 1/10 & 1/10 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

. Now, plug these back into the original equation, substituting r, s, and t for y, z, and w.

$$\begin{aligned} x &= 2/5\, z - 3/5\, w - 8/5 \\ y &= 2/5\, z - 1/10\, w - 1/10 \end{aligned} \quad \text{to get} \quad \begin{aligned} x &= -8/5 + 2/5\, s - 3/5\, t \\ y &= -1/10 + 2/5\, s - 1/10\, t \end{aligned} \quad \text{and} \quad \begin{aligned} z &= s \\ w &= t \end{aligned}.$$

3) Solve the following equations simultaneously.

$$\begin{aligned} 4x_1 - 7x_2 &= b_1 \\ x_1 + 2x_2 &= b_2 \end{aligned} \quad \text{for} \quad \begin{aligned} &a) b_1 = 0, b_2 = 1 \\ &b) b_1 = -4, b_2 = 6 \\ &c) b_1 = -1, b_2 = 3 \\ &d) b_1 = -5, b_2 = 1 \end{aligned}$$

Solution: (15):

Write this into an augmented matrix, then row reduce rows as follows.

$\left[\begin{array}{cc|c|c|c|c} 4 & 7 & 0 & -4 & -1 & -5 \\ 1 & 2 & 1 & 6 & 3 & 1 \end{array}\right]$, swap the two rows, and add to the bottom row by -4 times the top to get $\left[\begin{array}{cc|c|c|c|c} 1 & 2 & 1 & 6 & 3 & 1 \\ 0 & -15 & -4 & -28 & -13 & -9 \end{array}\right]$. Now extract -15 from the bottom row and add -2 times the bottom to the top row to get $\left[\begin{array}{cc|c|c|c|c} 1 & 0 & 7/15 & 34/15 & 45/15 & -3/15 \\ 0 & 1 & -4/15 & -28/15 & -13/15 & -9/5 \end{array}\right]$. From this extract the answers to each equation to get

$$\begin{aligned} &a) b_1 = 7/15, b_2 = -4/15 \\ &b) b_1 = 34/15, b_2 = -28/15 \\ &c) b_1 = 45/15, b_2 = -13/15 \\ &d) b_1 = -1/5, b_2 = 3/5 \end{aligned}.$$

4) To demonstrate your knowledge of inverses and transpose, prove the following:
a) If Ax=b, then $x = A^{-1}b$
b) If $A=A^T$, then AA^T is also symmetric.
c) $(A^T)^{-1} = (A^{-1})^T$.

Solution: (15)
a) Use the inverse properties ($AA^{-1}=I$) and also remember that matrix algebra is not commutative for multiplication.

$Ax=b$, $A^{-1}Ax=A^{-1}b$, $Ix=A^{-1}b$, $x=A^{-1}b$

b) To find if AA^T is symmetric, take the Transpose of both sides. (Remember that $A^{TT} = A$ and $(AB)^T= B^TA^T$).

$(AA^T)^T = A^{TT}A^T = AA^T$

c) To find if $(A^T)^{-1} = (A^{-1})^T$, show that $A^T(A^{-1})^T=(A^{-1})^TA^T= I$.

$A^T(A^{-1})^T=(A^{-1}A)^T = I^T = I$, and
$(A^{-1})^TA^T=(AA^{-1})^T= I^T = I$.

5. Show that if a square matrix satisfies $A^3 - 3A^2 + 4A - I = 0$, then
$A^{-1} = A^2 - 3A + 4I$.

Solution (10)

:

$$\begin{aligned} &A^3 - 3A^2 + 4A - I = 0 \rightarrow \\ &A^{-1}A^3 - 3A^{-1}A^2 + 4A^{-1}A - A^{-1}I = 0 \rightarrow \\ &IA^2 - 3IA + 4I - A^{-1} = 0 \rightarrow \\ &A^2 - 3A + 4I = A^{-1} \end{aligned}$$

6. Solve each of the following systems by Gauss-Jordan elimination:

a.
$$\begin{aligned} x - 4y + 3z &= 2 \\ 2x + 3y - z &= 5 \\ 2y - 5z &= 3 \end{aligned}$$

b.
$$\begin{aligned} 4x + y - z &= 1 \\ 3x \quad + \quad z &= 4 \\ x + 2y \quad &= 2 \end{aligned}$$

Solution: (20)

Solution a:

$$\begin{bmatrix} 1 & -4 & 3 & 2 \\ 2 & 3 & -1 & 5 \\ 0 & 2 & -5 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 & 3 & 2 \\ 0 & 11 & -7 & 1 \\ 0 & 2 & -5 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 & 3 & 2 \\ 0 & 1 & -7/11 & 1/11 \\ 0 & 1 & -5/2 & 3/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 & 3 & 2 \\ 0 & 1 & -7/11 & 1/11 \\ 0 & 0 & -41/22 & 31/22 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -4 & 3 & 2 \\ 0 & 1 & -7/11 & 1/11 \\ 0 & 0 & 1 & -31/41 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 & 0 & 175/41 \\ 0 & 1 & 0 & -16/41 \\ 0 & 0 & 1 & -31/41 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 111/41 \\ 0 & 1 & 0 & -16/41 \\ 0 & 0 & 1 & -31/41 \end{bmatrix} \qquad \begin{aligned} x &= 111/41 \\ y &= -16/41 \\ z &= -31/41 \end{aligned}$$

Solution b:

$$\begin{bmatrix} 4 & 1 & -1 & 1 \\ 3 & 0 & 1 & 4 \\ 1 & 2 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 12 & 3 & -3 & 3 \\ 12 & 0 & 4 & 16 \\ 12 & 24 & 0 & 24 \end{bmatrix} \rightarrow \begin{bmatrix} 12 & 3 & -3 & 3 \\ 0 & -3 & 7 & 13 \\ 0 & 21 & 3 & 21 \end{bmatrix} \rightarrow \begin{bmatrix} 12 & 3 & -3 & 3 \\ 0 & 21 & -49 & -91 \\ 0 & 21 & 3 & 21 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 12 & 3 & -3 & 3 \\ 0 & 21 & -49 & -91 \\ 0 & 0 & 52 & 112 \end{bmatrix} \rightarrow \begin{bmatrix} 12 & 3 & -3 & 3 \\ 0 & 1 & -49/21 & -91/21 \\ 0 & 0 & 1 & 28/13 \end{bmatrix} \rightarrow \begin{bmatrix} 12 & 3 & 0 & 123/13 \\ 0 & 1 & 0 & 9/13 \\ 0 & 0 & 1 & 28/13 \end{bmatrix} \rightarrow \begin{bmatrix} 12 & 0 & 0 & 96/13 \\ 0 & 1 & 0 & 9/13 \\ 0 & 0 & 1 & 28/13 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 8/13 \\ 0 & 1 & 0 & 9/13 \\ 0 & 0 & 1 & 28/13 \end{bmatrix} \qquad \begin{aligned} x &= 8/13 \\ y &= 9/13 \\ z &= 28/13 \end{aligned}$$

7.Compute the determinants of the following matrices using cofactor expansion;

a.
$$\begin{bmatrix} 3 & 1 & 5 \\ 2 & 4 & 0 \\ 1 & 6 & 3 \end{bmatrix}$$

b.
$$\begin{bmatrix} 5 & 1 & 2 & 4 \\ 2 & 0 & 1 & 3 \\ 4 & 8 & 2 & 3 \\ 0 & 5 & 6 & 7 \end{bmatrix}$$

Solution: (15)

Solution a:
$$\det\begin{pmatrix} 3 & 1 & 5 \\ 2 & 4 & 0 \\ 1 & 6 & 3 \end{pmatrix} = 3\det\begin{pmatrix} 4 & 0 \\ 6 & 3 \end{pmatrix} - \det\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} + 5\det\begin{pmatrix} 2 & 4 \\ 1 & 6 \end{pmatrix} = 36 - 6 + 40 = 80$$

Solution b:
$$\det\begin{pmatrix} 5 & 1 & 2 & 4 \\ 2 & 0 & 1 & 3 \\ 4 & 8 & 2 & 3 \\ 0 & 5 & 6 & 7 \end{pmatrix} = 5\det\begin{pmatrix} 0 & 1 & 3 \\ 8 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix} - 2\det\begin{pmatrix} 1 & 2 & 4 \\ 8 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix} + 4\det\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 5 & 6 & 7 \end{pmatrix}$$

$$= 5((-1)\det([\ 8\ 3\ ]) + 3\det([\ 8\ 2\ ])) - 2(\det([\ 2\ 3\ ]) - 2\det([\ 8\ 3\ ]) + 4\det([\ 8\ 2\ ]))$$

([5 7])   ([5 6])   ([6 7])   ([5 7])   ([5 6])

+ 4(det ([1 3]) + 5 det ([2 4])) = 5(-41 + 114) – 2(-4 – 82 +152) + 4(-11 + 10)
([6 7])   ([1 3])

= 365 – 132 – 4 = 229

8. a) Find the inverse of B= $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and C= $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ if it exists. If not prove that the matrix is singular

Solution: (15)

b) $$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4/3 & -1/3 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4/3 & -1/3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

because the third row is all zeros, B is singular.

c) $$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & -8 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -12 & -7 & 4 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 7/12 & -1/3 & -1/12 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -3/4 & 1 & 1/4 \\ 0 & 1 & 0 & -5/12 & 2/3 & -1/12 \\ 0 & 0 & 1 & 7/12 & -1/3 & -1/12 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/12 & -1/3 & 5/12 \\ 0 & 1 & 0 & -5/12 & 2/3 & -1/12 \\ 0 & 0 & 1 & 7/12 & -1/3 & -1/12 \end{array}\right] \text{ so } C^{-1} = \begin{bmatrix} 1/12 & -1/3 & 5/12 \\ -5/12 & 2/3 & -1/12 \\ 7/12 & -1/3 & -1/12 \end{bmatrix}$$

9. Let A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, B = $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, and C = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find (a) AB, (b) A+B, (c) AC, (d) A^t, and (e) tr(A).

Solution: (15)

(a) AB= $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+24 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 42 \end{bmatrix}$

(b) A+B= $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + \begin{bmatrix}5 & 6\\7 & 8\end{bmatrix} = \begin{bmatrix}6 & 8\\10 & 12\end{bmatrix}$

(c) AC $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} = \begin{bmatrix}1+8 & 2+10 & 3+12\\3+16 & 6+20 & 9+24\end{bmatrix} = \begin{bmatrix}9 & 12 & 15\\19 & 26 & 33\end{bmatrix}$

(d) $A^t = \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$

(e) tr(A) = (1)(4) = 4

10. Find det(A) if

(a) $A = \begin{bmatrix}a & b & c & d\\0 & e & f & g\\0 & 0 & h & i\\0 & 0 & 0 & j\end{bmatrix}$

(b) $A = \begin{bmatrix}a & 0 & 0 & 0 & 0\\0 & b & 0 & 0 & 0\\0 & 0 & c & 0 & 0\\0 & 0 & 0 & d & 0\\0 & 0 & 0 & 0 & e\end{bmatrix}$

(c) $A = \begin{bmatrix}a & 0 & 0 & 0 & 0\\b & f & 0 & 0 & 0\\c & g & 0 & 0 & 0\\d & h & j & l & 0\\e & i & k & m & n\end{bmatrix}$

Solution:(10)

In triangular and diagonal matrices the determinant is the product of the entries in the main diagonal.

(a) det(A) = a*e*h*i
(b) det(A) = a*b*c*d*e
(c) det(A) = a*f*0*l*n = 0

1) Solve the following systems of linear equations by reducing the augmented matrix to reduced row-echelon form:

(a)
$$\begin{aligned} w + x - y + 2z &= 10 \\ 3w - x + 7y + 4z &= 1 \\ -5w + 3x - 15y - 6z &= 9 \end{aligned}$$

(b)
$$\begin{aligned} 3x - y + 7z &= 0 \\ 2x - y + 4z &= 1/2 \\ x - y + z &= 1 \\ 6x - 4y + 10z &= 3 \end{aligned}$$

(c)
$$\begin{aligned} 2x + 3y - 4z &= 1 \\ 2y + 3z &= 4 \\ 2w + 2x - 5y + 2z &= 4 \\ 2w - 6y + 9z &= 7 \end{aligned}$$

Answer:

(a)

$$\begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 3 & -1 & 7 & 4 & 1 \\ -5 & 3 & -15 & -6 & 9 \end{bmatrix} \textit{Solving by Gauss-Jordan}$$

Elimination...

$$\begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 0 & -4 & 10 & -2 & -29 \\ 0 & 8 & -20 & 4 & 59 \end{bmatrix} \textit{Adding } (-3 \textit{ times row } 1)$$

to row 2 and (5 times row 1) to row 5

$$\begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} & \frac{29}{4} \\ 0 & 8 & -20 & 4 & 59 \end{bmatrix} \textit{Multiplying row 2 by } -\frac{1}{4}$$

$$\begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} & \frac{29}{4} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \textit{Adding } (-8 \textit{ times row } 2)$$

$\textit{to row } 3 \rightarrow \textit{Inconsistent!}$

(b)

$$\begin{bmatrix} 3 & -1 & 7 & 0 \\ 2 & -1 & 4 & \frac{1}{2} \\ 1 & -1 & 1 & 1 \\ 6 & -4 & 10 & 3 \end{bmatrix} \textit{Solving by } G-J \textit{ elimination}$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 4 & \frac{1}{2} \\ 3 & -1 & 7 & 0 \\ 6 & -4 & 10 & 3 \end{bmatrix} \textit{Swap rows 1 and 3}$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & -\frac{3}{2} \\ 0 & 2 & 4 & -3 \\ 0 & 2 & 4 & -3 \end{bmatrix}$$ *Add* $(-2$ *times row* $1)$ *to row* 2, $(-3$ *times row* $1)$ *to row* 3, $(-6$ *times row* $1)$ *to row* 4

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ *Add* $(-2$ *times row* $2)$ *to both rows* 3 *and* 4

$$\begin{bmatrix} 1 & 0 & 3 & -\frac{1}{2} \\ 0 & 1 & 2 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ *Add* $(1$ *times row* $2)$ *to row one*

$$\rightarrow \quad z = t, \quad y = \left(\frac{-3}{2}\right) - 2t, \quad x = \left(\frac{-1}{2}\right) - 3t$$

(c)

$$\begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$ *Solving by* $G - J$ *elimination*

$$\begin{bmatrix} 2 & 2 & -5 & 2 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix} \textit{Swap rows}: 1 \ \& \ 2, \ \textit{then} \ 2 \ \& \ 3,$$

then 3 & 4

$$\begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 2 & 0 & -6 & 9 & 7 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix} \textit{Multiply rows } 1 \textit{ and } 2 \textit{ by } \left(\frac{1}{2}\right)$$

$$\begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & -2 & -1 & 7 & 3 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix} \textit{Add } (-2 \textit{ times row } 1) \textit{ to row } 3$$

$$\begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix} \textit{Add } (2 \textit{ times row } 2) \textit{ to row } 3$$

$$\begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 1 & \frac{3}{2} & 2 \end{bmatrix} \textit{Multiply rows } 3 \ \& \ 4 \textit{ by } \left(\frac{1}{2}\right)$$

$$\begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \textit{Add } (-1 \textit{ times row } 3) \textit{ to row } 4$$

$$\begin{bmatrix} 1 & 1 & 0 & \frac{19}{4} & 7 \\ 0 & 1 & 0 & -\frac{17}{4} & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \textit{Add } \left(\left(\frac{-3}{2}\right) \textit{ times row } 3\right)$$

$$\textit{to row } 2 \textit{ and } \left(\left(\frac{5}{2}\right) \textit{times row3}\right) \textit{ to row } 1$$

$$\begin{bmatrix} 1 & 0 & 0 & 9 & \frac{19}{2} \\ 0 & 1 & 0 & -\frac{17}{4} & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad Add(-1\ times\ row\ 2)\ to\ row$$

$$1 \rightarrow z = t,\ y = 2 - \left(\frac{3}{2}\right) t,\ x = -\left(\frac{5}{2}\right) + \left(\frac{17}{4}\right) t,\ w = \left(\frac{19}{2}\right) - 9t$$

2)

$$Let\ A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

(a) Verify that $A^3 = 3A^2 - 3A + I^3$

(b) Calculate A^4 explicitly.

(c) Find A^{-1} explicitly.

Answers:

(a)

$$A^2 = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 2 \\ 6 & 4 & 3 \end{bmatrix}$$

$$A^2 \cdot A = A^3 = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 2 \\ 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -5 & -3 & -3 \\ 6 & 4 & 3 \\ 12 & 9 & 4 \end{bmatrix}.$$

$$3\begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 2 \\ 6 & 4 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} -5 & -3 & -3 \\ 6 & 4 & 3 \\ 12 & 9 & 4 \end{bmatrix}$$

(b) $$A^4 = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -11 & -8 & -4 \\ 12 & 9 & 4 \\ 20 & 16 & 5 \end{bmatrix}$$

(c) $$A^{-1} = \begin{bmatrix} -1 & -3 & 1 \\ 2 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

3. Let $A = I_n - X(X^TX)^{-1}X^T$.

(a) What are the dimensions of $X(X^TX)^{-1}X^T$?

(b) Must X^TX be square?

(c) Show that A is symmetric.

Answer:

(a) Since $X (X^T X)^{-1} X^T$ must be conformable with I_n for addition, it must be n x n.

(b) Since a matrix must be square to have an inverse, $X^T X$ must be square although X by itself need not be.

(c) For A to be symmetric, $A = A^T$
So:

$$[I_n - X (X^T X)^{-1} X^T]^T$$
$$= I_n^T - [X (X^T X)^{-1} X^T]^T$$
$$= I_n - [X \{(X^T X)^{-1}\}^T X^T]$$
$$= I_n - X (X^T X)^{-1} X^T$$

Therefore A is symmetric

4) Solve the following system of linear equations by using Gauss- Jordan elimination:

$$\begin{aligned} a + 3b + 2c + 6d &= 0 \\ -2b - 3d &= -1 \\ 3a - b + 4d &= 3 \\ -4a + 2b - 5d &= 17 \end{aligned}$$

Answer:

$$\begin{bmatrix} 1 & 3 & 2 & 6 & 0 \\ 0 & -2 & 0 & -3 & -1 \\ 3 & -1 & 4 & 0 & 3 \\ -4 & 2 & 0 & -5 & 17 \end{bmatrix}$$ Insert equations into a matrix

$$\begin{bmatrix} 1 & 3 & 2 & 6 & 0 \\ 0 & 1 & 0 & 3/2 & 1/2 \\ 0 & -10 & -2 & -18 & 3 \\ 0 & 14 & 8 & 19 & 17 \end{bmatrix}$$ Add (-3 times row 1) to row 2 and (4 times row 1) to row 3, and divide row 2 by -2

$$\begin{bmatrix} 1 & 3 & 2 & 6 & 0 \\ 0 & 1 & 0 & 3/2 & 1/2 \\ 0 & 0 & -2 & -3 & 8 \\ 0 & 0 & 8 & -2 & 10 \end{bmatrix}$$ Add (10 times row 2) to row 3 and (-14 times row 2) to row 4

$$\begin{bmatrix} 1 & 3 & 2 & 6 & 0 \\ 0 & 1 & 0 & 3/2 & 1/2 \\ 0 & 0 & 1 & 3/2 & -4 \\ 0 & 0 & 8 & -2 & 10 \end{bmatrix}$$ Divide row 3 by -2

$$\begin{bmatrix} 1 & 3 & 2 & 6 & 0 \\ 0 & 1 & 0 & 3/2 & 1/2 \\ 0 & 0 & 1 & 3/2 & -4 \\ 0 & 0 & 0 & -14 & 42 \end{bmatrix}$$ Add (-8 times row 3) to row 4

$$\begin{bmatrix} 1 & 3 & 2 & 6 & 0 \\ 0 & 1 & 0 & 3/2 & 1/2 \\ 0 & 0 & 1 & 3/2 & -4 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$ Divide row 4 by -14

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 18 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$ Add (-3/2 times row 4) to rows 2 & 3 and (-6 times row 4) to row 1

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 17 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$ Add (-2 times row 3) to row 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$ Add (-3 times row 2) to row 1

$\rightarrow$ a = 2, b = 5, c = 1/2, & d = -3

5) Solve the following systems:

(a).

$$\begin{aligned} r - s - 3t &= -2 \\ -4r + 5s + 7t &= -3 \end{aligned}$$

$$\begin{aligned} 3r + 4s - t &= 3 \end{aligned}$$

(b).

$$\begin{aligned} r - s - 3t &= 6 \\ -4r + 5s + 7t &= -33 \\ 3r + 4s - t &= -2 \end{aligned}$$

Answer:
Both (a) and (b) can be solved simultaneously.

$$\left[\begin{array}{ccc|c|c} 1 & -1 & -3 & -2 & 6 \\ -4 & 5 & 7 & -3 & -33 \\ 3 & 4 & -1 & 3 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|c} 1 & -1 & -3 & -2 & 6 \\ 0 & 1 & -5 & -11 & -9 \\ 0 & 7 & 8 & 9 & -20 \end{array}\right]$$ Add (4 times row 1) to row 2 and (-3 times row 1) to row 3

$$\left[\begin{array}{ccc|c|c} 1 & -1 & -3 & -2 & 6 \\ 0 & 1 & -5 & -11 & -9 \\ 0 & 0 & 43 & 86 & 43 \end{array}\right]$$ Add (-7 times row 2) to row 3

$$\left[\begin{array}{ccc|c|c} 1 & -1 & -3 & -2 & 6 \\ 0 & 1 & -5 & -11 & -9 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right]$$ Divide row 3 by 43.

$$\left[\begin{array}{ccc|c|c} 1 & -1 & 0 & 4 & 9 \\ 0 & 1 & 0 & -1 & -4 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right]$$ Add (5 times row 3) to row 2 and (3 times row 3) to row 1

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 3 & 5 \\ 0 & 1 & 0 & -1 & -4 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right]$$ Add row 2 to row 1

→ (a). r = 3, s = -1, & t = 2
(b). r = 5, s = -4, & t = 1.

6) Given the matrix

$$A = \begin{bmatrix} 0 & 5 & 7 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Find A^2, A^3, A^-1, and A^T (the transpose)

Answers:

$$A^2 = \begin{bmatrix} 0 & 5 & 7 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 5 & 7 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 5 & 7 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

A^-1 does not exist because there is a row and a column of zeros.

$$A^{\wedge}T = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 7 & 2 & 0 \end{bmatrix}$$

7) Evaluate Det(A) using co-factor expansion

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 7 & 5 & -1 \\ -2 & 0 & 6 \end{bmatrix}$$

Answer:
Co-factor expansion along the third row

$$\text{Det(A)} = \begin{bmatrix} 4 & 1 & 3 \\ 7 & 5 & -1 \\ -2 & 0 & 6 \end{bmatrix} = -2 \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix} + 0 \begin{bmatrix} 4 & 3 \\ 7 & -1 \end{bmatrix} + 6 \begin{bmatrix} 4 & 1 \\ 7 & 5 \end{bmatrix}$$

$= (-2) (-1 – 15) + (0) + (6) (20 – 7)$

$= 32 + 78$

$= 110$

8) Use row operations to find the inverse of

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 3 \\ -1 & 0 & 9 \end{bmatrix}$$

Answer:
Let r1 = Row 1, Let r2 = Row 2, Let r3 = Row 3

Add (r1 * -2) to r2 $\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 & 1 & 0 \\ -1 & 0 & 9 & 0 & 0 & 1 \end{array}\right]$

Add r1 to r3 $\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 & 1 & 0 \\ 0 & 3 & 12 & 1 & 0 & 1 \end{array}\right]$

Swap r2 with r3 $\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 3 & 12 & 1 & 0 & 1 \\ 0 & 0 & -3 & -2 & 1 & 0 \end{array}\right]$

Mult. R2 by (1/3) $\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1/3 & 0 & 1/3 \\ 0 & 0 & -3 & -2 & 1 & 0 \end{array}\right]$

Mult. R2 by (-1/3) $\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 2/3 & -1/3 & 0 \end{array}\right]$

Add (r3 * -4) to r2 $\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -7/3 & 4/3 & 1/3 \\ 0 & 0 & 1 & 2/3 & -1/3 & 0 \end{array}\right]$

Add (r3 * -3) to r1 $\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -7/3 & 4/3 & 1/3 \\ 0 & 0 & 1 & 2/3 & -1/3 & 0 \end{array}\right]$

Add (r2 * -3) to r1 $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -3 & -1 \\ 0 & 1 & 0 & -7/3 & 4/3 & 1/3 \\ 0 & 0 & 1 & 2/3 & -1/3 & 0 \end{array}\right]$

A^-1 = $\begin{bmatrix} 6 & -3 & -1 \\ -7/3 & 4/3 & 1/3 \\ 2/3 & -1/3 & 0 \end{bmatrix}$

9) Solve the system of linear equations by finding the inverse and solving $\mathbf{A}\mathbf{x} = \mathbf{b}$.

$2x + 4y + 8z = 4$
$3x + 4y + 4z = 5$
$1x + 8y + 8z = 15$

Answer:

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 8 & 1 & 0 & 0 \\ 3 & 4 & 4 & 0 & 1 & 0 \\ 1 & 8 & 8 & 0 & 0 & 1 \end{array}\right]$$ Multiply row 1 by 1/2

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1/2 & 0 & 0 \\ 3 & 4 & 4 & 0 & 1 & 0 \\ 1 & 8 & 8 & 0 & 0 & 1 \end{array}\right]$$ Add -3 times row 1 to row 2

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1/2 & 0 & 0 \\ 0 & -2 & -8 & -3/2 & 1 & 0 \\ 1 & 8 & 8 & 0 & 0 & 1 \end{array}\right]$$ Add -1 times row 1 to row 3

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1/2 & 0 & 0 \\ 0 & -2 & -8 & -3/2 & 1 & 0 \\ 0 & 6 & 4 & -1/2 & 0 & 1 \end{array}\right]$$ Add 3 times row 2 to row 3

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1/2 & 0 & 0 \\ 0 & -2 & -8 & -3/2 & 1 & 0 \\ 0 & 0 & -20 & -5 & 3 & 1 \end{array}\right]$$ Multiply row 2 by -1/2

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1/2 & 0 & 0 \\ 0 & 1 & 4 & 3/4 & -1/2 & 0 \\ 0 & 0 & -20 & -5 & 3 & 1 \end{array}\right]$$ Multiply row 3 by -1/20

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1/2 & 0 & 0 \\ 0 & 1 & 4 & 3/4 & -1/2 & 0 \\ 0 & 0 & 1 & 1/4 & -3/20 & -1/20 \end{array}\right]$$ Add -4 times row 3 to row 2

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/4 & 1/10 & 1/5 \\ 0 & 0 & 1 & 1/4 & -3/20 & -1/20 \end{array}\right]$$ Add -4 times row 3 to row 1

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1/2 & 3/5 & 1/5 \\ 0 & 1 & 0 & -1/4 & 1/10 & 1/5 \\ 0 & 0 & 1 & 1/4 & -3/20 & -1/20 \end{array}\right]$$ Add -2 times row 2 to row 1

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2/5 & -1/5 \\ 0 & 1 & 0 & -1/4 & 1/10 & 1/5 \\ 0 & 0 & 1 & 1/4 & -3/20 & -1/20 \end{array}\right]$$

So the inverse is $\begin{bmatrix} 0 & 2/5 & -1/5 \\ -1/4 & 1/10 & 1/5 \\ 1/4 & -3/20 & -1/20 \end{bmatrix}$

$$\mathbf{x} = \begin{bmatrix} 0 & 2/5 & -1/5 \\ -1/4 & 1/10 & 1/5 \\ 1/4 & -3/20 & -1/20 \end{bmatrix} * \begin{bmatrix} 4 \\ 5 \\ 15 \end{bmatrix} = \begin{bmatrix} -1 \\ -5/2 \\ -1/2 \end{bmatrix}$$

So x = -1, y = -5/2, and z = -1/2.

10) Find the determinate of matrix A by using cofactor expansion along the row or column of you choice.

$$A = \begin{bmatrix} 1 & 7 & 4 & 7 & 8 & 1 \\ 0 & 6 & 0 & 0 & 0 & 6 \\ 3 & 2 & 0 & 1 & 0 & 2 \\ 4 & 3 & 3 & 2 & 9 & 3 \\ 5 & 6 & 0 & 3 & 5 & 6 \\ 0 & 4 & 0 & 4 & 0 & 4 \end{bmatrix}$$

Answer:

Expand Along Column 3

$$\det \begin{bmatrix} 1 & 7 & 4 & 7 & 8 & 1 \\ 0 & 6 & 0 & 0 & 0 & 6 \\ 3 & 2 & 0 & 1 & 0 & 2 \\ 4 & 3 & 3 & 2 & 9 & 3 \\ 5 & 6 & 0 & 3 & 5 & 6 \\ 0 & 4 & 0 & 4 & 0 & 4 \end{bmatrix} =$$

$$4 \det \begin{bmatrix} 0 & 6 & 0 & 0 & 6 \\ 3 & 2 & 1 & 0 & 2 \\ 4 & 3 & 2 & 9 & 3 \\ 5 & 6 & 3 & 5 & 6 \\ 0 & 4 & 4 & 0 & 4 \end{bmatrix} - 3 \det \begin{bmatrix} 1 & 7 & 7 & 8 & 1 \\ 0 & 6 & 0 & 0 & 6 \\ 3 & 2 & 1 & 0 & 2 \\ 5 & 6 & 3 & 5 & 6 \\ 0 & 4 & 4 & 0 & 4 \end{bmatrix}$$

The first matrix det will be 0 because columns 4 & 5 are identical.

$$-3\det\begin{bmatrix} 1 & 7 & 7 & 8 & 1 \\ 0 & 6 & 0 & 0 & 6 \\ 3 & 2 & 1 & 0 & 2 \\ 5 & 6 & 3 & 5 & 6 \\ 0 & 4 & 4 & 0 & 4 \end{bmatrix}$$

Expand Along Column 4

$$(-3)\left(-8\det\begin{bmatrix} 0 & 6 & 0 & 6 \\ 3 & 2 & 1 & 2 \\ 5 & 6 & 3 & 6 \\ 0 & 4 & 4 & 4 \end{bmatrix} + 5\det\begin{bmatrix} 1 & 7 & 7 & 1 \\ 0 & 6 & 0 & 6 \\ 3 & 2 & 1 & 2 \\ 0 & 4 & 4 & 4 \end{bmatrix}\right)$$

The first matrix det will be 0 because columns 2 & 4 are identical.

$$(-3)\,(5)\det\begin{bmatrix} 1 & 7 & 7 & 1 \\ 0 & 6 & 0 & 6 \\ 3 & 2 & 1 & 2 \\ 0 & 4 & 4 & 4 \end{bmatrix}$$

Expand Along Column 1

$$(-3)\,(5)\left(\det\begin{bmatrix} 6 & 0 & 6 \\ 2 & 1 & 2 \\ 4 & 4 & 4 \end{bmatrix} + 3\det\begin{bmatrix} 7 & 7 & 1 \\ 6 & 0 & 6 \\ 4 & 4 & 4 \end{bmatrix}\right)$$

This first matrix will be 0 because columns 1 & 3 are identical.

$$(-3)\,(5)\,(3)\begin{bmatrix} 7 & 7 & 1 \\ 6 & 0 & 6 \\ 4 & 4 & 4 \end{bmatrix}$$

Expand Along Column 2

$$(-3)\,(5)\,(3)\left(-7\begin{bmatrix} 6 & 6 \\ 4 & 4 \end{bmatrix} - 4\begin{bmatrix} 7 & 1 \\ 6 & 6 \end{bmatrix}\right)$$

The first matrix det will be 0 because both ~~rows~~ cols are identical.

$$(-3)\,(5)\,(3)\,(-4)\begin{bmatrix} 7 & 1 \\ 6 & 6 \end{bmatrix}$$

$= (-3)\,(5)\,(3)\,(-4)\,(42-6)$

$= 6480$

1. Solve the following system of equations using Gaussian elimination.

$$x_1 + 5x_2 - 7x_3 + 6x_5 = 2$$
$$x_2 - 3x_3 = 12$$
$$2x_3 - 3x_5 = 6$$
$$2x_1 + 4x_2 + 7x_3 + 9x_4 + 3x_5 = 9$$

<u>Solution:</u> Write out the augmented matrix for the system and perform elementary row operations on the matrix to reduce it to row–echelon form.

Switch the 2nd and 4th row.

Add - 2 times the 1st row to the 2nd.

Switch the 3rd and 4th row.

Add 6 times the 2nd row to the 4th.

Switch the 3rd and 4th row.

Divide the 4th by 21.

Add - 1 times the 3rd row to the 4th.

Divide the 4th row by 3/7.

$$\begin{bmatrix} 1 & 5 & -7 & 0 & 6 & 2 \\ 0 & 1 & 0 & -3 & 0 & 12 \\ 0 & 0 & 2 & 0 & -3 & 6 \\ 2 & 4 & 7 & 9 & 3 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -7 & 0 & 6 & 2 \\ 2 & 4 & 7 & 9 & 3 & 9 \\ 0 & 0 & 2 & 0 & -3 & 6 \\ 0 & 1 & 0 & -3 & 0 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -7 & 0 & 6 & 2 \\ 0 & -6 & 21 & 9 & -9 & 5 \\ 0 & 0 & 2 & 0 & -3 & 6 \\ 0 & 1 & 0 & -3 & 0 & 12 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 5 & -7 & 0 & 6 & 2 \\ 0 & 1 & 0 & -3 & 0 & 12 \\ 0 & 0 & 2 & 0 & -3 & 6 \\ 0 & -6 & 21 & 9 & -9 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -7 & 0 & 6 & 2 \\ 0 & 1 & 0 & -3 & 0 & 12 \\ 0 & 0 & 2 & 0 & -3 & 6 \\ 0 & -6 & 21 & 9 & -9 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -7 & 0 & 6 & 2 \\ 0 & 1 & 0 & -3 & 0 & 12 \\ 0 & 0 & 2 & 0 & -3 & 6 \\ 0 & 0 & 21 & -9 & -9 & 77 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 5 & -7 & 0 & 6 & 2 \\ 0 & 1 & 0 & -3 & 0 & 12 \\ 0 & 0 & 21 & -9 & -9 & 6 \\ 0 & 0 & 2 & 0 & -3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -7 & 0 & 6 & 2 \\ 0 & 1 & 0 & -3 & 0 & 12 \\ 0 & 0 & 21 & -9 & -9 & 77 \\ 0 & 0 & 1 & 0 & -3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -7 & 0 & 6 & 2 \\ 0 & 1 & 0 & -3 & 0 & 12 \\ 0 & 0 & 1 & -3/7 & -3/7 & 11/3 \\ 0 & 0 & 1 & 0 & -3 & 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 5 & -7 & 0 & 6 & 2 \\ 0 & 1 & 0 & -3 & 0 & 12 \\ 0 & 0 & 1 & -3/7 & -3/7 & 11/3 \\ 0 & 0 & 0 & 3/7 & 3/7 & -2/3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -7 & 0 & 6 & 2 \\ 0 & 1 & 0 & -3 & 0 & 12 \\ 0 & 0 & 1 & -3/7 & -3/7 & 11/3 \\ 0 & 0 & 0 & 1 & 1 & -14/9 \end{bmatrix}$$

The corresponding system of equations is

$$x_1 + 5x_2 - 7x_3 + 6x_5 = 2$$
$$x_2 - 3x_4 = 12$$
$$x_3 - 3/7x_4 - 3/7x_5 = 11/3$$
$$x_4 + x_5 = -14/9$$

Solving for the leading variables we obtain.

$$x_1 = 2 - 5x_2 + 7x_3 - 6x_5$$
$$x_2 = 12 + 3x_4$$
$$x_3 = 11/3 + 3/7x_4 + 3/7x_5$$
$$x_4 = -14/9 - x_5$$

Assigning free variable t to x_5 the general solution is given

$$x_5 = t$$
$$x_4 = -14/9 - t$$
$$x_3 = 11/3 + 3/7(-14/9 - t) + 3/7t = 3$$
$$x_2 = 12 + 3(-14/9 - t) = 22/3 - 3t$$
$$x_1 = 2 - 5(22/3 - 3t) + 7 \cdot 3 - 6t = -41/3 + 9t$$

2. Given the following matrices ~~find the answers to the~~ Compute the following ~~equations~~ expressions.

$$A = \begin{bmatrix} 5 & 6 & 3 \\ 2 & 9 & -6 \\ 5 & 4 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 10 & 6 & -3 \\ 5 & 2 & 5 & 4 \\ -7 & 7 & -4 & 9 \end{bmatrix} \quad C = \begin{bmatrix} 8 & 9 \\ 10 & 3 \\ 2 & 1 \\ -4 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 7 & 5 & 1 \\ 2 & 8 & -4 \\ -3 & 2 & 9 \end{bmatrix}$$

a) AB
b) BA
c) 2BC
e) 5D + A
f) AD + A

Solution:

a) $$\begin{bmatrix} 5 & 6 & 3 \\ 2 & 9 & -6 \\ 5 & 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 10 & 6 & -3 \\ 5 & 2 & 5 & 4 \\ -7 & 7 & -4 & 9 \end{bmatrix} = \begin{bmatrix} 14 & 83 & 48 & 36 \\ 89 & -4 & 81 & -24 \\ 4 & 79 & 38 & 28 \end{bmatrix}$$

b) The product is undefined because the dimensions of the matrices.

c) $2\begin{bmatrix} 1 & 10 & 6 & -3 \\ 5 & 2 & 5 & 4 \\ -7 & 7 & -4 & 9 \end{bmatrix}\begin{bmatrix} 8 & 9 \\ 10 & 3 \\ 2 & 1 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 264 & 90 \\ 108 & 112 \\ -60 & -92 \end{bmatrix}$

d) $5\begin{bmatrix} 7 & 5 & 1 \\ 2 & 8 & -4 \\ -3 & 2 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 6 & 3 \\ 2 & 9 & -6 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 40 & 31 & 8 \\ 12 & 49 & -26 \\ -10 & 14 & 48 \end{bmatrix}$

e)

$AD \Rightarrow \begin{bmatrix} 5 & 6 & 3 \\ 2 & 9 & -6 \\ 5 & 4 & 3 \end{bmatrix}\begin{bmatrix} 7 & 5 & 1 \\ 2 & 8 & -4 \\ -3 & 2 & 9 \end{bmatrix} = \begin{bmatrix} 38 & 79 & 8 \\ 50 & 70 & -88 \\ 34 & 63 & 16 \end{bmatrix}$

$AD + D \Rightarrow \begin{bmatrix} 38 & 79 & 8 \\ 50 & 70 & -88 \\ 34 & 63 & 16 \end{bmatrix} + \begin{bmatrix} 5 & 6 & 3 \\ 2 & 9 & -6 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 43 & 85 & 11 \\ 52 & 79 & -94 \\ 39 & 67 & 19 \end{bmatrix}$

3. Use the given information to find A:

$(3A - I)^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Solution:

First of all the equality $(3A - I)^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ can be reduced by taking the inverse of both sides. The left hand side following the Law of Exponents reduces to 3A – I, and the right hand side becomes $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, using Theorem 1.4.5 from the book; which results in $3A - I = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. Next the identity matrix can be added to both sides resulting in 3A = $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The sum of these two matrices can be further multiplied by (1/3), as to the right to reach $A = \left(\frac{1}{3}\right)\begin{bmatrix} 3 & 3 \\ 1 & 3 \end{bmatrix}$, which when multiplied out gives us the solution, A = $\begin{bmatrix} 1 & 1 \\ 1/3 & 1 \end{bmatrix}$.

4. Find the inverse of A using row operations, where $A = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$.

Solution:
In reducing A to the identity matrix by row operations and at the same time applying these operations to I to produce A^{-1}. We will adjoin the two matrices as shown below and apply the operations.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right] \longleftarrow \text{We added } -1 \text{ times the first row to the third row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \longleftarrow \text{We added the second row to the third row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & -3 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \longleftarrow \text{We added } -3 \text{ times the second row to the first row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & -3 \\ 0 & 1 & 0 & 3 & -2 & -3 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \longleftarrow \text{We added } -3 \text{ times the third row to the second row.}$$

Thus,

$$A^{-1} = \begin{bmatrix} -2 & 0 & 3 \\ 3 & -2 & -3 \\ -1 & 1 & 1 \end{bmatrix}.$$

5. Find a corresponding elementary matrix such that $EA = \begin{bmatrix} 7 & -1 & 4 & 4 \\ -1 & 3 & 2 & -4 \\ 2 & -1 & 1 & 3 \end{bmatrix}$ where E is an elementary matrix and $A = \begin{bmatrix} 1 & 2 & 1 & 5 \\ -1 & 3 & 2 & -4 \\ 2 & -1 & 1 & 3 \end{bmatrix}$.

Solution:
By examination of EA and A, we see that the two matrices are identical with the exception of the first row. We can see that the first row has been summed with 3 times

the third row. Based on theorem 1.5.1, a correct matrix would be, $E = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. As multiplying E and A results in the given value of EA.

Good.

6. Solve the system of equations using $\mathbf{A}^{-1}$

$$x_1 + 3x_2 - x_3 = 20$$
$$-2x_1 + 3x_2 = 15$$
$$-x_1 + 3x_2 + x_3 = 22$$

Solution:

This system can be written in matrix form **Ax=b**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -1 \\ -2 & 3 & 0 \\ -1 & 3 & 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 20 \\ 15 \\ 22 \end{bmatrix}$$

A is invertible and

$$\mathbf{A}^{-1} = \left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 & 1 & 0 \\ -1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & \frac{-1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{2} & \frac{3}{4} \end{array}\right]$$

By Theorem 1.6.2, the solution of the system is

$$x = A - 1b = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{6} & 0 & \frac{1}{6} \\ -\frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 20 \\ 15 \\ 22 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$$

$$\text{Therefore } \begin{aligned} x_1 &= 3 \\ x_2 &= 7 \\ x_3 &= 4 \end{aligned}$$

7. Find the inverse of the given matrix **A** by using its adjoint and cofactor expansion.

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 2 & 0 \\ 1 & 3 & 3 \end{bmatrix}$$

Solution:

$$\mathbf{C}_{11} = \begin{vmatrix} 2 & 0 \\ 3 & 3 \end{vmatrix} = 6 \quad \mathbf{C}_{12} = -\begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} = 0 \quad \mathbf{C}_{13} = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -2$$

$$\mathbf{C}_{21} = -\begin{vmatrix} 4 & 1 \\ 3 & 3 \end{vmatrix} = -9 \quad \mathbf{C}_{22} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 \quad \mathbf{C}_{23} = -\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = -2$$

$$\mathbf{C}_{31} = \begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} = -2 \quad \mathbf{C}_{32} = -\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} = 0 \quad \mathbf{C}_{33} = \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} = 4$$

The first step is to find the adjoint for this matrix. To find the adjoint, first create a coefficient matrix for **A**.

$$\text{adj}(\mathbf{A}) = \begin{bmatrix} 6 & 0 & -2 \\ -9 & 5 & -2 \\ -2 & 0 & 4 \end{bmatrix}^T = \begin{bmatrix} 6 & -9 & -2 \\ 0 & 5 & 0 \\ -2 & -2 & 4 \end{bmatrix}$$

Next, take the transverse of the coefficient matrix for **A**.

$$\det(\mathbf{A}) = 2(\mathbf{C}_{11}) + 4(\mathbf{C}_{12}) + 1(\mathbf{C}_{13}) = 12 + 0 + (-2) = 10$$

Once the adjoint is calculated, find the determinant of the matrix.

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\text{adj}(\mathbf{A}) = \frac{1}{10}\begin{bmatrix} 6 & -9 & -2 \\ 0 & 5 & 0 \\ -2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{-9}{10} & \frac{-1}{5} \\ 0 & \frac{1}{2} & 0 \\ \frac{-1}{5} & \frac{-1}{5} & \frac{2}{5} \end{bmatrix}$$

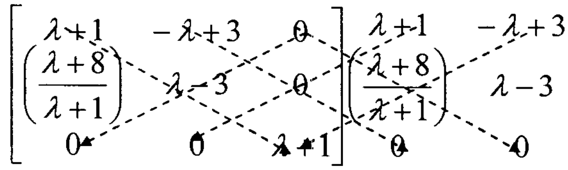
Then combine the adjoint and the determinant in this manner to find the inverse.

8. a) For the matrix *S*, using the definition of determinants (dealing with permutations), find for what values of λ det(S) = 0

$$S = \begin{bmatrix} \lambda+1 & -\lambda+1 & 0 \\ \left(\frac{\lambda+6}{\lambda+1}\right) & \lambda-3 & 0 \\ 0 & 0 & \lambda+1 \end{bmatrix}$$

b) Check your answer for part a. by substituting the value(s) for λ into *S* and evaluating det(*S*).

Solution: **a)** Since S is a 3 x 3 matrix, the figure below helps us calculate the determinant by subtracting the sum of the products of the factors along the left arrows from the sum products of the factors along the right arrows.



$$\det(S) = (\lambda+1)(\lambda-3) + (-\lambda+3)(0)(0) + (0)\left(\frac{\lambda+8}{\lambda+1}\right)(0) - \ldots$$

$$(0)(\lambda-3)(0) - (\lambda+1)(0)(0) - (-\lambda+3)\left(\frac{\lambda+8}{\lambda+1}\right)(0)$$

$$\det(S) = \lambda^3 - \lambda^2 - 5\lambda - 3 - (-\lambda^2 - 5\lambda + 24)$$

$$\det(S) = \lambda^3 - 27 = 0$$

$$0 = \lambda^3 - 27$$

$$\lambda^3 = 27$$

$$\lambda = 3$$

Note: the expression $\lambda = -1$ is not a solution, because in the entry of the first column and second row you would have to divide by zero

b) Substituting $\lambda = 3$ into S you obtain

$$S = \begin{bmatrix} 4 & -2 & 0 \\ \left(\frac{9}{4}\right) & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\det(S) = \begin{vmatrix} 4 & .2 & 0 \\ \frac{9}{4} & 0 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 0M_{31} + 0(-M_{32}) + 4M_{33} = 4\begin{vmatrix} 4 & -2 \\ \frac{9}{4} & 0 \end{vmatrix} = 4 \cdot 0$$

?

$(-2 \cdot \frac{9}{4}) = -18$

9. A tetrahedron has the vertices (0,0,0), (1,1,2), (2,0,0), (0,2,0). Find the volume of the tetrahedron using the determinate. Calculate that determinate using row operations. (Hint: the volume of the tetrahedron is the absolute value of the determinant of the matrix that contains the three vectors that begin at the origin and end at the vertices.)

Solution: First write the matrix that includes each vector as a row

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

Then perform the following row operations, writing down the appropriate numbers in the column on the right.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3 \quad \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad (-1)$$

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad (-1)$$

$$R_1 \rightarrow \frac{1}{2} R_1 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad \left(\frac{1}{2}\right)$$

$$R_2 \rightarrow \frac{1}{2} R_2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad \left(\frac{1}{2}\right)$$

$$R_3 = R_3 - R_1 - R_2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (1)$$

$$R_3 \rightarrow \frac{1}{2} R_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \left(\frac{1}{2}\right)$$

The matrix that you end up with has determinant equal to 1 and the numbers that you wrote down to the right have the product $\frac{1}{8}$, so the determinant of the vectors' matrix is $1/\left(\frac{1}{8}\right) = 8$.

Therefore, the volume is 8 units.

10. Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

a) Find the characteristic polynomial for A and multiply it by the characteristic polynomial for B

b) Find the characteristic polynomial for AB

c) Do the equations from part a and b have the same roots?

Solution: The characteristic polynomial for A is simply $\det(xI\text{-}A)$.

$$\det(xI - A) = \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} x-1 & -3 \\ -4 & x-2 \end{bmatrix}\right)$$

$$= (x-1)(x-2) - (-3)(-4)$$

$$= x^2 - 3x - 10$$

Similarly,

$$\det(xI - B) = \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} x-1 & -4 \\ -3 & x-2 \end{bmatrix}\right)$$

$$= (x-1)(x-2) - (-3)(-4)$$

$$= x^2 - 3x - 10$$

Which are the characteristic polynomials. Their product is:

$$(x^2 - 3x - 10)^2 = x^4 - 6x^3 - 11x^2 - 60x + 100$$

That is the solution for part a.

For part b, you first multiply the two matrices together.

$$AB = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1+9 & 4+6 \\ 4+6 & 16+4 \end{bmatrix} = \begin{bmatrix} 10 & 10 \\ 10 & 20 \end{bmatrix}$$

The characteristic polynomial for AB is simply det(xI-AB).

$$\det(xI - AB) = \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 10 & 10 \\ 10 & 20 \end{bmatrix}\right) = \det\left(\begin{bmatrix} x-10 & 10 \\ 10 & x-20 \end{bmatrix}\right)$$

$$= (x-10)(x-20) - (-10)(-10)$$

$$= x^2 - 30x - 300$$

For part c, you see that the equation from part one is simply (x-5)(x-5)(x+2)(x+2) =0 and setting that equal to 0 gives you roots at x = 5 and at x = -2. The equation from part b has roots at $x = \dfrac{30 \pm \sqrt{30^2 - 4(1)(-300)}}{4(1)(-300)} = -\dfrac{30 \pm 10\sqrt{21}}{600}$.

The answer is that, no, they do not have the same roots.