KEY

Math 343 Midterm 1
Instructor: Scott Glasgow
Sections: 1, 6 and 8
Dates: September 27th and 28th, 2005

Instructions: You will be docked points for *irrelevant* developments as well as patently wrong ones. So organize your efforts on two sets of papers, one set called "scratch" containing whatever *meanderings* you needed to get things clear in your mind, and which set I do not ever want to see, the other set on which you put your name and section number containing only ***relevant*** calculations and discussions leading ***directly*** to (hopefully) correct answers, and which you hand in to the testing center staff (so that I may see and grade them). Related to this idea is the fact that you will also be docked for handwriting and organization that is so poor that either I cannot determine your development, or I find it painful to determine it. In summary, your work that I ultimately see and grade should be a (logical) work of art.

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1. a) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix}$$

by row reducing $\left[A|I\right]$ to $\left[I|B\right] = \left[I|A^{-1}\right]$,

<u>7pts</u>

and then

b) check your proposed answer B by confirming that $AB = I$.

<u>8pts</u>

<u>Solution:</u>

a) Row reductions give

$$\left[A|I\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 5 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \mapsto 1R_3 - 5R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -5 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \mapsto \frac{1}{3} R_3]{R_2 \mapsto \frac{1}{2} R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -5/3 & 0 & 1/3 \end{array}\right]$$

so that

$$B = A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ -5/3 & 0 & 1/3 \end{bmatrix}.$$

b) Viewing both A and B "column wise" we get the multiplication AB proceeds as

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ -5/3 & 0 & 1/3 \end{bmatrix} = \left[1\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} - 5/3\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \quad 1/2\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \quad 1/3\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right]$$

$$= \left[\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

(so that $B = A^{-1}$ since the latter behaves this way and is unique).

2. a) Find all solutions to the following system of equations by row reducing an augmented matrix to reduced row-echelon form (and interpreting the resulting matrix as code for a simplified system of equations to be solved):

$$3x - y + 7z = 0,\ 2x - y + 4z = 1/2,\ x - y + z = 1,\ 6x - 4y + 10z = 3.$$

<u>7pts</u>

b) Check your proposed solutions to a) by demonstrating that they actually solve the four equations in a).

<u>8pts</u>

<u>Solution</u>

a) The information contained in the four equations is contained in the following augmented matrix (obtained with multiplying the second equation by 2 to eliminate fractions, and by ordering the equations differently), and which reduces as indicated:

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 3 & -1 & 7 & 0 \\ 4 & -2 & 8 & 1 \\ 6 & -4 & 10 & 3 \end{bmatrix} \xrightarrow{\substack{R_2 \mapsto 1R_2 - 3R_1 \\ R_3 \mapsto 1R_3 - 4R_1 \\ R_4 \mapsto 1R_4 - 6R_1}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & 4 & -3 \\ 0 & 2 & 4 & -3 \\ 0 & 2 & 4 & -3 \end{bmatrix} \xrightarrow{\substack{R_3 \mapsto 1R_3 - R_2 \\ R_4 \mapsto 1R_4 - R_2}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & 4 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \mapsto 2R_1 + R_2} \begin{bmatrix} 2 & 0 & 6 & -1 \\ 0 & 2 & 4 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\substack{R_1 \mapsto 1/2R_1 \\ R_2 \mapsto 1/2R_2}} \begin{bmatrix} 1 & 0 & 3 & -1/2 \\ 0 & 1 & 2 & -3/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The latter is in reduced row-echelon form, and is code for the simplified equations

$$\begin{aligned} 1x \quad +0y \quad +3z \quad &= -1/2 \\ 0x \quad +1y \quad +2z \quad &= -3/2, \end{aligned}$$

the obvious solutions to which being parameterized by the (vector) equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/2-3t \\ -3/2-2t \\ t \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}.$$

b) We can demonstrate that these solve the equations (for which the augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 3 & -1 & 7 & 0 \\ 4 & -2 & 8 & 1 \\ 6 & -4 & 10 & 3 \end{bmatrix}$$

is code) by noting that

$$\begin{bmatrix} 1 & -1 & 1 \\ 3 & -1 & 7 \\ 4 & -2 & 8 \\ 6 & -4 & 10 \end{bmatrix}\left(-\frac{1}{2}\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}\right) = -\frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 3 & -1 & 7 \\ 4 & -2 & 8 \\ 6 & -4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 & -1 & 1 \\ 3 & -1 & 7 \\ 4 & -2 & 8 \\ 6 & -4 & 10 \end{bmatrix}\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 1-3 \\ 3-3 \\ 4-6 \\ 6-12 \end{bmatrix} + t\begin{bmatrix} -3+2+1 \\ -9+2+7 \\ -12+4+8 \\ -18+8+10 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 \\ 0 \\ -2 \\ -6 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

3. a) List each disordered pair in the following permutations of $(1,2,3,4,5)$:

i) $(5,4,3,2,1)$
ii) $(2,4,3,1,5)$.

7pts

b) For each of the two permutations above find the associated signed elementary product (used in the construction of the determinant) of the matrix

$$A=\begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & q & r & s & t \\ u & v & w & x & y \end{bmatrix}.$$

<u>8pts</u>

<u>Solution</u>

a) For i) there are 10 disordered pairs, which are $(5,4)$, $(5,3)$, $(5,2)$, $(5,1)$, $(4,3)$, $(4,2)$, $(4,1)$, $(3,2)$, $(3,1)$, and $(2,1)$.

For ii) there are 4 disordered pairs, which are $(2,1)$, $(4,3)$, $(4,1)$, and $(3,1)$.

b) The signed elementary products are then $(-1)^{10} A_{15}A_{24}A_{33}A_{42}A_{51} = +1eimqu = eimqu$, and $(-1)^{4} A_{12}A_{24}A_{33}A_{41}A_{55} = +1bimpy = bimpy$.

4. a) If such exists, find an elementary matrix E such that $EC = Q$, where

$$C=\begin{bmatrix} 5 & 1 & 4 & 9 \\ 1 & 0 & 3 & 2 \\ 4 & 2 & 1 & 1 \end{bmatrix}, \text{ and } Q=\begin{bmatrix} 5 & 1 & 4 & 9 \\ 1 & 0 & 3 & 2 \\ 8 & 2 & 13 & 9 \end{bmatrix},$$

<u>7pts</u>

and

b) confirm that such is the case by computing this product EC.

<u>8pts</u>

<u>Solution</u>

a) Since the only difference between C and Q lies in their 3rd rows, the existence of E is possible, provided the 3rd row of Q is obtained as an elementary row operation on C which does not involve swapping. Since the second entry of the third row is the same between C and Q (yet the third rows do differ elsewhere), this row operation, if it exists, must actually be that of multiplying a row of C by a constant, and adding it to this third row, but with the row "added" having a 0 in its second entry. Thus the row of C "added", if such exists, must be row 2. So since the equation $4+k\cdot 1=C_{31}+kC_{21}=Q_{31}=8$ has only the solution $k=4$, the multiple of row 2 used in the elementary operation, if the operation exists, must be 4. One then notes that this operation does give the relevant row of Q, and then, according to theorem, one finds such an E by performing this operation on the relevant identity matrix I, here necessarily of size 3 by3:

$$I_{3\times 3}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\overset{R_3\mapsto 1R_3+4R_2}{\sim}\begin{bmatrix}1&0&0\\0&1&0\\0&4&1\end{bmatrix}=E.$$

b) One confirms that

$$EC=\begin{bmatrix}1&0&0\\0&1&0\\0&4&1\end{bmatrix}\begin{bmatrix}5&1&4&9\\1&0&3&2\\4&2&1&1\end{bmatrix}=\begin{bmatrix}5&1&4&9\\1&0&3&2\\4\cdot 1+1\cdot 4&1\cdot 2&4\cdot 3+1\cdot 1&4\cdot 2+1\cdot 1\end{bmatrix}$$
$$=\begin{bmatrix}5&1&4&9\\1&0&3&2\\8&2&13&9\end{bmatrix}=Q.$$

5. If necessary, give a condition on the constants b_1, b_2, and b_3 so that the system

$$x_1+x_3=b_1,\ 3x_1+x_2+x_3=b_2,\ 4x_1+x_2+2x_3=b_3$$

is consistent.

<u>8pts</u>

<u>Solution</u>

We proceed by row reducing to echelon form, as follows, the associated augmented matrix

$$\begin{bmatrix}1&0&1&b_1\\3&1&1&b_2\\4&1&2&b_3\end{bmatrix}\overset{\substack{R_2\mapsto 1R_2-3R_1\\R_3\mapsto 1R_3-4R_1}}{\sim}\begin{bmatrix}1&0&1&b_1\\0&1&-2&b_2-3b_1\\0&1&-2&b_3-4b_1\end{bmatrix}\overset{R_3\mapsto 1R_3-1R_2}{\sim}\begin{bmatrix}1&0&1&b_1\\0&1&-2&b_2-3b_1\\0&0&0&b_3-4b_1-(b_2-3b_1)\end{bmatrix}$$

which dictates for consistency the condition $0=b_3-4b_1-(b_2-3b_1)=b_3-4b_1-b_2+3b_1$ $=b_3-b_2-b_1$, i.e. the condition that $b_3=b_1+b_2$.

6. a) Solve the systems

$$\begin{aligned} x-y+2z&=5 \\ 2x+y-4z&=-6 \\ x+3y+z&=3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x-y+2z&=-2 \\ 2x+y-4z&=9 \\ x+3y+z&=4 \end{aligned}$$

simultaneously,

<u>14pts</u>

and

b) check your proposed solutions by making sure they do solve the indicated equations.

<u>8pts</u>

<u>Solution</u>

a) Proceeding with row reduction on the relevant (doubly) augmented matrix we get

$$\begin{bmatrix} 1 & -1 & 2 & 5 & -2 \\ 2 & 1 & -4 & -6 & 9 \\ 1 & 3 & 1 & 3 & 4 \end{bmatrix} \xrightarrow[R_3 \mapsto 1R_3 - 1R_1]{R_2 \mapsto 1R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 2 & 5 & -2 \\ 0 & 3 & -8 & -16 & 13 \\ 0 & 4 & -1 & -2 & 6 \end{bmatrix} \xrightarrow{R_3 \mapsto 3R_3 - 4R_2} \begin{bmatrix} 1 & -1 & 2 & 5 & -2 \\ 0 & 3 & -8 & -16 & 13 \\ 0 & 0 & 29 & 58 & -34 \end{bmatrix} \xrightarrow{R_3 \mapsto R_3/29}$$

$$\begin{bmatrix} 1 & -1 & 2 & 5 & -2 \\ 0 & 3 & -8 & -16 & 13 \\ 0 & 0 & 1 & 2 & -34/29 \end{bmatrix} \xrightarrow[R_2 \mapsto 1R_2 + 8R_3]{R_1 \mapsto 1R_1 - 2R_3} \begin{bmatrix} 1 & -1 & 0 & 1 & -2+2\cdot 34/29 \\ 0 & 3 & 0 & 0 & 13-8\cdot 34/29 \\ 0 & 0 & 1 & 2 & -34/29 \end{bmatrix} \xrightarrow{R_2 \mapsto R_2/3}$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 2(-29+34)/29 \\ 0 & 1 & 0 & 0 & (29\cdot 13-8\cdot 34)/29/3 \\ 0 & 0 & 1 & 2 & -34/29 \end{bmatrix} \xrightarrow{R_1 \mapsto 1R_1 + 1R_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 2(5)/29+105/29/3 \\ 0 & 1 & 0 & 0 & 105/29/3 \\ 0 & 0 & 1 & 2 & -34/29 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 10/29+35/29 \\ 0 & 1 & 0 & 0 & 35/29 \\ 0 & 0 & 1 & 2 & -34/29 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 45/29 \\ 0 & 1 & 0 & 0 & 35/29 \\ 0 & 0 & 1 & 2 & -34/29 \end{bmatrix}$$

so that the pair of solutions are

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix} = \begin{bmatrix} 1 & 45/29 \\ 0 & 35/29 \\ 2 & -34/29 \end{bmatrix}.$$

b) We can check that these solve the systems (for which

$$\begin{bmatrix} 1 & -1 & 2 & 5 & -1 \\ 2 & 1 & -4 & -6 & 9 \\ 1 & 3 & 1 & 3 & 4 \end{bmatrix}$$

was code) by noting that

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -4 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 45/29 \\ 0 & 35/29 \\ 2 & -34/29 \end{bmatrix} = \begin{bmatrix} 1\cdot 1+2\cdot 2 & 1\cdot 45/29-1\cdot 35/29-2\cdot 34/29 \\ 2\cdot 1-4\cdot 2 & 2\cdot 45/29+1\cdot 35/29+4\cdot 34/29 \\ 1\cdot 1+1\cdot 2 & 1\cdot 45/29+3\cdot 35/29-1\cdot 34/29 \end{bmatrix} = \begin{bmatrix} 5 & -58/29 \\ -6 & 261/29 \\ 3 & 116/29 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & -2 \\ -6 & 9 \\ 3 & 4 \end{bmatrix}.$$

.

7. a) Find the characteristic polynomial $P_Q(x)$ of the matrix

$$Q = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix},$$

<u>10pts</u>

and then

b) find the eigenvalues of Q.

<u>5pts</u>

<u>Solution</u>

a) By definition

$$P_Q(x) := \det[xI - Q] = \det\left[\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\right] = \det\begin{bmatrix} x-3 & -1 \\ -2 & x-2 \end{bmatrix}$$
$$= (x-3)(x-2) - (-1)(-2) = x^2 - 5x + 6 - 2 = x^2 - 5x + 4$$
$$= (x-1)(x-4).$$

b) By definition, the eigenvalues of Q are the values λ for which the system $Q\underline{x} = \lambda\underline{x} = \lambda I\underline{x} \Leftrightarrow (\lambda I - Q)\underline{x} = \underline{0}$ has a nontrivial solution $\underline{x}$. So by (easy) theorem, the eigenvalues λ of Q are (then) the zeroes λ of $P_Q(\lambda)$, which are 1 and 4 here.

8. Find the determinant of the following matrix by row reducing the matrix to a simple form, keeping in mind how the row operations change the determinant (do not use cofactor expansion):

$$\begin{bmatrix} 3 & 1 & 5 & 4 \\ 2 & 6 & 2 & 8 \\ 0 & 7 & 1 & 9 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

<u>**15pts**</u>

<u>**Solution**</u>

Row reducing and using the various theorems we get

$$\det\begin{bmatrix} 3 & 1 & 5 & 4 \\ 2 & 6 & 2 & 8 \\ 0 & 7 & 1 & 9 \\ 1 & 2 & 3 & 4 \end{bmatrix} = 2\det\begin{bmatrix} 3 & 1 & 5 & 4 \\ 1 & 3 & 1 & 4 \\ 0 & 7 & 1 & 9 \\ 1 & 2 & 3 & 4 \end{bmatrix} = -2\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 4 \\ 0 & 7 & 1 & 9 \\ 3 & 1 & 5 & 4 \end{bmatrix} = -2\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 7 & 1 & 9 \\ 0 & -5 & -4 & -8 \end{bmatrix}$$

$$= -2\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 15 & 9 \\ 0 & 0 & -14 & -8 \end{bmatrix} = -2\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -14 & -8 \end{bmatrix} = -2\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix} = -2(1)(1)(1)(6)$$

$$= -12.$$

9. a) Find a cubic $y = ax^3 + bx^2 + cx + d$ that passes through the points $\{(x, y)\} = \{(0,4),(1,3),(2,6),(-1,3)\}$,

<u>**7pts**</u>

and then

b) check that your proposed cubic does pass through these points.

<u>**8pts**</u>

Solution

a) The equations dictating a, b, c, and d are evidently given by

$$\begin{array}{l} \left. ax^3+bx^2+cx+d=y\right|_{(x,y)\to(0,4)} \\ \left. ax^3+bx^2+cx+d=y\right|_{(x,y)\to(1,3)} \\ \left. ax^3+bx^2+cx+d=y\right|_{(x,y)\to(2,6)} \\ \left. ax^3+bx^2+cx+d=y\right|_{(x,y)\to(-1,3)} \end{array} \Leftrightarrow \begin{array}{r} 0a+0b+0c+d=4 \\ 1a+1b+1c+d=3 \\ 8a+4b+2c+d=6 \\ -1a+1b-1c+d=3 \end{array} \Rightarrow \begin{array}{r} d=4 \\ 1a+1b+1c=3-4=-1 \\ 8a+4b+2c=6-4=2 \\ -1a+1b-1c=3-4=-1 \end{array}$$

which we may further solve for a, b, and c by reducing an augmented matrix as follows:

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 8 & 4 & 2 & 2 \end{bmatrix} \overset{\substack{R_2\mapsto 1R_2+1R_1 \\ R_3\mapsto 1R_3-8R_1}}{\sim} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 0 & -2 \\ 0 & -4 & -6 & 10 \end{bmatrix} \overset{R_3\mapsto -R_3/2}{\sim} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 0 & -2 \\ 0 & 2 & 3 & -5 \end{bmatrix} \overset{R_3\mapsto 1R_3-1R_2}{\sim} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

$$\overset{\substack{R_2\mapsto R_2/2 \\ R_3\mapsto R_3/3}}{\sim} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \overset{R_1\mapsto 1R_1-1R_3}{\sim} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \overset{R_1\mapsto 1R_1-1R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

the latter indicating that $a=1$, $b=-1$, and $c=-1$. Thus (since $d=4$), the cubic is $y=ax^3+bx^2+cx+d \;=1x^3-1x^2-1x+4 \;=x^3-x^2-x+4$.

b). We have $\left. x^3-x^2-x+4\right|_{x=0}=0-0-0+4=4$, $\left. x^3-x^2-x+4\right|_{x=1}=1-1-1+4=3$, $\left. x^3-x^2-x+4\right|_{x=2}=8-4-2+4=6$, and $\left. x^3-x^2-x+4\right|_{x=-1}=-1-1+1+4=3$, as required by the set $\{(x,y)\}=\{(0,4),(1,3),(2,6),(-1,3)\}$.

10. Prove or disprove the following statements (assuming the various matrices indicated exist):

a) $\left(A^T\right)^{-1}=\left(A^{-1}\right)^T$, b) A^TA is symmetric, c) $A+A^T$ is symmetric, d) $A-A^T$ is skew symmetric, e) the product of a symmetric matrix A and a skew symmetric matrix B is skew symmetric, f) $(A-B)^2=A^2-2AB+B^2$, and g) $(A+B)(A-B)=A^2-B^2$.

3, 1, 1, 1, 3, 3, and 3 pts

Solution

a) This is true since $A^T\left(A^{-1}\right)^T=\left(A^{-1}A\right)^T=\left(I\right)^T=I=\left(AA^{-1}\right)^T=\left(A^{-1}\right)^T A^T$, and since the inverse (which behaves this way) is unique.

b) This is true since $\left(A^T A\right)^T=A^T\left(A^T\right)^T=A^T A$.

c) This is true since $\left(A+A^T\right)^T=A^T+\left(A^T\right)^T=A^T+A=A+A^T$.

d) This is true since $\left(A-A^T\right)^T=A^T-\left(A^T\right)^T=A^T-A=-\left(A-A^T\right)$.

e) This is not always true since even if $A^T=A$ and $B^T=-B$, one will have that $\left(AB\right)^T=B^T A^T=-BA\neq -AB$ whenever A and B don't commute.

f) This is not always true since

$$
\begin{aligned}
\left(A-B\right)^2-\left(A^2-2AB+B^2\right)&=\left(A-B\right)\left(A-B\right)-AA+2AB-BB\\
&=\left(A-B\right)A-\left(A-B\right)B-AA+2AB-BB\\
&=AA-BA-AB+BB-AA+2AB-BB\\
&=-BA+AB\neq 0
\end{aligned}
$$

whenever A and B don't commute.

g) This is not always true since

$$
\begin{aligned}
\left(A+B\right)\left(A-B\right)-\left(A^2-B^2\right)&=\left(A+B\right)A-\left(A+B\right)B-AA+BB\\
&=AA+BA-AB-BB-AA+BB\\
&=BA-AB\neq 0
\end{aligned}
$$

whenever A and B don't commute.