9

1) Consider the bases A = {u, v} and A' = {u', v'} for R2, where (30 Points)

$u = (1, 0);\quad v = (0,1);\quad u' = (1, 1);\quad v' = (2, 1)$

(a) Find the transition matrix from A' to A.

(b) Use the equation $[v]_B = P[v]_{B'}$ to find $[v]_B$ if

$$[v]_{B'} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Solution:

(a) First we must find the coordinate vectors for the new basis vectors u' and v' relative to the old basis B. By inspection,

$$u' = u + v$$

$$v' = 2u + v$$

so

$$[u']_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad [v']_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus the transition matrix from A' to A is

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

(b) Using the equation $[v]_B = P[v]_{B'}$ and the transition matrix in part (a) yields

$$[v]_B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

Great!

2) Prove that the following statements are equivalent. (45 Points)

(a) A is orthogonal

(b) $\|\mathbf{Ax}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n.

(c) $\mathbf{Ax} \cdot \mathbf{Ay} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ in R^n.

Solution:

(a) ⇒ (b)

$\|\mathbf{Ax}\| = (\mathbf{Ax} \cdot \mathbf{Ax})^{1/2}$ (def. of magnitude)

$(\mathbf{Ax} \cdot \mathbf{Ax})^{1/2} = (\mathbf{x} \cdot \mathbf{A}^T\mathbf{Ax})^{1/2}$ By properties of Euclidean inner product ($\mathbf{Au} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{A}^T\mathbf{v}$)

$(\mathbf{x} \cdot \mathbf{A}^T\mathbf{Ax})^{1/2} = (\mathbf{x} \cdot \mathbf{A}^{-1}\mathbf{Ax})^{1/2}$ Because **A** is orthogonal $\mathbf{A}^T = \mathbf{A}^{-1}$.

$(\mathbf{x} \cdot \mathbf{A}^{-1}\mathbf{Ax})^{1/2} = (\mathbf{x} \cdot \mathbf{Ix})^{1/2} = (\mathbf{x} \cdot \mathbf{x})^{1/2}$ By property of an inverse matrix

$(\mathbf{x} \cdot \mathbf{x})^{1/2} = \|\mathbf{x}\|$ By definition of magnitude

(b) $\Rightarrow$ (c)

$\mathbf{Ax} \cdot \mathbf{Ay} = \frac{1}{4}\|\mathbf{Ax}+\mathbf{Ay}\|^2 - \frac{1}{4}\|\mathbf{Ax}-\mathbf{Ay}\|^2$ This is Theorem 4.1.6 with **Ax** in place of **u** and **Ay** in place of **v**. It can be derived simply by expanding $\|\mathbf{u} + \mathbf{v}\|^2$ and $\|\mathbf{u} - \mathbf{v}\|^2$ using the def. and then simple algebra.

$\frac{1}{4}\|\mathbf{Ax}+\mathbf{Ay}\|^2 - \frac{1}{4}\|\mathbf{Ax}-\mathbf{Ay}\|^2 = \frac{1}{4}\|\mathbf{A}(\mathbf{x}+\mathbf{y})\|^2 - \frac{1}{4}\|\mathbf{A}(\mathbf{x}-\mathbf{y})\|^2$ Left distributive law.

$\frac{1}{4}\|\mathbf{A}(\mathbf{x}+\mathbf{y})\|^2 - \frac{1}{4}\|\mathbf{A}(\mathbf{x}-\mathbf{y})\|^2 = \frac{1}{4}\|(\mathbf{x}+\mathbf{y})\|^2 - \frac{1}{4}\|(\mathbf{x}-\mathbf{y})\|^2$ (b) $\|\mathbf{Ax}\| = \|\mathbf{x}\|$

$\frac{1}{4}\|(\mathbf{x}+\mathbf{y})\|^2 - \frac{1}{4}\|(\mathbf{x}-\mathbf{y})\|^2 = \mathbf{x} \cdot \mathbf{y}$ Again by Theorem 4.1.6

(c) $\Rightarrow$ (a)

$\mathbf{Ax} \cdot \mathbf{Ay} = \mathbf{x} \cdot \mathbf{y}$ (b)

$\mathbf{x} \cdot \mathbf{A}^T\mathbf{Ay} = \mathbf{x} \cdot \mathbf{y}$ By properties of Euclidean inner product ($\mathbf{Au} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{A}^T\mathbf{v}$)
$\mathbf{x} \cdot \mathbf{A}^T\mathbf{Ay} - \mathbf{x} \cdot \mathbf{y} = 0$ Algebra (remember inner products produce scalers)

$\mathbf{x} \cdot (\mathbf{A}^T\mathbf{Ay} - \mathbf{y}) = 0$ Property of Euclidean inner product

$\mathbf{x} \cdot (\mathbf{A}^T\mathbf{A} - \mathbf{I})\mathbf{y} = 0$ (1) Right distributive law.

Choose $\mathbf{x} = (\mathbf{A}^T\mathbf{A} - \mathbf{I})\mathbf{y}$ (1) is true for any **x** so lets choose a particular **x**

$(\mathbf{A}^T\mathbf{A} - \mathbf{I})\mathbf{y} \cdot (\mathbf{A}^T\mathbf{A} - \mathbf{I})\mathbf{y} = 0$ Substituting the **x** we chose.

This implies: $(\mathbf{A}^T\mathbf{A} - \mathbf{I})\mathbf{y} = \mathbf{0}$ The inner product of a matrix with itself can only equal zero if the matrix is the zero matrix.

$(A^TA - I)y = 0$ is a homogeneous system of linear equations that is satisfied by every **y** in R^n. Because this is true for every **y** this implies that the coefficient matrix $(A^TA - I)$ must equal zero. Therefore

$A^TA = I$ why? (Choose y to be respective rows of $A^TA - I$).

$A^T = A^{-1}$ Which means **A** is orthogonal.

3) Find the least squares solution of the linear system Ax=**b**. (30 points)
Also find the orthogonal projection of **b** onto the column space of A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

Solution:

$$A^TA = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -2 & 4 \\ -2 & -9 & -3 \\ 4 & -3 & 3 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \\ 5 \end{bmatrix}$$

$A^TAx = A^T\mathbf{b}$

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \\ 5 \end{bmatrix}$$

Solving the system we find that the least squares solution is

$x_1 = 5/6$
$x_2 = 5/9$
$x_3 = 10/9$

Using the formula $proj_w\mathbf{b} = Ax = A(A^TA)^{-1} A^T\mathbf{b}$
we can solve for the orthogonal projection of **b** onto the column space of A.

$$Ax = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5/6 \\ 5/9 \\ 10/9 \end{bmatrix} = ?$$

?

4) (a) Derive the formula for writing a vector in terms of an orthonormal basis. (45 points

(b) Convert the formula to that of an orthogonal basis.

Good

Solution:

(a) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for the vector space V, and let $\mathbf{u}$ be a vector in V. Then we can write:

$$\mathbf{u} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n.$$

And for each vector $\mathbf{v}_i$, we see that:

$$< \mathbf{u}, \mathbf{v}_i > = < k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i >$$

$$< \mathbf{u}, \mathbf{v}_i > = k_1 < \mathbf{v}_1, \mathbf{v}_i > + k_2 < \mathbf{v}_2, \mathbf{v}_i > + \ldots + k_n < \mathbf{v}_n, \mathbf{v}_i >.$$

Since S is an orthonormal set, we know that:

$$< \mathbf{v}_i, \mathbf{v}_i > = \|\mathbf{v}_i^2\| = 1 \text{ and } < \mathbf{v}_j, \mathbf{v}_i > = 0 \text{ for } j \neq i.$$

Therefore, we see that:

$$< \mathbf{u}, \mathbf{v}_i > = k_i,$$

and so the vector $\mathbf{u}$ can be expressed:

$$\mathbf{u} = < \mathbf{u}, \mathbf{v}_1 > \mathbf{v}_1 + < \mathbf{u}, \mathbf{v}_2 > \mathbf{v}_2 + \ldots + < \mathbf{u}, \mathbf{v}_n > \mathbf{v}_n.$$

(b) We convert this formula to that for an orthogonal basis by normalizing the vectors of the basis, so that S becomes:

$$S = \{\mathbf{v}_1/\|\mathbf{v}_1\|, \mathbf{v}_2/\|\mathbf{v}_2\|, \ldots, \mathbf{v}_n/\|\mathbf{v}_n\|\},$$

and $\mathbf{u}$ becomes:

$$\mathbf{u} = < \mathbf{u}, \mathbf{v}_1/\|\mathbf{v}_1\| > \mathbf{v}_1/\|\mathbf{v}_1\| + < \mathbf{u}, \mathbf{v}_2/\|\mathbf{v}_2\| > \mathbf{v}_2/\|\mathbf{v}_2\| + \ldots + < \mathbf{u}, \mathbf{v}_n/\|\mathbf{v}_n\| > \mathbf{v}_n/\|\mathbf{v}_n\|$$

or rather:

$$\mathbf{u} = < \mathbf{u}, \mathbf{v}_1/\|\mathbf{v}_1\|^2 > \mathbf{v}_1 + < \mathbf{u}, \mathbf{v}_2/\|\mathbf{v}_2\|^2 > \mathbf{v}_2 + \ldots + < \mathbf{u}, \mathbf{v}_n/\|\mathbf{v}_n\|^2 > \mathbf{v}_n$$

Problem 1

Consider the bases B = { $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{U}_3$} and B' = {$\mathbf{U'}_1$, $\mathbf{U'}_2$, $\mathbf{U'}_3$} for R^3 where $\mathbf{U}_1$= (1,0,0) $\mathbf{U}_2$= (0,1,0) $\mathbf{U}_3$= (0,0,1) and $\mathbf{U'}_1$=(1,2,4) $\mathbf{U'}_2$=(2,5,8) $\mathbf{U'}_3$=(3,6,13). Find the transition matrix from B to B'.

Solution
30 Points

We shall solve this problem by finding the transition matrix from B' to B and then taking the inverse to find the transition matrix from B to B'. The solution could also have been found by directly finding the transition matrix from B to B'.

First we find the coordinate vectors for $\mathbf{U'}_1$, $\mathbf{U'}_2$ and $\mathbf{U'}_3$ relative to B. This can be done by inspection.

$\mathbf{U'}_1 = \mathbf{U}_1 + 2\mathbf{U}_2 + 4\mathbf{U}_3$
$\mathbf{U'}_2 = 2\mathbf{U}_1 + 5\mathbf{U}_2 + 8\mathbf{U}_3$
$\mathbf{U'}_3 = 3\mathbf{U}_1 + 6\mathbf{U}_2 + 13\mathbf{U}_3$

$$\Rightarrow \quad [\mathbf{U'}_1]_B = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \qquad [\mathbf{U'}_2]_B = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \qquad [\mathbf{U'}_3]_B = \begin{bmatrix} 3 \\ 6 \\ 13 \end{bmatrix}$$

Thus, the transition matrix from B' to B is $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 4 & 8 & 13 \end{bmatrix}$

So the transition matrix from B to B' is the inverse of that matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 4 & 8 & 13 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \text{ to } R_2 \\ R_3 - 4R_1 \text{ to } R_3 \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - 3R_3 \text{ to } R_1 \\ R_1 - 2R_2 \text{ to } R_1 \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 17 & -2 & -3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 0 & 1 \end{array}\right]$$

Thus the transition matrix from B to B' is $\begin{bmatrix} 17 & -2 & -3 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 4 & 8 & 13 \end{bmatrix} \begin{bmatrix} 17 & -2 & -3 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

Problem 2:

Find the orthogonal projection of u onto the subspace of R^4 spanned by the vectors v_1, v_2, v_3.

$u = (6, 1, 5, 2)$

$v_1 = (1, 2, 1, 2)$

$v_2 = (2, 3, 2, 1)$

$v_3 = (1, 3, 3, 1)$

Solution:
(40 points)

First we set the three vectors up in a matrix as column vectors.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}$$

Then we find what $A^T A$

$$A^T A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 12 & 12 \\ 12 & 18 & 18 \\ 12 & 18 & 20 \end{bmatrix}$$

Then we find what $A^T u$.

Problem 3

In $\mathbb{R}^3$, with the Euclidean inner product, use the Gram-Schmidt process to transform the basis vectors

$u_1 = (1, 1, 1)$
$u_2 = (0, 2, 2)$
$u_3 = (0, 0, 3)$

into an orthogonal basis
$\{v_1, v_2, v_3\}$ and **then** normalize the orthogonal basis vectors **to** obtain an orthonormal basis $\{q_1, q_2, q_3\}$

Solution
(40 points)

$$v_1 = u_1 = (1, 1, 1)$$

$$v_2 = u_2 - \frac{< u_2, v_1 >}{\| v_1 \|^2} v_1$$

$$= (0, 2, 2) - \frac{4}{3}(1, 1, 1) = \left(-\frac{4}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$v_3 = u_3 - \frac{< u_3, v_1 >}{\| v_1 \|^2} v_1 - \frac{< u_3, v_2 >}{\| v_2 \|^2} v_2$$

$$= (0, 0, 3) - (1, 1, 1) - \frac{3}{4}\left(\frac{-4}{3}, \frac{2}{3}, \frac{2}{3}\right) = \left(0, \frac{-3}{2}, \frac{3}{2}\right)$$

Producing:
$v_1 = (1, 1, 1)$
$v_2 = \left(-\frac{4}{3}, \frac{2}{3}, \frac{2}{3}\right)$
$v_3 = \left(0, \frac{-3}{2}, \frac{3}{2}\right)$

get rid of noninteger results a.s.a.p.

We must know the norms of these vectors to form the orthogonal basis. The norms are:

$$\| v_1 \| = \sqrt{3}$$

$$\| v_2 \| = \frac{2\sqrt{6}}{3}$$

$$\| v_3 \| = \frac{3\sqrt{2}}{2}$$

so an orthonormal basis is

$$q_1 = \frac{v_1}{\| v_1 \|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\| v_2 \|} = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$q_3 = \frac{v_3}{\| v_3 \|} = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Problem 4: (40 points)

Prove the Triangle inequality, $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$, where $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space V.

Solution:

By definition we know that

$$
\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle \\
&= \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\
&\le \langle \mathbf{u}, \mathbf{u} \rangle + 2|\langle \mathbf{u}, \mathbf{v} \rangle| + \langle \mathbf{v}, \mathbf{v} \rangle \qquad \text{(property of absolute value)} \\
&\le \langle \mathbf{u}, \mathbf{u} \rangle + 2\|\mathbf{u}\|\|\mathbf{v}\| + \langle \mathbf{v}, \mathbf{v} \rangle \qquad \text{(by Cauchy-Schwarz)} \\
&= \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2
\end{aligned}
$$

Now, taking the square root of both sides gives

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

3

Problem 1:
(40 Points)
Let $\mathbb{R}^3$, have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis $\{u_1,u_2,u_3\}$ into an orthonormal basis $\{v_1,v_2,v_3\}$. (W=Span ($\{v_1,v_2,v_3\}$))
$u_1 = (1, 1, 1)$, $u_2 = (-1, 1, 0)$, $u_3 = (1, 2, 1)$

nonsequitur?

Solution:
We will use the Gram-Schmidt procedure to find the desired basis

$$u_1' = u_1 = (1, 1, 1)$$

$$u_2' = u_2 - \text{proj}_{W1}\, u_2 = u_2 - \frac{\langle u_2, u_1' \rangle}{\|u_1'\|^2} * u_1' = (-1, 1, 0) - \frac{(-1,1,0) \bullet (1,1,1)}{(1,1,1) \bullet (1,1,1)} * (1, 1, 1)$$

$$= (-1, 1, 0) - 0 * (1, 1, 1) = (-1, 1, 0)$$

(Since u_2 and u_1' are already orthogonal)

$$u_3' = u_3 - \text{proj}_{W2}\, u_3 - \text{proj}_{W1}\, u_3 = u_3 - \frac{\langle u_3, u_2' \rangle}{\|u_2'\|^2} * u_2' - \frac{\langle u_3, u_1' \rangle}{\|u_1'\|^2} * u_1'$$

$$= (1, 2, 1) - \frac{(1, 2, 1) \bullet (-1,1,0)}{(-1,1,0) \bullet (-1,1,0)} * (-1, 1, 0) - \frac{(1,2,1) \bullet (1,1,1)}{(1,1,1) \bullet (1,1,1)} * (1, 1, 1)$$

$$= (1, 2, 1) - \frac{1}{2} (-1, 1, 0) - \frac{4}{3} (1, 1, 1) = \left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$$

$$v_1 = \frac{u_1'}{\|u_1'\|} = \frac{(1, 1, 1)}{((1,1,1) \bullet (1,1,1))^{½}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$v_2 = \frac{u_2'}{\|u_2'\|} = \frac{(-1, 1, 0)}{((-1,1,0) \bullet (-1,1,0))^{½}} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{0}{0}\right)$$

$$v_3 = \frac{u_3'}{\|u_3'\|} = \frac{\left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)}{\left(\left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right) \bullet \left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)\right)^{½}} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{3}\right)$$

must have same ratios

not correct

$\{v_1,v_2,v_3\}$ as defined above form an orthonormal basis for the subspace W.

Problem 2:
(40 Points)
Prove that if $S = \{v_1, v_2, \ldots, v_2\}$is an orthonormal basis for an inner product space V, and u is any vector in V, then

$$u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \ldots\ldots + \langle u, v_n\rangle v_n$$

Solution:

Since $S = \{v_1, v_2, \ldots, v_2\}$ is a basis, a vector us can be expresses in the form

$$u = k_1v_1 + k_2v_2 + \ldots + k_nv_n$$

For all i= 1, 2, ..., n we have

$\langle u, v_i\rangle = \langle k_1v_1 + k_2v_2 + \ldots + k_nv_n, v_i\rangle$ because the inner products of both sides with respect to the same vector will remain equal.

$$= k_1\langle v_1, v_i\rangle + k_2\langle v_2, v_i\rangle + \ldots\ldots + k_n\langle v_n, v_i\rangle \text{ by additivity axioms}$$

Since $S = \{v_1, v_2, \ldots, v_2\}$ is an orthonormal set we know that

$$\langle v_i, v_i\rangle = \|v_i\|^{½} = 1 \text{ and } \langle v_j, v_i\rangle = 0 \text{ if } j \neq i.$$

Therefore, the above expression for $\langle u, v_i\rangle = k_i$ — done?

extra?

So $\langle u, v_1\rangle, \langle u, v_2\rangle, \ldots, \langle u, v_n\rangle$ become scalers in the coordinate vecotor u relative to the orthonormal basis $S = \{v_1, v_2, \ldots, v_2\}$, and

$$(u)_s = (\langle u, v_1\rangle, \langle u, v_2\rangle, \ldots, \langle u, v_n\rangle)$$

is the coordinate vector of u relative to this basis. And by definition when

$(u)_s = (k_1, k_2, \ldots, k_n)$, the vector u will be $u = k_1v_1 + k_2v_2 + \ldots + k_nv_n$

So if $(u)_s = (\langle u, v_1\rangle, \langle u, v_2\rangle, \ldots, \langle u, v_n\rangle)$ then

$$u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \ldots\ldots + \langle u, v_n\rangle v_n$$

Finishing our proof.

$$A^T u = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 17 \\ 27 \\ 26 \end{bmatrix}$$

Now we can solve for x_1, x_2, x_3.

$$\begin{bmatrix} 10 & 12 & 12 \\ 12 & 18 & 18 \\ 12 & 18 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17 \\ 27 \\ 26 \end{bmatrix} \qquad x_1 = -\frac{1}{2} \quad x_2 = \frac{7}{3} \quad x_3 = -\frac{1}{2}$$

Now that we have our x-values we will multiply our original matrix A by x to find our b.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{7}{3} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{11}{3} \\ \frac{9}{2} \\ \frac{8}{3} \\ \frac{5}{6} \end{bmatrix}$$

Cool

Therefore the $proj_w v = \left(\frac{11}{3}, \frac{9}{2}, \frac{8}{3}, \frac{5}{6} \right)$

PROBLEM 3
(35 POINTS)

Find the least square solution of the linear system given by:

$$x_1 - 2x_2 = 4$$
$$3x_1 + 7x_2 = 0$$
$$-x_1 - 4x_2 = 1$$

And find the orthogonal projection of b on the column space of A

antecedent?

beginning of solution?

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 7 \\ -1 & -4 \end{bmatrix}, \; b = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 3 & -1 \\ -2 & 7 & -4 \end{bmatrix} * \begin{bmatrix} 1 & -2 \\ 3 & 7 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} 11 & 23 \\ 23 & 69 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} * \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -12 \end{bmatrix}$$

The normal system $A^T Ax = A^T b$ is

$$\begin{bmatrix} 11 & 23 \\ 23 & 69 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -12 \end{bmatrix}$$

Solving this system gives

$$x_1 = \frac{21}{10}$$
$$x_2 = \frac{-201}{230}$$

The orthogonal projection of b onto the column space of A is

$$Ax = \begin{bmatrix} 1 & -2 \\ 3 & 7 \\ -1 & -4 \end{bmatrix} * \begin{bmatrix} \frac{21}{10} \\ \frac{-201}{230} \end{bmatrix} = \begin{bmatrix} \frac{177}{46} \\ \frac{21}{115} \\ \frac{321}{230} \end{bmatrix}$$

4 — 35 *pts*

Given A, *an* $m \times n$ *matrix with row vectors* $r_1, \ldots, r_m$,

prove that $nul(A)$ *and row space of* A *are orthogonal complements in* R^n *with respect to the Euclidean inner product*

***answer*:**

?

Assume: $\exists\, v$ *s.t.* $< r_1, v >= \ldots.. =< r_m, v >= 0 \Rightarrow v$ *orthogonal to* $r_1 \ldots r_m$ *and lies in nullspace of* A

$Ax = 0$ — ideas — not just symbols

$$\begin{bmatrix} < r_1, x > \\ \ldots.. \\ < r_m, x > \end{bmatrix} = \begin{bmatrix} 0 \\ \ldots.. \\ 0 \end{bmatrix}$$

antecedent here? unheralded — have you made a recent conclusion?

Conversely, $v \in nul(A)$ *thus*

$$< r_1, v >= \ldots.. =< r_m, v >= 0$$

$\forall$ *vector* $\boldsymbol{b}$ *in the row space of* A:

$$\boldsymbol{b} = c_1 r_1 + \ldots + c_m r_m$$

thus:

$$< \boldsymbol{b}, v > \; = \; < c_1 r_1 + \ldots + c_m r_m, v >$$

$$= c_1 < r_1, v >= + \ldots.. + c_m < r_m, v >= 0$$

This is not a proof. Proofs involve precise communication of ideas, not just symbols that the reader will hopefully interpret correctly and use to produce their own correct ideas.

4

1. *Prove that* if A *is an* $n \times n$ *matrix* and A *is orthogonal*, then $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for *all* x in R^n.

Solution :

Assume that A is orthogonal, so that $A^T A = I$

$$\|Ax\|^2 = (Ax)\cdot(Ax) = (x\cdot(A^T Ax)) = (x\cdot x) = \|x\|^2$$

2. Compute $< u,\ v >$ using the inner product

— in the following?

$$< u, v > = tr(u^T\ v) = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4,$$

— same?

a.) $u = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, v = \begin{bmatrix} 6 & 7 \\ 8 & -9 \end{bmatrix}$

Solution : $\langle u,\ v\rangle = (1)(6) + (2)(7) + (3)(8) + 4(-9)$

$= 6 + 14 + 24 - 36$

$= 8$

b.) $u = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}, v = \begin{bmatrix} 7 & 2 \\ -1 & -20 \end{bmatrix}$

Solution : $\langle u,\ v\rangle = (3)(7) + (6)(2) + (9)(-1) + 12(-20)$

$= 21 + 12 - 9 - 240$

$= 216$

c.) $u = \begin{bmatrix} 0 & 3 \\ -4 & 12 \end{bmatrix}, v = \begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix}$

Solution : $\langle u,\ v\rangle = (0)(1) + (3)(4) + (-4)(0) + 12(-2)$

$= 12 - 24$

$= -12$

d.) $u = \begin{bmatrix} \sqrt{3} & -\sqrt{2} \\ \frac{10}{3} & 8 \end{bmatrix}$, $v = \begin{bmatrix} 2 & -1 \\ 0 & -3 \end{bmatrix}$

Solution : $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 2\sqrt{3} + \sqrt{2} - 24$

3. Given two orthogonal vectors :

$\boldsymbol{u} = (4, 1, 2, 2)$ and $\boldsymbol{v} = (-2, 4, 1, 1)$ Calculate $\|\boldsymbol{u} + \boldsymbol{v}\|^2$ and $\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$

Solution :

$\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \|(4 + -2, 1 + 4, 2 + 1, 2 + 1)\|^2 = \|(2, 5, 3, 3)\|^2 = (2^2 + 5^2 + 3^2 + 3^2) = 47$

$\|\boldsymbol{u}\|^2 = (4^2 + 1^2 + 2^2 + 2^2) = 25$

$\|\boldsymbol{v}\|^2 = (-2^2 + 4^2 + 1^2 + 1^2) = 22$

$\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 = 25 + 22 = 47$

4. *Let R^2 have the Euclidean inner product. Find the cosine of the angle between* **u** *and* **v**.
Find the cosine of the angle θ *between* **u** *and* **v** *where* $u = (1, -3)$ *and* $v = (2, 4)$.
θ *lies between* 0 *and* π.

Solution :
use the formula $\cos\theta = \dfrac{(\boldsymbol{u} \cdot \boldsymbol{v})}{\|\boldsymbol{u}\| \, \|\boldsymbol{v}\|}$
solve for the norms of **u** *and* **v**.
$\|\boldsymbol{u}\| = \sqrt{1^2 + (-3)^2} = \sqrt{1 + 9} = \sqrt{10}$, $\|\boldsymbol{v}\| = \sqrt{2^2 + 4^2} = \sqrt{4 + 16} = \sqrt{20}$

find the inner product of **u** *and* **v**. $\boldsymbol{u} \cdot \boldsymbol{v} = (2 + (-12)) = -10$

Therefore, $\cos\theta = \dfrac{-10}{\sqrt{10} \cdot \sqrt{20}} = \dfrac{(-1)}{\sqrt{2}} = \dfrac{-\sqrt{2}}{2}$

Question #1: (20 points for (a) and 30 points for (b)) – 50 points total

Suppose $S = \{u_1, u_2, u_3\}$ and $S' = \{v_1, v_2, v_3\}$ are bases in R^3 where

$$u_1 = \begin{bmatrix}1\\2\\1\end{bmatrix} \quad u_2 = \begin{bmatrix}2\\4\\6\end{bmatrix} \quad u_3 = \begin{bmatrix}-1\\3\\5\end{bmatrix} \quad v_1 = \begin{bmatrix}3\\2\\1\end{bmatrix} \quad v_2 = \begin{bmatrix}5\\7\\3\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\4\\2\end{bmatrix}$$

a) Find the transition matrix from S to S'.

b) Compute the coordinate vector $[w]_S$ where

$$w = \begin{bmatrix}0\\15\\10\end{bmatrix}$$

and use $[w]_S$ to compute $[w]_{S'}$.

Solution #1:

a) The transition matrix (P) =

$$\begin{bmatrix}3 & 5 & 1\\2 & 7 & 4\\1 & 3 & 2\end{bmatrix}^{-1} \begin{bmatrix}1 & 2 & -1\\2 & 4 & 3\\1 & 6 & 5\end{bmatrix}$$

$$= \begin{bmatrix}2/5 & 0 & -1/5\\-7/5 & 1 & -4/5\\13/5 & -2 & 11/5\end{bmatrix} \begin{bmatrix}1 & 2 & -1\\2 & 4 & 3\\1 & 6 & 5\end{bmatrix}$$

$$= \begin{bmatrix}1/5 & -2/5 & -7/5\\-1/5 & -18/5 & 2/5\\4/5 & 52/5 & 12/5\end{bmatrix}$$

b) Write w as a linear combination of the vectors in S.

$$w = k_1u_1 + k_2u_2 + k_3u_3$$

$$\begin{bmatrix}0\\15\\10\end{bmatrix} = k_1 \begin{bmatrix}1\\2\\1\end{bmatrix} + k_2 \begin{bmatrix}2\\4\\6\end{bmatrix} + k_3\begin{bmatrix}-1\\3\\5\end{bmatrix}$$

Solve for the coordinates (k_1, k_2, k_3) by writing and row-reducing the augmented matrix.

$$\left[\begin{array}{ccc|c}1 & 2 & -1 & 0\\2 & 4 & 3 & 15\\1 & 6 & 5 & 10\end{array}\right]$$

Row 2 – 2Row 1 in Row 2

Row 3 – Row 1 in Row 1

$$\left[\begin{array}{ccc|c}1 & 2 & -1 & 0\\0 & 0 & 5 & 15\\0 & 4 & 6 & 10\end{array}\right]$$

Switch Row 2 and Row 3

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 4 & 6 & 10 \\ 0 & 0 & 5 & 15 \end{array}\right]$$

(1/4) Row 2 in Row 2
(1/5) Row 3 in Row 3

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 3/2 & 5/2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

By this reduced augmented matrix we have

$k_1 = k_3 - 2k_2 = 7$
$k_2 = (5/2) - (3/2)k_3 = -2$
$k_3 = 3$

So $[w]_S = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$

$[w]_{S'} = P[w]_S$ using the P found in part a)

$$[w]_{S'} = \begin{bmatrix} 1/5 & -2/5 & -7/5 \\ -1/5 & -18/5 & 2/5 \\ 4/5 & 52/5 & 12/5 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/5 & 0 & -1/5 \\ -7/5 & 1 & -4/5 \\ 13/5 & -2 & 11/5 \end{bmatrix} \begin{bmatrix} 0 \\ 15 \\ 10 \end{bmatrix} \text{ (F.Y.I.)}$$

$$= \begin{bmatrix} -2 \\ 7 \\ -8 \end{bmatrix}$$

the following?

Question #2: **(35 points)**

Find the angle θ (from 0 to π) between $\mathbf{p}_1 = 1$ and $\mathbf{p}_2 = x$ in $P_1(0,1)$ with this inner product of p(x) and q(x):

$$\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)dx$$

Solution #2:

$$\cos\theta = \frac{\langle p(x), q(x)\rangle}{\|p(x)\|\|q(x)\|} = \frac{\langle 1, x\rangle}{\|1\|\|x\|} = \frac{\int_0^1 1x\,dx}{\sqrt{\int_0^1 1^2\,dx}\sqrt{\int_0^1 x^2\,dx}} = \frac{\frac{1}{2}}{1\times\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}}{2}$$

$$\cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$$

Question #3: **(40 points)** needs to be column vector

The following equation is inconsistent

[1 0 1] [x1 x2 x3] [3]
[1 2 1] [-3]
[1 2 2] [3]
(**Ax**=**b**)

need = symbol to be an equation

find the least squares solution for this equation (use least squares method)

Solution #3:

a.no — what is this?

[1 0 1 3]
[0 2 0 -6]
[0 0 1 0]

— what is this?

b. — are there parts to this problem? Not indicated as such in problem statement.

A^T=

[1 1 1]
[0 2 2]
[1 1 2]

$A^T\mathbf{b}$=[6]

try using M.S. Word's Equation Editor.

[0]
[6]
$A^T Ax = A^T b$
[3 4 4]=[6]
[4 8 6] [0]
[4 6 6] [6]
Row reduction leads to
[1 0 0 6]
[0 1 0 -3]
[0 0 1 0]

x1=6
X2=-3
X3=0
12/35=10x
x=12/350

24x=32/7

trivial - and done so 3 times.

Question #4: (25 points)

Find the eigenvalues of the following matrices:

$$a)\ A=\begin{bmatrix}5 & 1/2 & -2\\ 0 & -1 & 3\\ 0 & 0 & 2\end{bmatrix}\quad b)\ A=\begin{bmatrix}4 & 0 & 0\\ 3 & 3 & 0\\ -4 & 1 & 1/2\end{bmatrix}\quad c)\ A=\begin{bmatrix}-2 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Solution #4:
By Theorem 7.1.1, if A is an upper triangular, lower triangular, or diagonal matrix, then the eigenvalues of A are the entries on the main diagonal of A.

a) A is an upper triangular matrix. Therefore theorem 7.1.1 applies and the eigenvalues are simply the entries on the main diagonal of A.
Eigenvalues: $\lambda=5, \lambda=-1, \lambda=2$

b) A is a lower triangular matrix. Therefore theorem 7.1.1 applies and the eigenvalues are simply the entries on the main diagonal of A.
Eigenvalues: $\lambda=4, \lambda=3, \lambda=1/2$

c) A is a diagonal matrix. Therefore theorem 7.1.1 applies and the eigenvalues are simply the entries on the main diagonal of A.
Eigenvalues: $\lambda=-2, \lambda=5, \lambda=1$

6

1. Allow R^3 to have the Euclidean inner product. Use the Gram-Schmidt Process to transform the basis $\{u_1, u_2, u_3\}$ into an orthonormal basis $\{q_1, q_2, q_3\}$. First, find the orthogonal basis and then normalize to find the orthonormal basis.
$u_1 = (3, 1, -4)$, $u_2 = (2, 5, 6)$, $u_3 = (1, 4, 8)$

Solution

Let: $v_1 = u_1 = (3, 1, -4)$

$$v_2 = u_2 - proj_{w_1}u_2 = u_2 - \frac{<u_2, v_1>}{\|v_1\|^2}v_1 = (2, 5, 6) + \frac{1}{2}(3, 1, -4)$$

$$= (2, 5, 6) + (\frac{3}{2}, \frac{1}{2}, -2) = (\frac{7}{2}, \frac{11}{2}, 4)$$ — "integerize" by multiplying by 2

$$v_3 = u_3 - proj_{w_2}u_3 = u_3 - \frac{<u_3, v_1>}{\|v_1\|^2}v_1 - \frac{<u_3, v_2>}{\|v_2\|^2}v_2$$

$$= (1, 4, 8) + \frac{25}{26}(3, 1, -4) - \frac{115}{117}(\frac{7}{2}, \frac{11}{2}, 4)$$

$$= (1, 4, 8) + (\frac{75}{26}, \frac{25}{26}, \frac{-50}{13}) - (\frac{805}{234}, \frac{1265}{234}, \frac{460}{117}) = (\frac{4}{9}, \frac{-4}{9}, \frac{2}{9})$$ — multiply by 9

Let: $$q_1 = \frac{v_1}{\|v_1\|} = \frac{(3,1,-4)}{\sqrt{26}} = (\frac{3}{\sqrt{26}}, \frac{1}{\sqrt{26}}, \frac{-4}{\sqrt{26}})$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{(\frac{7}{2}, \frac{11}{2}, 4)}{\frac{117}{2}} = (\frac{7}{117}, \frac{11}{117}, \frac{8}{117})$$

not normalized:
49+121+64 = 234
$117^2 > 234$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{(\frac{4}{9}, \frac{-4}{9}, \frac{2}{9})}{\frac{2}{3}} = (\frac{2}{3}, \frac{-2}{3}, \frac{1}{3})$$

2. Consider the bases $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $B' = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ for $\mathbb{R}^2$ where $\boldsymbol{u}_1 = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$, $\boldsymbol{u}_2 = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$, $\boldsymbol{v}_1 = \begin{vmatrix} 1 \\ 3 \end{vmatrix}$, and $\boldsymbol{v}_2 = \begin{vmatrix} -1 \\ -1 \end{vmatrix}$. (a) Find the transition matrix from B' to B and (b) find the transition matrix from B to B'.

Solution

(a) $\boldsymbol{v}_1 = \boldsymbol{u}_1 + 3\boldsymbol{u}_2$

$\boldsymbol{v}_2 = -\boldsymbol{u}_1 - \boldsymbol{u}_2$

$[\boldsymbol{v}_1]_B = \begin{vmatrix} 1 \\ 3 \end{vmatrix}$ and $[\boldsymbol{v}_2]_B = \begin{vmatrix} -1 \\ -1 \end{vmatrix}$

$$P_{BB'} = \begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix}$$

(b) Using Theorem 6.5.1, $P_{B'B} = P_{BB'}^{-1} = \dfrac{1}{det(P_{BB'})}\begin{vmatrix} -1 & 1 \\ -3 & 1 \end{vmatrix} = \dfrac{1}{2}\begin{vmatrix} -1 & 1 \\ -3 & 1 \end{vmatrix}$

$$= \begin{vmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{-3}{2} & \frac{1}{2} \end{vmatrix}$$

3. Using the inner product $<\boldsymbol{f}, \boldsymbol{g}> = \int_{-1}^{1} f(x)g(x)dx$, find $<\boldsymbol{f}, \boldsymbol{g}>$ for $\boldsymbol{f} = 1 + x - 2x^2 + 3x^3$ and $\boldsymbol{g} = 1 + 4x^3$.

Solution

$$<f, g> = \int_{-1}^{1} f(x)g(x)dx = \int_{-1}^{1} (1 + x - 2x^2 + 3x^3)(1 + 4x^3)dx =$$

$$\int_{-1}^{1} (1 + x - 2x^2 + 7x^3 + 4x^4 - 8x^5 + 12x^6)dx =$$

$$\left.(x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{7x^4}{4} + \frac{4x^5}{5} - \frac{4x^6}{3} + \frac{12x^7}{7})\right|_{-1}^{1} = (-1 + \frac{1}{2} + \frac{7}{4} + \frac{4}{5} + \frac{12}{7}) -$$

$$(-1 + \frac{1}{2} - \frac{2}{3} + \frac{7}{4} - \frac{4}{5} - \frac{12}{7}) = \mathbf{\frac{598}{105}}$$

4. (a) Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$ and (b) find the orthogonal projection of $\mathbf{b}$ onto the column space of A.

$$A = \begin{bmatrix} -2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 6 \\ 5 \\ 3 \\ 2 \end{bmatrix}$$

Solution

(a) $$A^TA = \begin{vmatrix} -2 & 0 & 1 & 2 \\ 1 & 1 & -1 & 2 \\ -1 & 2 & 1 & 0 \end{vmatrix} \begin{vmatrix} -2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 9 & 1 & 3 \\ 1 & 7 & 0 \\ 3 & 0 & 6 \end{vmatrix}$$

$$A^T\mathbf{b} = \begin{vmatrix} -2 & 0 & 1 & 2 \\ 1 & 1 & -1 & 2 \\ -1 & 2 & 1 & 0 \end{vmatrix} \begin{vmatrix} 6 \\ 5 \\ 3 \\ 2 \end{vmatrix} = \begin{vmatrix} -5 \\ 12 \\ 7 \end{vmatrix}$$

$$A^TA\mathbf{x} = A^T\mathbf{b} = \begin{vmatrix} 9 & 1 & 3 \\ 1 & 7 & 0 \\ 3 & 0 & 6 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} -5 \\ 12 \\ 7 \end{vmatrix}$$

$$\mathbf{x} = (A^TA)^{-1}(A^T\mathbf{b}) = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} \frac{14}{103} & \frac{-2}{103} & \frac{-7}{103} \\ \frac{-2}{103} & \frac{15}{103} & \frac{1}{103} \\ \frac{-7}{103} & \frac{1}{103} & \frac{62}{309} \end{vmatrix} \begin{vmatrix} -5 \\ 12 \\ 7 \end{vmatrix} = \begin{vmatrix} \frac{-143}{103} \\ \frac{197}{103} \\ \frac{575}{309} \end{vmatrix}$$

(b) $$A\mathbf{x} = \begin{vmatrix} -2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{vmatrix} \begin{vmatrix} \frac{-143}{103} \\ \frac{197}{103} \\ \frac{575}{309} \end{vmatrix} = \begin{vmatrix} \frac{874}{309} \\ \frac{1741}{309} \\ \frac{-445}{309} \\ \frac{108}{103} \end{vmatrix}$$

is this the "orthogonal projection"? (Communicate in the terms specified in the problem statement.)

PROBLEM #1 40 points Let W be the subspace of R^5 spanned by the following vectors. W1=(2,2,-1,0,1), W2=(-1,-1,2,-3,1), W3=(1,1,-2,0,-1), W4=(0,0,1,1,1). Find a basis for the orthogonal complement of W.

SOLUTION 20 pts for finding the correct matrix that will allow them to use theorem 6.2.6 and 20 pts for finding the correct nullspace

The space spanned by w1, w2, w3, w4 is the same as the row space of the matrix

$$A = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Theorem 6.2.6 says the nullspace of A and the row space of A are orthogonal complements in R^n with respect to the Euclidean inner product.

This means when we row reduce the above matrix we can then find the nullspace and that is the orthogonal compliment for the basis W.

When the above Matrix is row reduced we get the following matrix.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This gives us the following values **x1**=-s-t **x2**=s **x3**=-t **x4**=0 **x5**=t
Which can be written as follows

$$S \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + T \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Which gives us the following basis for the nullspace which will be written in the same form as W1, W2, W3, W4.

V1 = (-1,1,0,0,0)   V2 = (-1,0,-1,0,1)

Which form a basis for the orthogonal complement of W.

PROBLEM #2 40 points Consider the bases B = {u1, u2} and B' = {v1, v2} for R^2, where:

no antecedent

$$u1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad v1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad v2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad W = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

A) Find the transition matrix from B' to B.
B) Find the transition matrix from B to B'.
C) Compute the coordinate vector $[w]_B$ and use it to compute the coordinate vector $[w]_{B'}$.

SOLUTION 10 points for part A and B 20 points for part C.

A) First we need to find the coordinate vectors for the new basis vectors v1 and v2 relative to the old basis B. By Inspection,

$$v1 = u1 + 3u2$$
$$v2 = -u1 - u2 \qquad \text{SO} \qquad [v1]_B = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad [v2]_B = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Thus the transition matrix from B' to B is

$$P = \begin{pmatrix} 1 & -1 \\ 3 & -1 \end{pmatrix}$$

B) There are several ways we can get the transition matrix from B to B' We know that this transition matrix is just the inverse of Matrix P from part A. We can also get the transition matrix by equating corresponding components and solving the linear system. The latter method leads to the following augmented matrix.

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right) \text{ by row reducing you get } \left(\begin{array}{cc|cc} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & -3/2 & 1/2 \end{array}\right)$$

Thus the transition matrix from B to B' is

$$P^{-1} = \begin{pmatrix} -1/2 & 1/2 \\ -3/2 & 1/2 \end{pmatrix}$$

C) By inspection we can see that the coordinate vector $[w]_B$ is:

what does that mean?

$[w]_B = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ This was found by solving the linear system Bx = W Where B was the the base of u1, u2 and W is the 2 x 1 matrix given.

By using the following equation we can find $[w]_{B'}$: $[w]_{B'} = p^{-1} * [w]_B$

$$[w]_{B'} = \begin{pmatrix} -1/2 & 1/2 \\ -3/2 & 1/2 \end{pmatrix} * \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$$

PROBLEM # 3 30 points Let R^3 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis {u1, u2, u3} into an orthonormal basis.

u1 = (1,1,1) **u2 = (-1,1,0)** **u3 = (1,2,1)**

SOLUTION 10 points for each orthonormal basis.

First we want to find the orthogonal basis by following the below mentioned steps. Then to find the orthonormal basis you divide the orthogonal basis by the norm denoted ||v1||, ||v2||, and ||v3||

Step 1) Let **v1 = u1**
Step 2) **v2 = u2**-PROJ$_{w1}$ **u2**
Step 3) **v3 = u3** –PROJ$_{w2}$ **u3**

– what's the point of writing this twice?

Step 1) **v1 = u1 = (1,1,1)**

$\|v1\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

Step 2) $v2 = u2 - (\langle u2, v1\rangle/\|v1\|^2)v1$

$= (-1,1,0) - [\langle(-1,1,0),(1,1,1)\rangle/3](1,1,1)$

$= (-1,1,0) - 0$

$v2 = (-1,1,0)$

$\|v2\| = \sqrt{(-1)^2+1^2} = \sqrt{2}$

Step 3) $v3 = u3 - (\langle u3,v1\rangle/\|v1\|^2)v1 - (\langle u3,v2\rangle/\|v2\|^2)v2$

$= (1,2,1) - [\langle(1,2,1),(1,1,1)\rangle/3](1,1,1) - [\langle(1,2,1),(-1,1,0)\rangle/2](-1,1,0)$

$= (1,2,1) - 4/3(1,1,1) - 1/2(-1,1,0)$

$= (1,2,1) - (4/3, 4/3, 4/3) - (-1/2,1/2,0)$

$v3 = (1/6,1/6,-1/3)$

$\|v3\| = \sqrt{(1/6)^2+(1/6)^2+(1/3)^2} = 1/\sqrt{6}$

So the orthonormal basis for R3 are:

Q1 = v1/||v1|| = (1/√3, 1/√3, 1/√3)
Q2 = v2/||v2|| = (-1/√2, 1/√2, 0)
Q3 = v3/||v3|| = (1√6, 1/√6, -2/√6)

PROBLEM #4 40 points Find a matrix P that diagonalizes a Matrix A

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

SOLUTION 10 points for finding the characteristic equation 10 points for finding the eigenvalues 10 points for finding the bases for the eigenspaces and 10 points for finding the diagonal matrix p

To find the matrix P that diagonalizes the above mentioned matrix A we need to first solve for the characteristic equation. This can be found by performing the following operation det(λI-A) which will give us the following matrix

$$A = \begin{pmatrix} \lambda & 0 & -2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{pmatrix}$$ = λI − A (hopefully)

When you take the determinant of this matrix you will end up with the following characteristic equation

$\lambda^3-5\lambda^2+8\lambda-4$ or in factored form $(\lambda-1)(\lambda-2)2$ — exponent?

This means that the eigenvalues are $\lambda = 1$ and $\lambda = 2$ This means that there are 2 eigenspaces of A.

By definition $x = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ is an eigenvector of A corresponding to λ if and only if x is a non trival solution of **(λI-A)x = 0.**

If $\lambda = 1$ then $\begin{pmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{pmatrix} * \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ if $\lambda = 2$ then $\begin{pmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix} * \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

When we row reduce these two matrices and solve for x we will see that we end up with 3 eigenspace bases namely

when $\lambda = 1$ $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ when $\lambda = 2$ $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ — confusing notation

Matrix A is only diagonalizable if there are three basis vectors, since there are three column vectors in Matrix A; which in our case there is therefore the following matrix P is the "diagonalizable of matrix A" — punctuation? what does that mean?

$$P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

NOTE: There is no preferred order for the columns of P. Since the ith diagonal entry of $P^{-1}AP$ is an egenvalue for the ith column vector of P, changing the order of the columns of P just changes the order of the eigenvalues on the diagonal of $P^{-1}AP$. — good

(2)

1. Let
   $v1 = (0,1,0)$, $v2 = (-4,0,3)$, $v3 = (3, 0, 4)$.

   a. Prove that the set $S = \{v1,v2,v3\}$ is an orthogonal basis for R3 with the Euclidean inner product. (10 pts)

To be an orthogonal basis for the inner product space of R3, all the distinct pairs of vectors in the set are orthogonal, meaning that their inner product must be equal to 0. So…

$\langle v1,v2\rangle = 0(-4) + 1(0) + 0(3) = 0 + 0 + 0 = 0$
$\langle v1,v3\rangle = 0(3) + 1(0) + 0(4) = 0 + 0 + 0 = 0$
$\langle v2,v3\rangle = -4(3) + 0(0) + 3(4) = -12 + 0 + 12 = 0$

   b. Convert the orthogonal set S into an orthonormal set O. (10 pts)

An orthonormal set is one in which each vector has a norm of 1. So, we must find the norms of each vector…

$||v1|| = (0^2 + 1^2 + 0^2)^{(1/2)} = (1)^{(1/2)} = 1$
$||v2|| = (-4^2 + 0^2 + 3^2)^{(1/2)} = (16 + 9)^{(1/2)} = 5$
$||v3|| = (3^2 + 0^2 + 4^2)^{(1/2)} = (9 + 16)^{(1/2)} = 5$

and to normalize any vector u, we know that $u = u/||u||$

so we may say that
$u1 = v1/||v1|| = (0,1,0)/1 = (0/1, 1/1, 0/1) = (0,1,0)$
$u2 = v2/||v2|| = (-4,0,3)/5 = (-4/5,0/5,3/5) = (-4/5,0,3/5)$
$u3 = v3/||v3|| = (3,0,4)/5 = (3/5, 0/5, 4/5) = (3/5,0,4/5)$

and the set $O = \{u1,u2,u3\}$ is the orthonormal set of the orthogonal set S.

   c. Express the vector $t = (2,2,2)$ as a linear combination of the vectors in O and find the coordinate vector $(t)_S$. (10 pts)

We can find t by knowing that in a general vector u expressed by a set $S = \{v1\ldots vn\}$,
$u = \langle u,v1\rangle v1 + \ldots + \langle u,vn\rangle vn$ — should be u's, not v's

So, we find that — right!
$\langle t,u1\rangle = 2(0) + 2(1) + 2(0) = 2$
$\langle t,u2\rangle = 2(-4/5) + 2(0) + 2(3/5) = -2/5$
$\langle t,u3\rangle = 2(3/5) + 2(0) + 2(4/5) = 14/5$

So by the previous theorem,

t = 2u1 -2/5u2 + 14/5v3 or
(2,2,2) = 2(0,1,0) – 2/5(-4/5, 0, 3/5) + 14/5(3/5,0,4/5)

And the coordinate vector relative to O is
(t)s = (<t,u1>,<t,u2>,<t,u3>) = (2,-2/5,14/5)

d. Find the cos of the angle between the vectors t and u1. (10 pts)

We know that
cosø = <t,u1>/(||t||||u1||) with 0<=ø<=π

<t,u1> = 2 from part c.
||t|| = (2^2 + 2^2 + 2^2)^(1/2) = (4 + 4 + 4)^(1/2) = 12^(1/2)
||u1|| = (0^2 + 1^2 + 0^2)^(1/2) = (1)^(1/2) = 1

So the cosø = (2)/(12)^(1/2) = 1/3^(1/2)

#2 (40 *points*)

First, show that the solution to the system $\mathbf{Ax} = \mathbf{b}$ is inconsistent. Then use the normal system associated with $\mathbf{Ax} = \mathbf{b}$ to find the consistent least squares solution of the linear system, and the orthogonal projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

$$\text{Augmented Matrix of } \mathbf{Ax} = \mathbf{b} \Rightarrow \begin{bmatrix} 2 & 1 & -2 & 2 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & -1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & -4 \\ 0 & -1 & 0 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & -2 \end{bmatrix}$$

$\Rightarrow x_2 = 0 = (-2) \Rightarrow$ *inconsistent*

Normal system $\Rightarrow \mathbf{A}^{\mathbf{T}}\mathbf{Ax} = \mathbf{A}^{\mathbf{T}}\mathbf{b}$

$$\mathbf{A}^{\mathbf{T}}\mathbf{Ax} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathbf{A}^{\mathbf{T}}\mathbf{b} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ -7 \end{bmatrix}$$

Augmented Matrix of $\mathbf{A^T A x} = \mathbf{A^T b} \Leftrightarrow \begin{bmatrix} 7 & 4 & -6 & 10 \\ 4 & 3 & -3 & 7 \\ -6 & -3 & 6 & -7 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 7 & 4 & -6 & 10 \\ -2 & 0 & 3 & 0 \\ 0 & -5 & -3 & -9 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 7 & 4 & -6 & 10 \\ 0 & 8 & 9 & 20 \\ 0 & -5 & -3 & -9 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 7 & 4 & -6 & 10 \\ 0 & 8 & 9 & 20 \\ 0 & -5 & -3 & -9 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & 4 & -6 & 10 \\ 0 & 8 & 9 & 20 \\ 0 & -7 & 0 & -7 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & 4 & -6 & 10 \\ 0 & 8 & 9 & 20 \\ 0 & 1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & 4 & -6 & 10 \\ 0 & 0 & 9 & 12 \\ 0 & 1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & 4 & -6 & 10 \\ 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 7 & 0 & 0 & 14 \\ 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{4}{3} \end{bmatrix} \Rightarrow \boldsymbol{x_1 = 2, x_2 = 1, x_3 = \frac{4}{3}}$$

$$\textbf{Proj b} = \mathbf{Ax} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{2}{3} \\ 3 \\ \frac{5}{3} \end{bmatrix}$$

3. (40 points)
Consider the bases B = {p1, p2} and B* = {q1, q2} for P1.

p1 = 6+3x   p2 = 10+2x
q1 = 2   q2 = 3+2x

(A) Symbolically "show the matrix from B* to B" } – difference?

(B) Find the transition matrix from B* to B.

(C) Symbolically show the matrix from B to B*

(D) Find the transition matrix from B to B*

not a sentence
- information free
subscript? $(P)_B \neq [P]B$

SOLUTION:

(A)

There is a matrix [B] which multiplied by [p]B which is the coordinate vector of p relative to the basis B. It is equal to [B*] multiplied by [p]B* (the coordinate vector of p relative to B*) so [B][p]B= [B*][p]B*. So to relate [p]B to [p]B* we must multiply on the lift by the inverse of [B]. Giving, [p]B = [B]^-1[B*][p]B* The matrix is [B]^-1[B*]

(B) First we must find the inverse of [B]. "Convert p1 and p2 into column vectors."

explain

[B] = [6 10]
[3 2]

Using Row operations, [B]^-1 is found to be [-1/9 5/9]
[1/6 -1/3]

Then multiply by [B*] using q1 and q2 and column vectors. [B*] = [2 3]
[0 2]

[B]^-1[B*] = [-2/9 7/9]
[1/3 -1/6]

(C) The matrix that converts [p]B to [p]B* can be found as described in part A.

The matrix is [B*]^-1[B].

(D) First find [B]^-1. Use q1 and q2 and the column vectors in [B*]. [B*] = [2 3]
[0 2]

Using row operations [B*]^-1 is found to be [½ -¾]
[0 ½]

This can then be multiplied by [B] = [6 10]
[3 2]

[B*]^-1[B] = [¾ 14/4]
[3/2 1]

4. i) Find the eigenvalues of matrix A (6 points each)

a) $A = \begin{bmatrix} -1 & 6 \\ 0 & 5 \end{bmatrix}$

b) $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

ii) Find bases for the eigenspaces with given eigenvalues. (10 points each)

a) $A = \begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}$ $\lambda = -8$

b) $A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$ $\lambda = 2$

(Total of up to 38 points)

Solution:

i) Theorem 7.1.1 states that if A is an nxn triangular matrix, then the eigenvalues of A are the entries on the main diagonal of A. So, by inspection,

a) $\lambda = -1$, $\lambda = 5$
b) $\lambda = 3$, $\lambda = 7$, $\lambda = 1$
c) $\lambda = -\frac{1}{3}$, $\lambda = 1$, $\lambda = \frac{1}{2}$

ii) By definition, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A corresponding to λ iff $\mathbf{x}$ is a nontrivial solution of $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

a) $(\lambda I - A)\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} \lambda+2 & 0 & -1 \\ 6 & \lambda+2 & 0 \\ -19 & -5 & \lambda+4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ if $\lambda = -8$,

$$\begin{bmatrix} -6 & 0 & -1 \\ 6 & -6 & 0 \\ -19 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this by Gaussian elimination yields

$x_1 = -1/6$

$x_2 = -1/6$

$x_3 = 1$

How about $\begin{bmatrix} -1 \\ -1 \\ 6 \end{bmatrix}$?

Thus the eigenvectors of A corresponding to $\lambda = -8$ are the nonzero vectors of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/6 \\ -1/6 \\ 1 \end{bmatrix}$. Since $\begin{bmatrix} -1/6 \\ -1/6 \\ 1 \end{bmatrix}$ is linearly independent, it forms a basis for the eigenspace corresponding to $\lambda = -8$.

b) $(\lambda I - A)\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} \lambda+1 & 0 & -1 \\ 1 & \lambda-3 & 0 \\ 4 & -13 & \lambda+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{if } \lambda = 2,$$

$$\begin{bmatrix} 3 & 0 & -1 \\ 1 & -1 & 0 \\ 4 & -13 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this by Gaussian elimination yields

$x_1 = 1/3$

$x_2 = 1/3$

$x_3 = 1$

How about $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$?

Thus the eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix}$. Since $\begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix}$ is linearly independent, it forms a basis for the eigenspace corresponding to $\lambda = 2$.

8

1. (25 points)

Find a basis of the orthogonal complement of the column space of A, where A is

$$A= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Solution:

The orthogonal complement of cola is also the nullspace of A^T.

$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

By Gauss-Jordan elimination the reduced row echelon form is,

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

By defining my leading variables in terms of the free variables, I get a basis of nul(A^T):

$$\begin{aligned} x_1 &= s \\ x_2 &= -2s \\ x_3 &= s \end{aligned} \longrightarrow \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} s$$

Since the orthogonal complement of col(A) is the same as nul(A^T) a basis for the orthogonal complement is,

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$ or (1, -2, 1)

2. (50 points)

Find the least squares solution of the linear system Ax=b given by

x1 -x2 =4

3x1+2x2 =1

-2x1+4x2 =3

And the orthogonal projection of b on the column space of A.

antecedent?

Solution

- Notice that A has linearly independent column vectors, so we know in advance that there is a unique least squares solution.

$$A=\begin{bmatrix}1 & -1\\ 3 & 2\\ -2 & 4\end{bmatrix} \qquad b=\begin{bmatrix}4\\1\\3\end{bmatrix}$$

$$A^T A=\begin{bmatrix}1 & 3 & -2\\ -1 & 2 & 4\end{bmatrix}\begin{bmatrix}1 & -1\\ 3 & 2\\ -2 & 4\end{bmatrix}=\begin{bmatrix}14 & -3\\ -3 & 21\end{bmatrix} \qquad A^T b=\begin{bmatrix}1 & 3 & -2\\ -1 & 2 & 4\end{bmatrix}\begin{bmatrix}4\\1\\3\end{bmatrix}=\begin{bmatrix}1\\10\end{bmatrix}$$

- The normal system of $A^T Ax = A^T b$ is

$$\begin{bmatrix}14 & -3\\ -3 & 21\end{bmatrix}\begin{bmatrix}x1\\x2\end{bmatrix}=\begin{bmatrix}1\\10\end{bmatrix}$$

- To solving this system, start with putting into a matrix and then row reducing

$$\begin{bmatrix}14 & -3 & 1\\ -3 & 21 & 10\end{bmatrix} \begin{matrix}R1/14\\ R2/3\end{matrix} \begin{bmatrix}1 & -3/14 & 1/14\\ -1 & 7 & 10/3\end{bmatrix}$$

$$\overset{R1+R2\rightarrow R2}{\begin{bmatrix}1 & -3/14 & 1/14\\ 0 & 95/14 & 143/42\end{bmatrix}} \quad \overset{R2*14/95}{\begin{bmatrix}1 & -3/14 & 1/14\\ 0 & 1 & 143/285\end{bmatrix}}$$

- Thus yields the least squares solution

$$x2=\frac{143}{285} \qquad x1-\frac{3}{14}*x2=1/14 \qquad x1=\frac{17}{95}$$

- The orthogonal projection of **b** on the column space of A is

$$Ax=\begin{bmatrix}1 & -1\\ 3 & 2\\ -2 & 4\end{bmatrix}\begin{bmatrix}\frac{17}{95}\\ \frac{143}{285}\end{bmatrix}=\begin{bmatrix}-92/285\\ 439/285\\ 94/57\end{bmatrix}$$

perfect

3. (50 points)

Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 2 & -7 \\ 3 & 1 & 0 \\ -5 & 0 & 0 \end{bmatrix}$$

Solution:

To find the eigenvalues, we find the solutions to the characteristic polynomial, $\det(\lambda I - A)$.

So the characteristic polynomial of A is

$$\det(\lambda I - A) = \det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 2 & -7 \\ 3 & 1 & 0 \\ -5 & 0 & 0 \end{bmatrix} \right) = \det \begin{bmatrix} \lambda & -2 & 7 \\ -3 & \lambda - 1 & 0 \\ 5 & 0 & \lambda \end{bmatrix}$$

We take the determinant of this by cross multiplying:

$$\det \begin{bmatrix} \lambda & -2 & 7 \\ -3 & \lambda - 1 & 0 \\ 5 & 0 & \lambda \end{bmatrix} \begin{matrix} \lambda & -2 \\ -3 & \lambda - 1 \\ 5 & 0 \end{matrix} = \lambda(\lambda-1)(\lambda) + (-2)(0)(5) + (7)(-3)(0) - (7)(\lambda-1)(5) -$$

$$(\lambda)(0)(0) - (-2)(-3)(\lambda) = \lambda^3 - \lambda^2 + 0 + 0 - 35\lambda + 35 - 0 - 6\lambda = \lambda^3 - \lambda^2 - 41\lambda + 35$$

To find the solution, and therefore the eigenvalues, we solve for:

$$\lambda^3 - \lambda^2 - 41\lambda + 35 = 0$$

Solving this on a calculator, we find the solutions to be:

$$\lambda = -6.33896,\ \lambda = .851027,\ \lambda = 6.48794$$

These are the eigenvalues of A.

(Try "reverse engineering" to get whatever eigenvalues you want: compute A as $P \Lambda P^{-1}$ for your favorite diagonal Λ and invertible P.)

4. (25 points)

Determine whether the following matrix is orthogonal

$$A=\begin{bmatrix} 1 & 0 & 5 \\ 2 & 3 & 1 \\ 0 & -2 & 4 \end{bmatrix}$$

Solution:

We know this matrix to be orthogonal if $A^{-1} = A^T$. So first we write down what we know A^T to be:

$$A^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \\ 5 & 1 & 4 \end{bmatrix}$$

very inefficient – just check columns are orthonormal, equivalently see whether $A^T A = I$.

Now we will solve for A^{-1} by row reducing and performing the same operation on the identity matrix of A. After doing this we get A^{-1} to be:

$$A^{-1} = \begin{bmatrix} 3 & -2 & 2 \\ -2 & 2 & -1 \\ 2 & -1 & 5/3 \end{bmatrix}$$

We can see that $A^{-1} \neq A^T$. Therefore, this matrix is not orthogonal.

10

QUESTION #1:

Let $\mathbf{u}=(u_1,u_2)$ and $\mathbf{v}=(v_1,v_2)$ and let $\mathbf{u}$, $\mathbf{v}$ be contained in $\mathbf{R}^2$. Let

$$\langle \mathbf{u},\mathbf{v}\rangle = u_1v_1 - u_2v_2.$$

Prove using the **Four Inner Product Space Axioms** whether this is an inner product or not.

Good

SOLUTION:

Axiom 1: Does $\langle \mathbf{u},\mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle$?

$\langle \mathbf{v}, \mathbf{u}\rangle$
$= v_1u_1 - v_2u_2$
$= u_1v_1 - u_2v_2$
$= \langle \mathbf{u},\mathbf{v}\rangle$

Yes.

Axiom 2: Does $\langle \mathbf{u}+\mathbf{v},\mathbf{w}\rangle = \langle \mathbf{u},\mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$?

$\langle \mathbf{u}+\mathbf{v},\mathbf{w}\rangle$
$= (u_1+v_1)w_1 + (u_2+v_2)w_2$
$= u_1w_1 + v_1w_1 - u_2w_2 - v_2w_2$
$= (u_1w_1 - u_2w_2) + (v_1w_1 - v_2w_2)$
$= \langle \mathbf{u},\mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$

Yes.

Axiom 3: Does $\langle k\mathbf{u},\mathbf{v}\rangle = k\langle \mathbf{u},\mathbf{v}\rangle$?

$\langle k\mathbf{u},\mathbf{v}\rangle$
$= (ku_1)v_1 - (ku_2)v_2$
$= ku_1v_1 - ku_2v_2$
$= k(u_1v_1 - u_2v_2)$
$= k\langle \mathbf{u},\mathbf{v}\rangle$

Yes.

Axiom 4: Is $\langle \mathbf{v},\mathbf{v}\rangle \geq 0$ and does $\langle \mathbf{v},\mathbf{v}\rangle = 0$ if and only if $\mathbf{v}=\mathbf{0}$?

$\langle \mathbf{v},\mathbf{v}\rangle$
$= v_1v_1 - v_2v_2$
$= (v_1)^2 - (v_2)^2$

Fails, because $(v_1)^2 - (v_2)^2 < 0$, when $|v_2|>|v_1|$. It also fails, because when $(v_1)^2 - (v_2)^2 = 0$, $v_1 = v_2 = 0$ is not necessarily true.

So, $\langle \mathbf{u},\mathbf{v}\rangle = u_1v_1 - u_2v_2$ is not an inner product on $\mathbf{R}^2$.

Q, Consider the bases $B = \{u_{1,}u_2\}$ $and\_B' = \{v_1, v_2\}\_forR^2$ where

$u_1 = (1,0) : u_2 = (0,1) : v_1 = (1,1) : v_2 = (2,1)$

a) Find the transition matrix from B' to B.
b) Use $[v]_B = P[v]_{B'}$ to find $[v]_B$ if

$$[v]_{B'} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Solution

Part (a) =>First we must find the coordinate vectors for the new basis vectors $v_1\_and\_v_2$ relative to the old basis B, by inspection,

$$v_1 = u_1 + u_2$$
$$v_2 = 2u_1 + u_2$$

so

$$[v_1]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\_and\_[v_2]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus the transition matrix from B' to B is

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Part (b) =>Using $[v]_B = P[v]_{B'}$ and the transition matrix in part (a) yields

$$[v]_B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

As check, we should be able to recover the vector v either from $[v]_B\_or\_[v]_{B'}$.

So

$$-3v_1 + 5v_2 = 7u_1 + 2u_2 = (7,2)$$

3. Let $A = \begin{pmatrix} -2 & 4 \\ 1 & 1 \\ 2 & -1 \end{pmatrix}$.

(a) (10 points) Use the Gram-Schmidt algorithm to find an orthonormal basis $\mathbf{q}_1, \mathbf{q}_2$ for the column space of A.

Solution: To find $\mathbf{q}_1$ we need only normalize the length of the first column $\mathbf{a}_1$:

$$\mathbf{q}_1 = \frac{\mathbf{A}_1}{\|\mathbf{A}_1\|} = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

To find a vector $\mathbf{A}_2$ that is orthogonal to $\mathbf{q}_1$, we subtract the projection of $\mathbf{a}_2$ onto $\mathbf{q}_1$ from $\mathbf{a}_2$:

$$\mathbf{A}_2 = \mathbf{a}_2 - (\mathbf{q}_1^T\mathbf{a}_2)\mathbf{q}_1 = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{9}(-8+1-2)\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

Finally we normalize the length of $\mathbf{A}_2$ to obtain $\mathbf{q}_2$:

$$\mathbf{q}_2 = \frac{\mathbf{A}_2}{\|\mathbf{A}_2\|} = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

(b) (10 points) There is no solution to $A\mathbf{x} = \mathbf{b}$ when $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Find the best approximate solution $\hat{\mathbf{x}}$, which minimizes $\|A\hat{\mathbf{x}} - \mathbf{b}\|$.

Good

Solution: We need to solve the equation $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$. We compute

$$A^TA = \begin{pmatrix} 9 & -9 \\ -9 & 18 \end{pmatrix} \quad A^T\mathbf{b} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

This system can be solved in several ways. For instance, we can apply the formula for the inverse of a 2×2 matrix to obtain:

$$\hat{\mathbf{x}} = (A^TA)^{-1}\mathbf{b} = \begin{pmatrix} 9 & -9 \\ -9 & 18 \end{pmatrix}^{-1}\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2}{9} \end{pmatrix}$$

Problem #4:

Show that if A is orthogonal, then A^T is orthogonal as well.

Solution:

Given: A is orthogonal, which means $AA^T = A^TA = I$.

Prove: A^T is orthogonal.

If A^T is orthogonal, then by definition, these two equations, $(A^T)(A^T)^T = I$ and $(A^T)^T(A^T) = I$, must be true. If we can prove they are, then A^T is orthogonal.

Rewriting these equations:

$$(A^T)^T(A^T) = I$$
$$(A)(A^T) = I$$
$$AA^T = I$$

and

$$(A^T)(A^T)^T = I$$
$$(A^T)(A) = I$$
$$A^TA = I$$

We already know both of these are true, because it was given that the original matrix A is orthogonal. Therefore, A^T is also orthogonal.

II

Problem (40 points):

Suppose $A\underline{x} = \underline{b}$ is inconsistent. Form $\min_{x\in\Re} \|A\underline{x} - \underline{b}\|_E$ using the concept of a gradient.

Solution:

Define the function $f : \Re^n \to \Re$ as $f(\underline{x}) = \|A\underline{x} - \underline{b}\|^2$
$= (A\underline{x} - \underline{b}) \cdot (A\underline{x} - \underline{b}) = (A\underline{x} - \underline{b})^T (A\underline{x} - \underline{b}) = (\underline{x}^T A^T - \underline{b}^T)(A\underline{x} - \underline{b}) = \underline{x}^T A^T A\underline{x} - \underline{x}^T A^T \underline{b} - \underline{b}^T A\underline{x} + \underline{b}^T \underline{b}$
And because $\underline{x}^T A^T \underline{b}$ and $\underline{b}^T A\underline{x}$ are both one by one matrices and are each the transpose of the other $\underline{x}^T A^T \underline{b} = \underline{b}^T A\underline{x}$. So the above function can be written as
$f(\underline{x}) = \underline{x}^T A^T A\underline{x} - 2\underline{x}^T A^T \underline{b} + \underline{b}^T \underline{b}$

Cool

We now take the gradient of *f*, written $D_{\underline{x}} f(\underline{x})$; using the fact that $D_{\underline{x}} f(\underline{x}) \cdot \underline{u} = \left.\frac{df(\underline{x} + \varepsilon \underline{u})}{d\varepsilon}\right|_{\varepsilon=0}$
The function *f* evaluated at $\underline{x} + \varepsilon \underline{u}$ can be written as
$f(\underline{x} + \varepsilon \underline{u}) = (\underline{x}^T + \varepsilon \underline{u}^T) A^T A(\underline{x} + \varepsilon \underline{u}) - 2(\underline{x}^T + \varepsilon \underline{u}^T) A^T \underline{b} + \underline{b}^T \underline{b}$
So
$D_{\underline{x}} f(\underline{x}) \cdot \underline{u} = \underline{u}^T A^T A\underline{x} + \underline{x}^T A^T A\underline{u} - 2\underline{u}^T A^T \underline{b}$
And because $\underline{u}^T A^T A\underline{x}$ and $\underline{x}^T A^T A\underline{u}$ are both one by one matrices and are each the transpose of the other $\underline{u}^T A^T A\underline{x} = \underline{x}^T A^T A\underline{u}$. So the inner product of $\underline{u}$ and the gradient of *f* can be written as
$D_{\underline{x}} f(\underline{x}) \cdot \underline{u} = 2\underline{u}^T A^T A\underline{x} - 2\underline{u}^T A^T \underline{b} = 2\underline{u}^T (A^T A\underline{x} - A^T \underline{b})$
Which means that $D_{\underline{x}} f(\underline{x}) = 2(A^T A\underline{x} - A^T \underline{b})$.
Now, to form the $\min_{x\in\Re} \|A\underline{x} - \underline{b}\|_E$ all we need to do is set the gradient equal to zero and solve, which yields the "Normal Equations":

$$A^T A\underline{x} = A^T \underline{b}$$

Problem (35 points):

p= $3 + 4x + 5x^2 - 6x^3$ and **q** = $2+4x+6x^2+x^3$ really?
are two vectors in P_3, the following formula defines an inner product on P_3:

for **p**= $a0 + a1x + a2x^2 - a3x^3$ and **q** = $b0 + b1x + b2x^2 + b3x^3$
$\langle \mathbf{p},\mathbf{q} \rangle = a0b0 + a1b1 + a2b2 + a3b3$

Please calculate $\langle \mathbf{p},\mathbf{q} \rangle$ and $\|\mathbf{p}\|$ relative to the above inner product

Solution:

⟨**p**,**q**⟩ = 3(2) + 4(4) + 5(6) + (-6)(1) = 46

||**p**|| = ⟨**p**,**p**⟩^1/2 = sqrt(3² + 4² + 5² + (-6)²) = sqrt(86)

Problem (35 points):

Determine if Matrix A is an Orthogonal Matrix.

The Matrix A is:

$$\begin{bmatrix} -2 & -4 & 1 \\ 0 & -2 & 4 \\ -4 & 1 & 2 \end{bmatrix}$$

Solution:

If the Matrix is orthogonal, then:

1) $\det(A) = \pm 1$, (Its determinant is 1 or -1).

2) $A^T = A^{-1}$ (Its inverse is equal to its transpose).

not ~~necessary~~ sufficient, but certainly necessary

Method 1 – Is its determinant 1 or -1?

$$\det\begin{bmatrix} -2 & -4 & 1 \\ 0 & -2 & 4 \\ -4 & 1 & 2 \end{bmatrix}$$

= (-2)(-2)(2) + (-4)(4)(-4) + (1)(0)(1) – (-2)(4)(1) – (-4)(0)(2) – (1)(-2)(-4)

= (8) + (64) + (0) – (-8) – (0) – (8)

= 72

Since $\det\begin{bmatrix} -2 & -4 & 1 \\ 0 & -2 & 4 \\ -4 & 1 & 2 \end{bmatrix} \neq$ 1 or -1, it is not an orthogonal matrix.

good (what if it was = 1?)

Method 2 - Is its inverse equal to its transpose?

Inverse:

$$\begin{bmatrix}-2 & -4 & 1\\ 0 & -2 & 4\\ -4 & 1 & 2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\xrightarrow{R_3-2R_1}\begin{bmatrix}-2 & -4 & 1\\ 0 & -2 & 4\\ 0 & 9 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}\xrightarrow{R_3\leftrightarrow R_2}$$

$$\begin{bmatrix}-2 & -4 & 1\\ 0 & 9 & 0\\ 0 & -2 & 4\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\xrightarrow{R_3+\frac{2}{9}R_2}\begin{bmatrix}-2 & -4 & 1\\ 0 & 9 & 0\\ 0 & 0 & 4\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 0 & 1\\ -\frac{4}{9} & 1 & \frac{2}{9}\end{bmatrix}\xrightarrow{\frac{-1}{2}R_1,\frac{1}{9}R_2,\frac{1}{4}R_3}$$

$$\begin{bmatrix}1 & 2 & \frac{-1}{2}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-\frac{1}{2} & 0 & 0\\ -\frac{2}{9} & 0 & \frac{1}{9}\\ -\frac{1}{9} & \frac{1}{4} & \frac{1}{18}\end{bmatrix}\xrightarrow{R_1-2R_2+\frac{1}{2}R_3}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-\frac{1}{9} & -\frac{1}{8} & -\frac{7}{3}\\ -\frac{2}{9} & 0 & \frac{1}{9}\\ -\frac{1}{9} & \frac{1}{4} & \frac{1}{18}\end{bmatrix}$$

The inverse of Matrix A: $\begin{bmatrix}-2 & -4 & 1\\ 0 & -2 & 4\\ -4 & 1 & 2\end{bmatrix}^{-1}=\begin{bmatrix}-\frac{1}{9} & -\frac{1}{8} & -\frac{7}{3}\\ -\frac{2}{9} & 0 & \frac{1}{9}\\ -\frac{1}{9} & \frac{1}{4} & \frac{1}{18}\end{bmatrix}$

The transpose of Matrix A: $\begin{bmatrix}-2 & -4 & 1\\ 0 & -2 & 4\\ -4 & 1 & 2\end{bmatrix}^{T}=\begin{bmatrix}-2 & 0 & -4\\ -4 & -2 & 1\\ 1 & 4 & 2\end{bmatrix}$

Since $\begin{bmatrix}-2 & 0 & -4\\ -4 & -2 & 1\\ 1 & 4 & 2\end{bmatrix}\neq\begin{bmatrix}-\frac{1}{9} & -\frac{1}{8} & -\frac{7}{3}\\ -\frac{2}{9} & 0 & \frac{1}{9}\\ -\frac{1}{9} & \frac{1}{4} & \frac{1}{18}\end{bmatrix}$ ($A^T \neq A^{-1}$), Matrix A is NOT orthogonal.

very inefficient — just see whether $A^TA=I$ or, better, see if A's columns (or rows) are orthonormal.

Problem (40 points):

Prove that for all orthogonal bases $S = \{\underline{v}_1, \ldots, \underline{v}_n\}$ of a vector space V the coordinate vector for any vector $\underline{u} \in V$ with respect to the basis S can be expressed as

$$\left(\frac{\langle \underline{u}, \underline{v}_1 \rangle}{\|\underline{v}_1\|^2}, \ldots, \frac{\langle \underline{u}, \underline{v}_n \rangle}{\|\underline{v}_n\|^2} \right) \in \cancel{X}\ \mathbb{R}^n.$$

Solution:

Because S is a basis for V, we know that for every vector $\underline{u} \in V$ there exists a coordinate vector $(C_1, \ldots, C_n) \in \cancel{K}\ \mathbb{R}^n$ such that $\underline{u} = C_1\underline{v}_1 + \ldots + C_n\underline{v}_n$.

Take the inner product of $\underline{u} = C_1\underline{v}_1 + \ldots + C_n\underline{v}_n$ with $\underline{v}_j$

$\langle \underline{u}, \underline{v}_j \rangle = \langle C_1\underline{v}_1 + \ldots + C_j\underline{v}_j + \ldots + C_n\underline{v}_n, \underline{v}_j \rangle$

Which, by the properties of inner products, is equal to:

$C_1\langle \underline{v}_1, \underline{v}_j \rangle + \ldots + C_j\langle \underline{v}_j, \underline{v}_j \rangle + \ldots + C_n\langle \underline{v}_n, \underline{v}_j \rangle$

But because S is an orthogonal basis, the inner product of $\underline{v}_j$ with any other vector in S besides itself is zero (By the definition of orthogonal). So the following is true:

$\langle \underline{u}, \underline{v}_j \rangle = C_j\langle \underline{v}_j, \underline{v}_j \rangle = C_j\|\underline{v}_j\|^2$

And solving for C_j we get $C_j = \dfrac{\langle \underline{u}, \underline{v}_j \rangle}{\|\underline{v}_j\|^2}$

Question:

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Determine which of the following are inner products on R^3. For those that are not, list the axioms that do not hold.

a) <**u**, **v**> = $u_1v_1 + u_3v_3$
b) <**u**, **v**> = $u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2$
c) <**u**, **v**> = $2u_1v_1 + u_2v_2 + 4u_3v_3$

Solution:

The four axioms are

1) <**u**, **v**> = <**v**, **u**>
2) <**u** + **v**, **z**> = <**u**, **z**> + <**v**, **z**>
3) <k**u**, **v**> = k<**u**, **v**>
4) <**v**, **v**> ≥ 0
   And <**v**, **v**> = 0 iff **v=0**

a)
-Axiom 1 does not fail because $u_1v_1 + u_3v_3 = v_1u_1 + v_3u_3$
-Axiom 2 does not fail because $(u_1 + v_1)z_1 + (u_3 + v_3)z_3 = u_1z_1 + v_1z_1 + u_3z_3 + v_3z_3$
-Axiom 3 does not fail because $(ku_1)v_1 + (ku_3)v_3 = ku_1v_1 + ku_3v_3 = k(u_1v_1 + u_3v_3)$
-Axiom 4 fails because although <**v**, **v**> is always greater than or equal to zero, when it equals zero, **v** does not necessarily equal the zero vector, because v_2 can be any value.

Therefore, because Axiom 4 fails, this is <u>not</u> an inner product.

b)
-Axiom 1 does not fail because $u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 = v_1^2u_1^2 + v_2^2u_2^2 + v_3^2u_3^2$
-Axiom 2 fails because
$(u_1 + v_1)^2 z_1^2 + (u_2 + v_2)^2 z_2^2 + (u_3 + v_3)^2 z_3^2 \neq u_1^2z_1^2 + v_1^2z_1^2 + u_2^2z_2^2 + v_2^2z_2^2 + u_3^2z_3^2 + v_3^2z_3^2$
-Axiom 3 fails because $(ku_1)^2v_1^2 + (ku_2)^2v_2^2 + (ku_3)^2v_3^2 \neq ku_1^2v_1^2 + ku_2^2v_2^2 + ku_3^2v_3^2$
-Axiom 4 does not fail because <**v**, **v**> is always greater than or equal to zero, and when it equals zero, **v=0**.

Therefore, because Axioms 2 and 3 fail, this is not an inner product.

c)

-Axiom 1 does not fail because $2u_1v_1 + u_2v_2 + 4u_3v_3 = 2v_1u_1 + v_2u_2 + 4v_3u_3$

-Axiom 2 does not fail because

$2(u_1 + v_1)z_1 + (u_2 + v_2)z_2 + 4(u_3 + v_3)z_3 = 2u_1z_1 + 2v_1z_1 + u_2z_2 + v_2z_2 + 4u_3z_3 + 4v_3z_3$

-Axiom 3 does not fail because

$2(ku_1)v_1 + (ku_2)v_2 + 4(ku_3)v_3 = 2k(u_1v_1) + k(u_2v_2) + 4k(u_3v_3)$

-Axiom 4 does not fail because <**v**, **v**> is always greater than or equal to zero, and when it equals zero, **v**=**0**.

Therefore, this is an inner product.

Question: : Recall Theorem 6.2.4 (Generalized Theorem of Pythagoras) which states:
If **u** *and* **v** *are orthogonal vectors in an inner product space, then*

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are pairwise orthogonal vectors in an inner product space V, prove the following:

$$\|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_r\|^2 = \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + \ldots + \|\mathbf{v}_r\|^2.$$

Solution: The orthogonality of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ implies that $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ where $i \neq j$, so

$\|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_r\|^2 = \langle \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_r, \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_r \rangle$

$= \mathbf{v}_1(\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_r) + \mathbf{v}_2(\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_r) + \cdots + \mathbf{v}_r(\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_r)$

$= [\mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_1 \cdot \mathbf{v}_2 + \ldots + \mathbf{v}_1 \cdot \mathbf{v}_r] + [\mathbf{v}_2 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_2 + \ldots + \mathbf{v}_2 \cdot \mathbf{v}_r] + \ldots + [\mathbf{v}_r \cdot \mathbf{v}_1 + \mathbf{v}_r \cdot \mathbf{v}_2 + \ldots + \mathbf{v}_r \cdot \mathbf{v}_r]$

$= \mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_2 + \ldots + \mathbf{v}_r \cdot \mathbf{v}_r$ — this is $\langle v_1, v_r \rangle$?

$= \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + \ldots + \|\mathbf{v}_r\|^2.$

Question:

Consider the bases B = {**u1**, **u2**} and B' = {**v1**, **v2**} in R² where

u1 = [2]	**u2 = [4]**	**v1 = [1]**	**v2 = [-1]**
[2]	**[-1]**	**[3]**	**[-1]**

Find the transition matrix from B to B'.

Solution:

Theorem 6.5.1 states that *If P is the transition matrix from a basis B' to a basis B for a finite-dimensional vector space V, then P is invertible, and P^-1 is the transition matrix from B to B'.* We can use this theorem to find the transition matrix from B to B'.

why use it?

First, we must find the coordinate vectors for the new basis vectors **u1** and **u2** relative to the old basis B'.

u1 = k1v1 + k2v2
u2 = k3v1 + k4v2

k1 (1,3) + k2 (-1,-1) = (2,2)

[1 -1 2] (Augmented matrix)
[3 -1 2]

k1 = 0
k2 = -2

k3 (1,3) + k4 (-1,-1) = (4,-1)

k3 = -5/2
k4 = -13/2

The solution to the coordinate vector equations yields P the transition matrix..

$$P = \begin{bmatrix} k1 & k3 \\ k2 & k4 \end{bmatrix} = \begin{bmatrix} 0 & -5/2 \\ -2 & -13/2 \end{bmatrix}$$

You didn't use theorem — but that's o.k. - it doesn't pertain

Question:

Let R^4 have the Euclidean inner product, and let W be the subspace spanned by the orthogonal vectors **v1**=(4,3,2,1), **v2**=(1,-2,3,-4), and **v3**=(-3,4,1,-2). Find the orthogonal projection of the vector **u**=(5,7,-1,6) onto W and the component of **u** orthogonal to W.

Solution:
Orthogonal projection of **u** onto W:

$$\text{Proj}_{W}\mathbf{u} = \frac{\langle \mathbf{u},\mathbf{v1}\rangle \mathbf{v1}}{\|\mathbf{v1}\|} + \frac{\langle \mathbf{u},\mathbf{v2}\rangle \mathbf{v2}}{\|\mathbf{v2}\|} + \frac{\langle \mathbf{u},\mathbf{v3}\rangle \mathbf{v3}}{\|\mathbf{v3}\|}$$

$$= \frac{(4*5+3*7+2*-1+6*1)(4,3,2,1)}{4^2+3^2+2^2+1^2} + \frac{(5*1+7*-2+-1*3+6*-4)(1,-2,3,-4)}{1^2+-2^2+3^2+-4^2} + \frac{(5*-3+7*4+-1*1+6*-2)(-3,4,1,-2)}{-3^2+4^2+1^2+-2^2}$$

$$= \frac{(20+21-2+6)(4,3,2,1)}{16+9+4+1} + \frac{(5-14-3-24)(1,-2,3,-4)}{1+4+9+16} + \frac{(-15+28-1-12)(-3,4,1,-2)}{9+16+1+4}$$

$$= \frac{45(4,3,2,1)}{30} + \frac{-36(1,-2,3,-4)}{30} + \frac{0(-3,4,1,-2)}{30}$$

= (180/30, 135/30, 90/30, 45/30) + (-36/30, 72/30, -108/30, 144/30) + (0,0,0,0)

$\text{Proj}_{W}\mathbf{u}$ = (144/30, 207/30, -18/30, , 189/30)

Component of **u** orthogonal to W:

$\text{Proj}_{Wperp}\mathbf{u}$ = **u** – Proj_{W}u = (5,7,-1,6) – (144/30, 207/30, -18/30, 189/30)

= (150/30, 210/30, -30/30, 180/30) – (144/30, 207/30, -18/30, 189/30)

= (6/30, 3/30, -12/30, -3/30)

$\text{Proj}_{Wperp}\mathbf{u}$= (1/5, 1/10, -2/5, -3/10)

Good

And just to verify that proj$_{wperp}$**u** is orthogonal to **v**1, **v**2 and **v**3:

$\langle \text{proj}_{wperp}\mathbf{u},\mathbf{v1}\rangle = 1/5*4 + 1/10*3 + -2/5*2 + -3/10*1 = 0$

$\langle \text{proj}_{wperp}\mathbf{u}, \mathbf{v2}\rangle = 1/5*1 + 1/10*-2 + -2/5*3 + -3/10*-4 = 0$

$\langle \text{proj}_{wperp}\mathbf{u}, \mathbf{v3}\rangle = 1/5*-3 + 1/10*4 + -2/5*1 + -3/10*-2 = 0$

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1. (40 points) If points S and P have respective coordinates $(1,3)$ and $(2,1)$ with respect to the standard normal basis, find the coordinates of S and P with respect to the basis $B = \{(1,1),(-1,1)\}$.

Solution:

The new coordinates can be found by finding S, P as a combination of vector in B. To do this place the coordinates in an augmented matrix with respect to B and find the reduced row echelon form.

S P

$$\begin{bmatrix} 1 & -1 & | & 1 & | & 2 \\ 1 & 1 & | & 3 & | & 1 \end{bmatrix} -R_1 + R_2 \to R_2$$

$$\begin{bmatrix} 1 & -1 & | & 1 & | & 2 \\ 0 & 2 & | & 2 & | & -1 \end{bmatrix} \frac{1}{2}R_2 \to R_2$$

$$\begin{bmatrix} 1 & -1 & | & 1 & | & 2 \\ 0 & 1 & | & 1 & | & -\frac{1}{2} \end{bmatrix} R_1 + R_2 \to R_1$$

$$\begin{bmatrix} 1 & 0 & | & 2 & | & \frac{3}{2} \\ 0 & 1 & | & 1 & | & -\frac{1}{2} \end{bmatrix}$$

$$S \neq \frac{3}{2}(1,1) + (-\frac{1}{2})(-1,1) = P$$

$$S = 2\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}-1\\1\end{pmatrix}$$

$$S \neq (\frac{3}{2},\frac{3}{2}) + (\frac{1}{2},-\frac{1}{2})$$

$$S \neq (2,1)$$

$$P \neq 2(1,1) + 1(-1,1)$$

$$P \neq (2,2) + (-1,1)$$

$$P \neq (1,3)$$

2. (40 points) Find the orthogonal projection of $\mathbf{u} = (5,6,7,2)$ onto the solution space of the homogeneous linear system

$$x_1 + x_2 + x_3 = 0$$

$$2x_2 + x_3 + x_4 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} +x_3/2 + x_4/2 - x_3 \\ -x_3/2 - x_4/2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_3 + \frac{x_4}{2} \\ -\frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{2}x_3\begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} + \frac{x_4}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix}$$

It's not clear to me that any of this is relevant. It might be.

update: I think you should find $A\underline{x}$ where $\underline{x}$ solves $A^TA\underline{x} = A^T\underline{u}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 0 \\ 0 & 2 \end{bmatrix}$.

$$A^T\mathbf{u} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 5 \\ 6 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 18 \\ 21 \end{bmatrix}$$

$$A^TA\mathbf{x} = A^T\mathbf{u}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 18 \\ 21 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 3 & 18 \\ 3 & 6 & 21 \end{array}\right] -R_1 + R_2 \to R_2$$

$$\left[\begin{array}{cc|c} 3 & 3 & 18 \\ 0 & 3 & 3 \end{array}\right] \frac{1}{3}R_2 \to R_2$$

$$\left[\begin{array}{cc|c} 3 & 3 & 18 \\ 0 & 1 & 1 \end{array}\right] -3R_2 + R_1 \to R_1$$

$$\left[\begin{array}{cc|c} 3 & 0 & 15 \\ 0 & 1 & 1 \end{array}\right] \frac{1}{3}R_1 \to R_1$$

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$proj_w\mathbf{u} = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 6 \\ 1 \end{bmatrix}$$

$$proj_w\mathbf{u} = (5, 7, 6, 1)$$ $\notin$ Nul A (your A).

inconsistent

3. (60 points) Prove the Best Approximation Theorem: If W is a finite-dimensional subspace of an inner product space V, and if $\mathbf{u}$ is a vector in V, then $proj_W\mathbf{u}$ is the best approximation to $\mathbf{u}$ from W in the sense that

$$||\mathbf{u} - proj_w\mathbf{u}|| \leq ||\mathbf{u} - \mathbf{w}||$$

ok, I see.

for every vector $\mathbf{w}$ in W that is different from $proj_W\mathbf{u}$.

Good

Solution:

For every vector $\mathbf{w}$ in W, we can write

$$\mathbf{u} - \mathbf{w} = (\mathbf{u} - proj_W\mathbf{u}) + (proj_W\mathbf{u} - \mathbf{w})$$

But $proj_W\mathbf{u} - \mathbf{w}$, being a difference of vectors in W is in W; and $\mathbf{u} - proj_W\mathbf{u}$ is orthogonal to W, so the two terms $(\mathbf{u} - proj_W\mathbf{u})$ and $(proj_W\mathbf{u} - \mathbf{w})$ are orthogonal. Thus, by the Theorem of Pythagoras,

$$||\mathbf{u} - \mathbf{w}||^2 = ||\mathbf{u} - proj_W\mathbf{u}||^2 + ||proj_W\mathbf{u} - \mathbf{w}||^2$$

If $\mathbf{w} \neq proj_W\mathbf{u}$, then the second term in this sum will be positive, so

$$||\mathbf{u} - \mathbf{w}||^2 > ||\mathbf{u} - proj_W\mathbf{u}||^2$$

$$||\mathbf{u} - \mathbf{w}|| > ||\mathbf{u} - proj_W\mathbf{u}||$$

4. (60 points) Show that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space $\mathbf{V}$ then

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

Solution:

By definition,

$$||\mathbf{u} + \mathbf{v}||^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle$$
$$= \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle$$
$$\leq \langle \mathbf{u}, \mathbf{u} \rangle + 2|\langle \mathbf{u}, \mathbf{v} \rangle| + \langle \mathbf{v}, \mathbf{v} \rangle \quad \text{algebra}$$
$$\leq \langle \mathbf{u}, \mathbf{u} \rangle + 2||\mathbf{u}||||\mathbf{v}|| + \langle \mathbf{v}, \mathbf{v} \rangle \quad \text{C.S.}$$
$$= ||\mathbf{u}||^2 + 2||\mathbf{u}||||\mathbf{v}|| + ||\mathbf{v}||^2 \quad \text{definition}$$
$$(||\mathbf{u}|| + ||\mathbf{v}||)^2$$

Taking the square root gives $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$

40 points

6.1-6.2

Problem

C[-1,1]?

Let **p**= p(x) and **q**= q(x) be two function in C[a, b] and define <**p** , **q**> = $\int_{-1}^{1} p(x)q(x)dx$

A) Find $\|p\|$ for **p**= 1, **p**= x, and **p**=x^2.

B) Find d(p,q) if **p**=1, **q**=x

Solution

A) $\|p\|$ for **p**= 1

Norm of p can also be written as: <**1, 1**>$^{1/2}$ which can also be written as: $\sqrt{\int_{-1}^{1} dx}$

After integrating, . evaluated from (-1, 1). Resulting in $\sqrt{1-(-1).}$ Answer: $\sqrt{2}$

$\|p\|$ for **p**= x

Norm of p can also be written as: <**x, x**>$^{1/2}$ which can also be written as: $\sqrt{\int_{-1}^{1} x\text{^}2dx}$

After integrating, $\sqrt{(x\text{^}3)/3}$. evaluated from (-1, 1). Resulting in $\sqrt{1/3-(-1/3).}$ Answer: $\sqrt{2/3}$

$\|p\|$ for **p**= x^2

Norm of p can also be written as: $< x^2, x^2>^{1/2}$ which can also be written as: $\sqrt{\int_{-1}^{1} x\text{^}4dx}$.

After integrating, $\sqrt{(x\text{^}5)/5}$. evaluated from (-1, 1). Resulting in $\sqrt{1/5-(-1/5).}$ Answer: $\sqrt{2/5}$

B) d(p,q)

Distance between p and q can also be written as < p-q, **p-q**>$^{1/2}$ or < 1-x, **1-x**>$^{1/2}$ which can also be written as: $\sqrt{\int_{-1}^{1} (1-x)\text{^}2dx}$. After integrating, $\sqrt{(-1(1-x)\text{^}3)/3}\Big|_{-1}^{1}$ Evaluated from (-1, 1). Resulting in $\sqrt{-1(0-(8/3)}$ Answer: $\sqrt{8/3}$

Evaluate √ after Fundamental theorem.

Prove the when A is an orthogonal nxn matrix then the row vectors of A form an orthonormal set in R^n.

Solution

By definition we know that if $A^{-1}=A^T$ than A is orthogonal. This definition implies that $A A^T=I$. We can use this identity to show that when A is orthogonal the row vectors of A form an orthonormal set in R^n. By matrix multiplication we know that row i and column j in the product will be the inner product of the row vector i from A and column vector j A^T. The row vectors $r_1, r_2, \ldots r_n$ of A are also the column vectors of A^T. Therefore

$$A A^T = \begin{bmatrix} \langle r_1, r_1\rangle & \langle r_1, r_2\rangle \ldots & \langle r_1, r_n\rangle \\ \langle r_2, r_1\rangle & \langle r_2, r_2\rangle \ldots & \langle r_2, r_n\rangle \\ \vdots & \vdots & \vdots \\ \langle r_n, r_1\rangle & \langle r_n, r_2\rangle \ldots & \langle r_n, r_n\rangle \end{bmatrix}$$

In order for $A A^T=I$ to hold true $\langle r_1, r_1\rangle=\langle r_2, r_2\rangle=\ldots=\langle r_n, r_n\rangle=1$ and $\langle r_i, r_j\rangle=0$ when $i\neq j$ which does hold true if and only if $\{r_1, r_2, \ldots r_n\}$ form an orthonormal set in R^n.

30 points

this sentence does not not carry much information

Worth 40 Pts.

Consider the bases B= $\{u_1, u_2\}$ and B'= $\{v_1, v_2\}$ for R^2, where

$$u_1 = \begin{bmatrix}1\\0\end{bmatrix},\ u_2 = \begin{bmatrix}2\\2\end{bmatrix},\ v_1 = \begin{bmatrix}1\\3\end{bmatrix},\ v_2 = \begin{bmatrix}3\\-1\end{bmatrix}$$

a. Find the transition matrix from B' to B.
b. Find the transition matrix from B to B'.
c. Computer the coordinate vector $[w]_B$ where, $w = \begin{bmatrix}2\\-2\end{bmatrix}$.

SOLUTION:

a.

Find the coordinate vector for B' = $[w]_{B'}$ — *why? (i.e. why do that?) unnecessary, even for part c.*

$$\begin{bmatrix}1 & 3 & 2\\3 & -1 & -2\end{bmatrix} \Rightarrow \text{rref} \Rightarrow \begin{bmatrix}1 & 0 & -2/5\\0 & 1 & 4/5\end{bmatrix}$$

$$[w]_{B'} = \begin{bmatrix}-2/5\\4/5\end{bmatrix}$$

$B[w]_B = B'\,[w]_{B'}$

$I\,[w]_B = B^{-1}B'\,[w]_{B'}$

$$B^{-1}B' \neq P = \begin{bmatrix}1 & 2 & 1 & 3\\0 & 2 & 3 & -1\end{bmatrix} \Rightarrow \text{rref} \Rightarrow \begin{bmatrix}1 & 0 & -2 & 4\\0 & 1 & 3/2 & -1/2\end{bmatrix} \Rightarrow \begin{bmatrix}-2 & 4\\3/2 & -1/2\end{bmatrix}$$

$$[w]_B = P\,[w]_{B'} = \begin{bmatrix}-2 & 4\\3/2 & -1/2\end{bmatrix}\begin{bmatrix}-2/5\\4/5\end{bmatrix} = \begin{bmatrix}4\\-1\end{bmatrix}$$

b. — *same matrix P?*

$B'\,[w]_{B'} = B[w]_B$

$I\,[w]_{B'} = B'^{-1}B[w]_B$

$$B'^{-1}B \neq P = \begin{bmatrix}1 & 3 & 1 & 2\\3 & -1 & 0 & 2\end{bmatrix} \Rightarrow \text{rref} \Rightarrow \begin{bmatrix}1 & 0 & 1/10 & 4/5\\0 & 1 & 3/10 & 2/5\end{bmatrix} \Rightarrow \begin{bmatrix}1/10 & 4/5\\3/10 & 2/5\end{bmatrix}$$

$$[w]_{B'} = P\,[w]_B = \begin{bmatrix} 1/10 & 4/5 \\ 3/10 & 2/5 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 4/5 \end{bmatrix}$$

c.

Find the coordinate vector for B = $[w]_B$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -2 \end{bmatrix} \Rightarrow \text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix}$$

$$[w]_B = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

2. If A is an n×n matrix, and if $T_A: R^n \rightarrow R^n$ is multiplication by A, then which of the following are equivalent? Modify the statements that aren't equivalent so they become one of the equivalent statements.

key word – otherwise ambiguous question (in that there are at least 2 answers)

a. A is invertible.
b. T_A is one-to-one.
c. A has nullity n.
d. The column vectors of A are linearly independent.
e. The row vectors of A are linearly independent.
f. The column vectors of A span R^n.
g. The row vectors of A span R^n.
h. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
i. The orthogonal complement of the nullspace of A is $\{\mathbf{0}\}$.
j. A^TA is invertible.
k. A has rank 0.

Solution

a, b, d, e, f, g, h, and j are equivalent.

To correct the unequivalent statements.
c. A has nullity 0.

i. The orthogonal complement of the nullspace of A is R^n.

OR

The orthogonal complement of the rowspace is $\{\mathbf{0}\}$.

k. A has rank n.

(problem is out of 40 points. Subtract 4 points for each alphabetical letter they place in the wrong category (because, with 11 statements, they must start with at least 2 to have any statements be equivalent).)

_____/40

Let W be the subspace of R^5 spanned by the vectors

$w_1 = (2, 2, -1, 0, -1)$, $w_2 = (-1, -1, 2, -3, 1)$,

$w_3 = (1, 1, -2, 0, -1)$, $w_4 = (0, 0, 1, 1, 1)$

Find a basis for the orthogonal complement of W.

Solution

The space W spanned by w_1, w_2, w_3, w_4 is the same as the row space of the matrix

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and we know that the nullspace of A is the orthogonal complement of W.
Row-reducing the matrix and solving it we get

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Those two vectors form a basis for this nullspace. Expressing these vectors in the same notation as w_1, w_2, w_3, and w_4, we conclude that the vectors

$\mathbf{v}_1 = (-1, 1, 0, 0, 0)$ and $\mathbf{v}_2 = (-1, 0, -1, 0, 1)$

form a basis for the orthogonal complement of W.

Problem:

Find the coordinate vector for **v** relative to S = $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
$\mathbf{v} = (2, -1, 3)$; $\mathbf{v}_1 = (1, 0, 0)$; $\mathbf{v}_2 = (2, 2, 0)$; $\mathbf{v}_3 = (3, 3, 3)$

Solution:

(2, -1, 3) = a(1, 0, 0) + b(2, 2, 0) + c(3, 3, 3)

2 = a +2b +3c
-1 = 2b + 3c
3 = 3c

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & 3 & 3 \end{array}\right] \text{(1/2)R2} \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 1 & 1 \end{array}\right] \text{(-3/2)R3→R2, -3R3→R1}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \text{-2R2→R1} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

So, $[\mathbf{v}]_S = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

Let R^2 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ into an orthonormal basis given that $\mathbf{u}_1 = (1,-3)$, $\mathbf{u2} = (2, 2)$.

Solution:

Using the steps of the Gram-Schmidt process gives the following:

$\mathbf{v}_1 = \mathbf{u}_1 = (1,-3)$

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_W\, \mathbf{u}_2 = \mathbf{u}_2 - \frac{<\mathbf{u}_1, \mathbf{u}_2>}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$

$$= (2, 2) - \left(-\frac{4}{10}\right)(1,-3)$$

$$= (2,2) + \left(\frac{2}{5}, -\frac{6}{5}\right)$$

$$= \left(\frac{12}{5}, \frac{4}{5}\right) \propto (3,1)$$

(multiply by 5 immediatley - and divide by 4)

$$\|\mathbf{v}_1\| = \sqrt{10}$$

$$\|\mathbf{v}_2\| = \frac{4\sqrt{2}}{\sqrt{5}} = \frac{4\sqrt{10}}{5}$$

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$$

$$\mathbf{q}_2 = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$$

Find the transition matrix **P** for the following "transition of **B** to **B'**." not a standard term

B= $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ **B'**= $\begin{bmatrix} 8 & 7 \\ 9 & 9 \end{bmatrix}$

SOLUTION:

$\text{rref}\begin{bmatrix} 8 & 7 & 1 & 0 \\ 9 & 9 & 0 & 1 \end{bmatrix}$ r2-r1->r1,r1<->r2

$\text{rref}\begin{bmatrix} 1 & 2 & -1 & 1 \\ 8 & 7 & 1 & 0 \end{bmatrix}$ r2-8r1->r2

$\text{rref}\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -9 & 9 & -8 \end{bmatrix}$ r2/-9->r2,r1-(2)r2->r1

$\text{rref}\begin{bmatrix} 1 & 0 & 1 & -7/9 \\ 0 & 1 & -1 & 8/9 \end{bmatrix}$ need B here?

$\begin{bmatrix} 1 & -7/9 \\ -1 & 8/9 \end{bmatrix}\begin{bmatrix} 8 & 7 \\ 9 & 9 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 3 \end{bmatrix}$?

P= $\begin{bmatrix} 2 & 1 \\ 3 & 3 \end{bmatrix}$

(16)

1. Consider the bases B={$\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3$} and B'={$\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$} for $\mathbf{R}^3$, where

$$\mathbf{u}_1=\begin{bmatrix}1\\0\\3\end{bmatrix}\quad \mathbf{u}_2=\begin{bmatrix}-1\\4\\-2\end{bmatrix}\quad \mathbf{u}_3=\begin{bmatrix}-3\\0\\2\end{bmatrix}\quad \mathbf{v}_1=\begin{bmatrix}-1\\1\\0\end{bmatrix}\quad \mathbf{v}_2=\begin{bmatrix}2\\5\\0\end{bmatrix}\quad \mathbf{v}_3=\begin{bmatrix}4\\3\\-2\end{bmatrix}$$

a)Find the transition matrix from B' to B.

b)Compute the coordinate vector $[\mathbf{w}]_{B'}$ where $\mathbf{w}=\begin{bmatrix}84\\147\\-22\end{bmatrix}$

c)Compute the coordinate vector $[\mathbf{w}]_B$.

inefficient - try row reducing $[\,u_1\ u_2\ u_3 \,|\, v_1\ v_2\ v_3\,]$

Solution:

a) $\mathbf{v}_1=k_1\mathbf{u}_1+ k_2\mathbf{u}_2+ k_3\mathbf{u}_3$

$(-1,1,0)= k_1(1,0,3)+ k_2(-1,4,-2)+ k_3(-3,0,2)$

$-1= k_1- k_2-3k_3$

$1=4k_2 \quad => \quad k_2= 1/4$

$0=3k_1-2k_2+2k_3$

Take the third equation and subtract 3 of the first equation to get,

$3= k_2+11k_3$

Then putting in our k_2 we get $k_3=1/4$. Putting k_2 and k_3 into the first equation we get $k_1=0$. So $\mathbf{v}_1=0\mathbf{u}_1+ 1/4\mathbf{u}_2+ 1/4\mathbf{u}_3$.

$\mathbf{v}_2=k_1\mathbf{u}_1+ k_2\mathbf{u}_2+ k_3\mathbf{u}_3$

$(2,5,0)= k_1(1,0,3)+ k_2(-1,4,-2)+ k_3(-3,0,2)$

$2= k_1- k_2-3k_3$

$5=4k_2 \quad => \quad k_2= 5/4$

$0=3k_1-2k_2+2k_3$

Take the third equation and subtract 3 of the first equation to get,

$-6= k_2+11k_3$

Then putting in our k_2 we get $k_3= -29/44$. Putting k_2 and k_3 into the first equation we get $k_1=115/22$. So $\mathbf{v}_2=115/22\mathbf{u}_1+ 5/4\mathbf{u}_2-29/44\mathbf{u}_3$.

$\mathbf{v}_3=k_1\mathbf{u}_1+ k_2\mathbf{u}_2+ k_3\mathbf{u}_3$

$(4,3,-2)= k_1(1,0,3)+ k_2(-1,4,-2)+ k_3(-3,0,2)$

$4= k_1- k_2-3k_3$

$3=4k_2 \quad => \quad k_2= 3/4$

$-2=3k_1-2k_2+2k_3$

Take the third equation and subtract 3 of the first equation to get,

$-14= k_2+11k_3$

Then putting in our k_2 we get $k_3= -59/44$. Putting k_2 and k_3 into the first equation we get $k_1=8/11$. So $\mathbf{v}_3=8/11\mathbf{u}_1+ 3/4\mathbf{u}_2- 59/44\mathbf{u}_3$.

$$[\mathbf{v}_1]_B=\begin{bmatrix}0\\1/4\\1/4\end{bmatrix}\quad [\mathbf{v}_2]_B=\begin{bmatrix}115/22\\5/4\\-29/44\end{bmatrix}\quad [\mathbf{v}_3]_B=\begin{bmatrix}8/11\\3/4\\-59/44\end{bmatrix}\qquad P=\begin{bmatrix}0 & 115/22 & 8/11\\1/4 & 5/4 & 3/4\\1/4 & -29/44 & -59/44\end{bmatrix}$$

30 points

b) $\mathbf{w}= k_1\mathbf{v}_1+ k_2\mathbf{v}_2+ k_3\mathbf{v}_3$

$(84,147,-22)= k_1(-1,1,0)+ k_2(2,5,0)+ k_3(4,3,-2)$

$84= -k_1+ 2k_2+ 4k_3$

$147= k_1+ 5k_2+ 3k_3$

$-22= -2k_3 \quad => \quad k_3= 11$

Then adding the first equation to the second we get,

$231=7k_2+7k_3$

Then putting in our k_3 we get $k_2=22$. Then putting in our k_2 and k_3 into the first equation we get $k_1=4$. So **w**= $4\mathbf{v}_1+22\mathbf{v}_2+11\mathbf{v}_3$. Then $[\mathbf{w}]_{B'}=\begin{bmatrix}4\\22\\11\end{bmatrix}$

10 points

c) $[\mathbf{w}]_B=P[\mathbf{w}]_{B'}$

$$[\mathbf{w}]_B=\begin{bmatrix}0 & 115/22 & 8/11\\ 1/4 & 5/4 & 3/4\\ 1/4 & -29/44 & -59/44\end{bmatrix}\begin{bmatrix}4\\22\\11\end{bmatrix}$$

$$[\mathbf{w}]_B=\begin{bmatrix}123\\147/4\\-113/4\end{bmatrix}$$

10 points

2. Let the inner product in M_{22} be defined as $<U,V>=tr(U^T V)=tr(V^T U)=u_1v_1+u_2v_2+u_3v_3+u_4v_4$. The norm of a matrix U relative to this inner product is:

$\sqrt{u_1^2+u_2^2+u_3^2+u_4^2}$.

Find the cosine of the angle between A and B, where

$$A=\begin{bmatrix}7 & -1\\ 4 & -6\end{bmatrix} \quad B=\begin{bmatrix}-4 & 7\\ 1 & 6\end{bmatrix}$$

Solution:

$$\cos\theta=(<A,B>)/(\|A\|\,\|B\|)=(7(-4)+(-1)(7)+4(1)+(-6)(6))/(\sqrt{7^2+(-1)^2+4^2+(-6)^2} * \sqrt{(-4)^2+7^2+1^2+6^2})$$

$$=(-28-7+4-36)/(\sqrt{102} * \sqrt{102})$$

$$= -67/102$$

20 points

3. Problem: Find the least squares solution of A**x** = **b**, where $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \\ 2 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$.

Solution:

Given A and **b**, $\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$

$$\mathbf{x} = \left(\begin{pmatrix} 3 & 5 & 2 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 5 & 2 \\ 2 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 3 & 5 & 2 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 38 & 13 \\ 13 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 5 & 2 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5/21 & -13/21 \\ -13/21 & 38/21 \end{pmatrix} \begin{pmatrix} 3 & 5 & 2 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2/21 & -1/21 & 10/21 \\ -1/21 & 11/21 & -26/21 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

Least squares solution $\mathbf{x} = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$

(40 points total)

4. Problem: Suppose that the characteristic polynomial of some matrix A is found to be

$$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3.$$

In each part, answer the question and explain your reasoning.

a. What is the size of A?
b. Is A invertible?

Solution:

a. If A is an n x n matrix, then the characteristic polynomial of A has degree n and the coefficient of λ^n is 1; that is, the characteristic polynomial $p(\lambda)$ of an n x n matrix has the form

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + \ldots + c_n .$$

It follows from the Fundamental Theorem of Algebra that the characteristic equation

$$\lambda^n + c_1\lambda^{n-1} + \ldots + c_n = 0$$

has at most n distinct solutions, so an n x n matrix has at most n distinct eigenvalues.

In expanded form, unnecessary
$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3 = \lambda^6 - 19\lambda^5 + 147\lambda^4 - 589\lambda^3 + 1276\lambda^2 - 1392\lambda + 576$.

Since the characteristic polynomial has degree 6 and the coefficent of λ^6 is 1, A is a 6 x 6 matrix. so?

b. Yes, A is invertible. Theorem 7.1.4, states that "A square matrix is invertible if and only if
$\lambda = 0$ is not an eigenvalue of A". In this problem, the eigenvalues are 1, 3, and 4 (setting the characteristic equation equal to zero and solving for λ).

(40 points total—20 for each part)

15

Given the following,
u = (13/2, 3/2, 2); $\mathbf{a}_1$ = (-1, 3, 2); $\mathbf{a}_2$ = (1, 5, -2)

Find the orthogonal projection of **u** onto the subspace of R^3 spanned by the vectors $\mathbf{a}_1$, and $\mathbf{a}_2$.

=y?

Solution:

a.) We know that to form an orthogonal projection onto a subspace, we can use the equation $A^TA\mathbf{x} = A^T\mathbf{b}$ because this is the normal system, associated with the projection of **u** onto the column space of A. Therefore,

$$A = \begin{bmatrix} -1 & 1 \\ 3 & 5 \\ 2 & -2 \end{bmatrix} \qquad A^T = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 5 & -2 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Using matrix multiplication we get $A^TA = \begin{bmatrix} 14 & 10 \\ 10 & 30 \end{bmatrix}$ and $A^T\mathbf{u} = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$

Next we use row reduction to solve for **x** in the equation.

$$\begin{bmatrix} 14 & 10 \\ 10 & 30 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \end{bmatrix} \underset{-14/10\, R_2 + R_1}{\overset{1/10\, R_2}{\Rightarrow}} \begin{bmatrix} 0 & -32 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -12 \\ 1 \end{bmatrix}$$

$$\overset{-1/32\, R_1}{\Rightarrow} \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/8 \\ 1 \end{bmatrix} \underset{R_1 \Leftrightarrow R_2}{\overset{-3R_1 + R_2}{\Rightarrow}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/8 \\ 3/8 \end{bmatrix}$$

Therefore, $x_1 = -1/8$; $x_2 = 3/8$. From this we can find the projection onto $\mathbf{a}_1$ and $\mathbf{a}_2$, by multiplying the column matrix **x** by A.

$$\text{proj}_{\text{span}\{a_1,a_2\}}\ \mathbf{u} = \begin{bmatrix} -1 & 1 \\ 3 & 5 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1/8 \\ 3/8 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \\ -1 \end{bmatrix}$$

Section 7.1-7.2

Prove that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \}$ is a linearly independent set.

Proof:

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. We shall assume that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent and obtain a contradiction. We can then conclude that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.

Since an eigenvector is a non-zero by definition, $\{\mathbf{v}_1\}$ is linearly independent. Let r be the largest integer such that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r \}$ is linearly independent. Since we are assuming that $\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \}$ is linearly dependent, r satisfies $1 \leq r < n$. Moreover, by definition of r, $\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{r+1} \}$ is linearly dependent. Thus there are scalars $c_1, c_2, \ldots, c_{r+1}$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_{r+1}\mathbf{v}_{r+1} = \mathbf{0}$$

Multiplying both sides of the equation by A and using the fact that

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \quad A\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \ \ldots, \quad A\mathbf{v}_{r+1} = \lambda_{r+1}\mathbf{v}_{r+1}$$

We obtain

$$c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \ldots + c_{r+1}\lambda_{r+1}\mathbf{v}_{r+1} = \mathbf{0}$$

multiplying both sides of the first equation by $\lambda_{r+1}$and subtracting the resulting equation from the last equation yields

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{v}_1 + c_2(\lambda_2 - \lambda_{r+1})\mathbf{v}_2 + \ldots + c_r(\lambda_r - \lambda_{r+1})\mathbf{v}_r = \mathbf{0}$$

Since $\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r \}$is a linearly independent set, this equation implies that

$$c_1(\lambda_1 - \lambda_{r+1}) = c_2(\lambda_2 - \lambda_{r+1}) = \ldots = c_r(\lambda_r - \lambda_{r+1}) = 0$$

and since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct by hypothesis, it follows that

$$c_1 = c_2 = \ldots = c_r = 0$$

Substituting these values into the first equation gives us

$$c_{r+1}\mathbf{v}_{r+1} = \mathbf{0}$$

Since the eigenvector $\mathbf{v}_{r+1}$ is non-zero, it follows that

$$c_{r+1} = 0$$

These last three equations show a contradiction in the fact that $c_1, c_2, \ldots, c_{r+1}$ are not all zero, this completes the proof.

cool

Question:

What conditions must be satisfied for the following matrix to be orthogonal?

$$A = \begin{bmatrix} a & \lambda \\ a & a \end{bmatrix}$$

Solution:

It follows from the definition of an orthogonal matrix that "a square matrix A is orthogonal if and only if $AA^T = A^TA = I$."

From this statement we know that:

$$\begin{bmatrix} a & \lambda \\ a & a \end{bmatrix}\begin{bmatrix} a & a \\ \lambda & a \end{bmatrix} = \begin{bmatrix} a & a \\ \lambda & a \end{bmatrix}\begin{bmatrix} a & \lambda \\ a & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can now write equations to set our conditions for a and λ:

$$a^2 + \lambda^2 = 1 \qquad\qquad a^2 + \lambda a = 0$$
$$a^2 + a^2 = 1$$

We have three unique equations and two unknowns, so we have enough equations to solve for each unknown. We begin by solving for a:

$$a^2 + a^2 = 1$$
$$2a^2 = 1$$
$$a^2 = \frac{1}{2}$$
$$a = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \quad \text{or } a = -\frac{1}{\sqrt{2}}$$

Next, we solve for λ:

$$a^2 + \lambda a = 0$$
$$a(a + \lambda) = 0$$

$$a \neq 0 \quad \text{or} \quad a + \lambda = 0$$
$$\lambda = -a = -\frac{1}{\sqrt{2}} \quad \text{or } \lambda = +\frac{1}{\sqrt{2}}$$

The trivial solution will not produce the identity matrix, so we know that for matrix A to be orthogonal,

$$\boxed{a = \frac{1}{\sqrt{2}}, \text{ and } \qquad \lambda = -\frac{1}{\sqrt{2}}}$$

Check: These solutions also satisfy our third unique equation:

$$a^2 + \lambda^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

Question

Let W be the subspace of R^5 spanned by the vectors

$w_1 = (3, 3, 0, 1, 2)$
$w_2 = (0, 0, 3, -2, 2)$
$w_3 = (2, 2, -1, 1, 0)$
$w_4 = (1, 1, 2, 2, 2)$

Find a basis for the orthogonal complement of W

Solution

The space W spanned by w_1, w_2, w_3 and w_4 is the same as the row space of the matrix

$$A = \begin{bmatrix} 3 & 3 & 0 & 1 & 2 \\ 0 & 0 & 3 & -2 & 2 \\ 2 & 2 & -1 & 1 & 0 \\ 1 & 1 & 2 & 2 & 2 \end{bmatrix}$$

According to Theorem 6.2.6 (a), the nullspace of matrix A and the row space of matrix A are orthogonal complements in R^n with respect to the Euclidean inner product.

To find the nullspace of matrix A, we begin by reducing A to its row reduced echelon form.

$$A= \begin{bmatrix} 3 & 3 & 0 & 1 & 2 \\ 0 & 0 & 3 & -2 & 2 \\ 2 & 2 & -1 & 1 & 0 \\ 1 & 1 & 2 & 2 & 2 \end{bmatrix} \text{swap (R1, R4)}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 3 & -2 & 2 \\ 2 & 2 & -1 & 1 & 0 \\ 3 & 3 & 0 & 1 & 2 \end{bmatrix} \text{R3 – 2*R1 -> R3; R4 – 3*R1 -> R4}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 3 & -2 & 2 \\ 0 & 0 & -5 & -3 & -4 \\ 0 & 0 & -6 & -5 & -2 \end{bmatrix} \text{R3 + 2*R2 -> R3; R4 + 2*R2 -> R4}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 3 & -2 & 2 \\ 0 & 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & -9 & 2 \end{bmatrix} \text{swap (R2, R3)}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & -7 & 0 \\ 0 & 0 & 3 & -2 & 2 \\ 0 & 0 & 0 & -9 & 2 \end{bmatrix} \text{R3} - 3\text{*R2} \text{ -> R3}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 19 & 2 \\ 0 & 0 & 0 & -9 & 2 \end{bmatrix} \text{R3} + 2\text{*R4} \text{ -> R3}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -9 & 2 \end{bmatrix} \text{R4} + 9\text{*R3} \text{ -> R4}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 56 \end{bmatrix} \text{R4 * 1/56 -> R4}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{R1} - 2\text{*R4 -> R1;} \;\; \text{R3} - 6\text{*R4 -> R3}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{R1} - 2\text{*R3 -> R1;} \;\; \text{R2} + 7\text{*R3 -> R2}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} R1 - 2*R2 \rightarrow R1$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, the RREF of A = $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

The nullspace of A is the solution space of the homogeneous system:

$X_1 + X_2 = 0$
$X_3 = 0$
$X_4 = 0$
$X_5 = 0$

Solving for the leading variables we get:

$X_1 = -X_2$
$X_3 = 0$
$X_4 = 0$
$X_5 = 0$

It follows that the general system for the solution is:

$X_1 = -s$
$X_2 = s$
$X_3 = 0$
$X_4 = 0$
$X_5 = 0$

This is equivalent to

$$\begin{bmatrix} X1 \\ X2 \\ X3 \\ X4 \\ X5 \end{bmatrix} = s^* \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The vector on the right hand side forms a basis for the solution space, giving us:

$$V_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Expressing this vector in the same notation as w_1, w_2, w_3 and w_4, we conclude that the vector

$V_1 = (-1, 1, 0, 0, 0)$

forms a basis for the orthogonal complement of W.

NOTE: As a check, one can verify that V_1 is orthogonal to w_1, w_2, w_3 and w_4 by calculating the necessary dot products.

(14)

Problem 1:

Question: Prove the Cauchy-Schwartz Inequality

Answer:

Cauchy-Schwartz is

Good

$|(\underline{u}, \underline{v})| \leq ||\underline{v}||\ ||\underline{u}||$

if one of the vectors, v say, equals zero, then

$<\underline{v},\underline{v}> = ||\underline{v}||^2 = 0$ ô $\underline{v} = \underline{0}$ ← put analogous statement here

in such case

$0 = |<\underline{u},\underline{v}>| \leq ||\underline{u}||\ ||\underline{v}|| = 0$

Cauchy-Schwartz holds if one of the vectors is 0 .

Now, if $\underline{v}$ does not equal 0 ô $<\underline{v},\underline{v}> > 0$ — ideas - not just symbols

$0 \leq f(\lambda) := ||\underline{u} + \lambda \underline{v}||^2 = <\underline{u} + \lambda \underline{v}, \underline{u} + \lambda \underline{v}>$

$= <\underline{u},\underline{u}> + 2<\underline{u},\underline{v}> \lambda + <\underline{v},\underline{v}> \lambda^2$

Looking at a graph, our worst-case scenario would be if this were equal to zero, or even just its minimum, so that's what we'll look for

$0 \leq \lambda^2 + 2(<\underline{u},\underline{v}> / <\underline{v},\underline{v}>) + <\underline{u},\underline{u}>/<\underline{v},\underline{v}> =: g(\lambda)$

now to find its minimum

$0 = g'(\lambda) = 2\lambda + 2(<\underline{u},\underline{v}>/<\underline{v},\underline{v}>)$

$\lambda_{min} = - <\underline{u},\underline{v}>/<\underline{v},\underline{v}>$

$0 \leq g(\lambda_{min}) = - <\underline{u},\underline{v}>^2/<\underline{v},\underline{v}>^2 + <\underline{u},\underline{u}>/<\underline{v},\underline{v}>$

$<\underline{u},\underline{v}>^2 \leq <\underline{v},\underline{v}> <\underline{u},\underline{u}> = ||\underline{v}||^2 ||\underline{u}||^2$

$|<\underline{u},\underline{v}>| \leq ||\underline{v}||\ ||\underline{u}||$

Use the Gram-Schmidt process to transform the basis $\{u_1,u_2,u_3\}$ into an orthonormal basis.

$u_1 = (1,0,0)$

$u_2 = (3,7,-2)$

$u_3 = (0,4,1)$

Solution:

Following the steps of the Gram-Schmidt process,

Step 1:

$v_1 = u_1 = (1,0,0)$

Step 2:

$v_2 = u_2 - \text{proj}_{w1}\, u_2 = u_2 - (\langle u_2,v_1\rangle * v_1) / (||v_1||\text{^}2)$

$v_2 = (3,7,-2)-(3+0+0)(1,0,0)=(3,7,-2)-(3,0,0)$

$v_2 = (0,7,-2)$

Step 3:

$v_3 = u_3 - \text{proj}_{w2}\, u_3 = u_3 - (\langle u_3,v_1\rangle v_1)/||v_1||\text{^}2 - (\langle u_3,v_2\rangle v_2)/||v_2||\text{^}2$

$v_3 = (0,4,1) - (0+0+0)-((0+28+-2)(0,7,-2))/53$

$v_3 = (0,4,1) - 0 - (26(0,7,-2))/53= (0,4,1) - (0,182/53, -52/53)$

$v_3 = (0, 30/53, 105/53)$ — multiply by 53, divide by at least 5 — makes life easier

v_1, v_2, and v_3 form an orthogonal basis for R^3. In order to find the orthonormal basis now, we must…

$q_1 = v_1/||v_1|| = (1, 0, 0) / (1) = (1, 0, 0)$

$q_2 = v_2/||v_2|| = (0, 7, -2) / (\text{sqrt}(53)) = (0, 7/\text{sqrt}(53), -2/\text{sqrt}(53))$

$q_3 = v_3/||v_3|| = (0, 30/53, 105/53) / (15/\text{sqrt}(53)) = (0, 2/\text{sqrt}(53), 7/\text{sqrt}(53))$

Find the angle y_1, w between $\underline{P}_1=1$ & $\underline{P}_2=x$ in P (0,1) with the usual notion of inner product.

$$\text{Cos } x_1 = \langle 1,x \rangle / \|1\| \, \|x\| = \int_0^1 1 \cdot x \, dx \Big/ \left[\left(\sqrt{\int_0^1 1 \cdot x \, dx}\right)\left(\sqrt{\int_0^1 x \cdot x \, dx}\right)\right]$$

$$= \tfrac{1}{2} / (1) (\sqrt{1/3})$$

$$= \sqrt{3}/2$$

$$[0, \underline{\Pi}] \; X_{1w} = \underline{\Pi}/6$$

what's going on here?

Problem 4:

Prove that the following statements are equivalent if A is an n x n matrix:

1. A is orthogonal.
2. $||A\mathbf{x}|| = ||\mathbf{x}||$ for all $\mathbf{x}$ in R^n.
3. $A\mathbf{x} * A\mathbf{y} = \mathbf{x} * \mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ in R^n.

good

In order to prove that these are all equivalent we shall prove a sequence of implications, namely that 1 implies 2, 2 implies 3, and 3 implies 1.

1 implies 2. Assume that A is orthogonal, so $A^TA = I$. Then by using the formula for calculation magnitude: $||A\mathbf{x}|| = (A\mathbf{x} * A\mathbf{x})^{1/2} = (\mathbf{x} * A^TA\mathbf{x})^{1/2} = (\mathbf{x} * \mathbf{x})^{1/2} = ||\mathbf{x}||$.

2 implies 3. Assume that $||A\mathbf{x}|| = ||\mathbf{x}||$ for all x in R^n. By Theorem $A\mathbf{x} * A\mathbf{y} = ¼||A\mathbf{x} – A\mathbf{y}||^2 – ¼||A\mathbf{x} – A\mathbf{y}||^2 = ¼||A(\mathbf{x} + \mathbf{y})||^2 – ¼||A(\mathbf{x} - \mathbf{y})||^2 = ¼||\mathbf{x} + \mathbf{y}||^2 – ¼||\mathbf{x} – \mathbf{y}||^2 = \mathbf{x} * \mathbf{y}$.

3 implies 1. Assume that $A\mathbf{x} * A\mathbf{y} = \mathbf{x} * \mathbf{y}$ for all x and y in R^n. Then by normal system formula $\mathbf{x} * \mathbf{y} = \mathbf{x} * A^TA\mathbf{y}$. This can be rewritten in the form $x * (A^TAy – y) = 0$ or $x * (A^TA – I)\mathbf{y} = 0$. Since this holds in R^n it is sure to hold ~~as~~ for $x = (A^TA – I)\mathbf{y}$, so $(A^TA – I)\mathbf{y} = 0$. From this we can conclude that $(A^TA – I)\mathbf{y} = \mathbf{0}$. Thus the last equation we produced ($(A^TA – I)\mathbf{y} = \mathbf{0}$) is a homogeneous system of linear equations that is satisfied by every $\mathbf{y}$ in R^n. This implies the coefficient matrix must be zero, thus $A^TA = I$ and, consequently, A is orthogonal.

why?

nothing to do with it - same justification as above when $(A\underline{x} \cdot A\underline{x})^{1/2} = (\underline{x} \cdot A^TA\underline{x})$

try using $y =$ ~~column~~ row of $A^TA - I$