Math 343 Final KEY
Fall 2006
sections 002 and 003
Instructor: Scott Glasgow

Please do NOT write on this exam. It must be used multiple times by many students. Rather write in a blue book, or on your own paper, preferably engineering.

Warning: check your solutions to each problem via a method independent of the one used to obtain your initial solution.

Multiple Choice section:

1) Which of the following best describes a determinant?

   a. It is a function of a matrix that produces numbers.
   b. It is an object used in a formula for computing the inverse of a matrix.
   c. It is a measure of the independence of a matrix's row vectors.
   d. It is a measure of the independence of a matrix's column vectors.

2) Which of the following best describes the coordinate vector of a given vector (with respect to a given basis of a vector space)?

   a. It is a list of weights needed to reconstruct the vector from the elements of a basis.
   b. It is a concrete/Euclidean representation of an otherwise abstract object.
   c. It is an object that can be used to turn abstract vector-space calculations into concrete matrix multiplications.
   d. It is an "address" with respect to the indicated basis, i.e. it records the distances and directions that should be traversed along the axes determined by the basis elements in order to arrive at the vector in question.

Essay section:

1. Assuming A and B are invertible matrices of the same size, prove that

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{1.1}$$

25 points

<u>Solution</u>

$B^{-1}A^{-1}$ is the inverse of AB if and only if

$$(AB)\left(B^{-1}A^{-1}\right) = \left(B^{-1}A^{-1}\right)(AB) = I, \tag{1.2}$$

to whit we first note that, by the associative property of matrix multiplication,

$$\begin{gathered}(AB)\left(B^{-1}A^{-1}\right) = A\left(BB^{-1}\right)A^{-1} \\ \text{and} \\ \left(B^{-1}A^{-1}\right)(AB) = B^{-1}\left(A^{-1}A\right)B.\end{gathered} \tag{1.3}$$

Then, by the definition of the inverses, in particular that an inverse is both a right and a left inverse, we have, respectively, that

$$\begin{gathered}(AB)\left(B^{-1}A^{-1}\right) = A\left(BB^{-1}\right)A^{-1} = AIA^{-1} = A\left(IA^{-1}\right) \\ \text{and} \\ \left(B^{-1}A^{-1}\right)(AB) = B^{-1}\left(A^{-1}A\right)B = B^{-1}IB = B^{-1}(IB).\end{gathered} \tag{1.4}$$

In (1.4) we have also used the associative property of matrix multiplication again. Using now the fact that the identity matrix is in fact the “multiplicative identity” we get

$$\begin{gathered}(AB)\left(B^{-1}A^{-1}\right) = A\left(IA^{-1}\right) = AA^{-1} \\ \text{and} \\ \left(B^{-1}A^{-1}\right)(AB) = B^{-1}(IB) = B^{-1}B.\end{gathered} \tag{1.5}$$

Finally we use again the definition of the inverses. In particular, using that an inverse is both a right and a left inverse, we have, respectively, that

$$\begin{gathered}(AB)\left(B^{-1}A^{-1}\right) = AA^{-1} = I \\ \text{and} \\ \left(B^{-1}A^{-1}\right)(AB) = B^{-1}B = I,\end{gathered} \tag{1.6}$$

which is the required (1.2)

2. Prove that, if the matrix A is invertible, the system $A\mathbf{x} = \mathbf{b}$ has one and only one solution $\mathbf{x}$, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

25 points

<u>Solution</u>

If the system has a solution $\mathbf{x}$, then, for any such $\mathbf{x}$,we may write

$$A\mathbf{x} = \mathbf{b} \tag{1.7}$$

(without implicitly lying), and then, by left application of A^{-1} to (1.7), as well as by the associative property of matrix multiplication, obtain that

$$A^{-1}\mathbf{b} = A^{-1}(A\mathbf{x}) = \left(A^{-1}A\right)\mathbf{x} = I\mathbf{x} = \mathbf{x}. \tag{1.8}$$

In (1.8) we also used that A^{-1} is a left inverse of A, as well as the fact that the so-called identity matrix I is in fact a "multiplicative identity". Here then we have just showed that if (1.7) has a solution, it's got to be $\mathbf{x} = A^{-1}\mathbf{b}$. Thus we have showed that (1.7) has at most one solution. But our demonstration does not yet preclude their being no solution. To do that, we confirm that, for the only promising candidate $\mathbf{x} = A^{-1}\mathbf{b}$, we get

$$A\mathbf{x} = A\left(A^{-1}\mathbf{b}\right) = \left(AA^{-1}\right)\mathbf{b} = I\mathbf{b} = \mathbf{b}, \tag{1.9}$$

so that our candidate was successful. (Here we used the associative property of matrix multiplication, the fact that A^{-1} is a right inverse of A, as well as the fact that the so-called identity matrix I is in fact a "multiplicative identity".) Thus we have showed that the system has one and only one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

3. Assume that both the matrix B and the matrix C are inverses of the matrix A. Show that B and C are just two aliases for the same matrix, i.e. show that in fact $B = C$.

25 points

Solution

The descriptions of B and C demand that

$$AB = BA = I = AC = CA. \tag{1.10}$$

Using the associative property of matrix multiplication in two different ways on the product BAC we get

$$\begin{gathered} BAC = B\left(AC\right) = BI = B \\ \text{and} \\ BAC = \left(BA\right)C = IC = C, \end{gathered} \tag{1.11}$$

so that indeed

$$\begin{gathered} B = BAC = C \\ \Rightarrow \\ B = C \end{gathered} \tag{1.12}$$

as claimed. Note that in (1.11) we also used that a) C is a right inverse of A, b) B is a left inverse of A, and that c) the identity matrix acts as both a right and left multiplicative identity.

4. Determine the standard matrix for the linear operator $T:\mathbb{R}^3 \to \mathbb{R}^3$ that first rotates a vector counterclockwise about the y axis through an angle θ, then reflects the resulting vector about the xy-plane, and then projects the latter resulting vector orthogonally onto the xz plane. If you do not recall the form of the standard matrices for these three transformations, recall the theorem (mnemonic)

$$[T] = [T\mathbf{e}_1 \ \ T\mathbf{e}_2 \ \ T\mathbf{e}_3], \tag{1.13}$$

where the indicated (boldface column) vectors are the standard basis elements of $\mathbb{R}^3$.

25 points

<u>Solution</u>

The linear transformation T can be expressed as the composition $T_3 \circ T_2 \circ T_1$, where T_1 is the rotation, T_2 is the reflection, and T_3 is the projection. Thus the standard matrix $[T]$ is the following product of the standard matrices of the other transformations: $[T_3][T_2][T_1]$. By considering the action of these three transformations on the standard basis elements, one finds that

$$[T_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ [T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ \text{and}\ [T_1] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}. \tag{1.14}$$

Thus

$$[T] = [T_3][T_2][T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & -\cos\theta \end{bmatrix}. \tag{1.15}$$

5. Let $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ be a basis for a vector space V. Prove that for any vector $\mathbf{v}\in V$ there exists one AND only one coordinate vector $(c_1,c_2,\ldots,c_n)\in\mathbb{R}^n$ such that $\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n$.

25 points

Solution

By definition of basis we immediately have that for any $\mathbf{v}\in V$ there exists at least one coordinate vector $(c_1,c_2,\ldots,c_n)\in\mathbb{R}^n$ such that $\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n$. As for uniqueness of this coordinate vector, and by way of contradiction, assume that there is another one, say $(k_1,k_2,\ldots,k_n)\in\mathbb{R}^n$. Then

$$\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n=k_1\mathbf{v}_1+k_2\mathbf{v}_2+\ldots+k_n\mathbf{v}_n \tag{1.16}$$

so that

$$\begin{aligned}(c_1-k_1)\mathbf{v}_1+(c_2-k_2)\mathbf{v}_2+\ldots+(c_n-k_n)\mathbf{v}_n&=\mathbf{0}\\ &\Leftrightarrow\\ c_1-k_1=c_2-k_2=\ldots=c_n-k_n&=0\\ &\Leftrightarrow\\ c_1=k_1,c_2=k_2,\ldots,c_n&=k_n,\end{aligned} \tag{1.17}$$

the first equivalence holding since S is independent. The last statement indicates then that $(c_1,c_2,\ldots,c_n)=(k_1,k_2,\ldots,k_n)$, i.e. the "second" coordinate vector was only an alias for the first.

6. Let $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{v}$ denote elements of any vector space. Solve the equation $\mathbf{x}+\mathbf{u}=\mathbf{v}$ for $\mathbf{x}$, stating at each step which vector space axiom or algebraic property or definition you used to arrive at that step.

25 points

Solution

$$
\begin{aligned}
\mathbf{x}+\mathbf{u} &= \mathbf{v} && \\
(\mathbf{x}+\mathbf{u})+(-\mathbf{u}) &= \mathbf{v}+(-\mathbf{u}) && \text{Add } -\mathbf{u} \text{ to both sides of the above equation} \\
\mathbf{x}+(\mathbf{u}+(-\mathbf{u})) &= \mathbf{v}+(-\mathbf{u}) && \text{Associative Axiom of Vector Addition} \\
\mathbf{x}+\mathbf{0} &= \mathbf{v}+(-\mathbf{u}) && \text{Additive Inverse Axiom} \\
\mathbf{x} &= \mathbf{v}+(-\mathbf{u}) && \text{Additive Identity Axiom} \\
\mathbf{x} &= \mathbf{v}-\mathbf{u} && \text{Definition of subtraction}
\end{aligned}
\tag{1.18}
$$

7. Find the orthogonal projection of $\mathbf{u}=(6,1,5,2)$ onto the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1=(1,2,1,2)$, $\mathbf{v}_2=(2,3,2,1)$ and $\mathbf{v}_3=(1,3,3,1)$.

25 points

<u>Solution</u>

The projection $\mathbf{u}_{\|}$ is $\mathbf{u}=(6,1,5,2)$ iff $\mathbf{u}\in Span\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, otherwise it is the vector $\mathbf{u}_{\|}\in Span\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ nearest $\mathbf{u}$. Actually, in either case, we have

$$\mathbf{u}_{\|}=A\mathbf{x} \tag{1.19}$$

where $\mathbf{x}$ is *any* solution of

$$A^T A\mathbf{x}=A^T\mathbf{u}\ ,^1 \tag{1.20}$$

and where

[1] The vector $\mathbf{u}_{\|}=A\mathbf{x}$ is unique even if $\mathbf{x}$ solving (1.20) isn't: Let $\mathbf{x}_1$ and $\mathbf{x}_2$ both solve (1.20). Then $\mathbf{d}=\mathbf{x}_1-\mathbf{x}_2$ satisfies $A^T A\mathbf{d}=\mathbf{0}$, so that either $\mathbf{d}\in Nul(A)$ or $A\mathbf{d}\in Nul(A^T)=(ColA)^{\perp}$. In the first case $\mathbf{u}_{\|1}-\mathbf{u}_{\|2}=A\mathbf{x}_1-A\mathbf{x}_2=A\mathbf{d}=\mathbf{0}$ and we are done. In the second case $A\mathbf{d}=\mathbf{0}\in(ColA)^{\perp}$ also since otherwise

$$\|\mathbf{u}_{\perp 1}\|^2=\|\mathbf{u}-\mathbf{u}_{\|1}\|^2=\|(\mathbf{u}-\mathbf{u}_{\|2})+(\mathbf{u}_{\|2}-\mathbf{u}_{\|1})\|^2=\|\mathbf{u}_{\perp 2}+(\mathbf{u}_{\|2}-\mathbf{u}_{\|1})\|^2=\|\mathbf{u}_{\perp 2}\|^2+\|\mathbf{u}_{\|2}-\mathbf{u}_{\|1}\|^2$$
$$\|\mathbf{u}_{\perp 2}\|^2+\|A\mathbf{d}\|^2>\|\mathbf{u}_{\perp 2}\|^2,$$

so that $\|\mathbf{u}_{\perp 1}\|^2$ and $\|\mathbf{u}_{\perp 2}\|^2$ are not both least.

$$A=\left[\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\right]=\begin{bmatrix}1&2&1\\2&3&3\\1&2&3\\2&1&1\end{bmatrix}. \tag{1.21}$$

We solve (1.20) by row reducing $\left[A^TA\middle|A^T\mathbf{u}\right]$ after calculating A^TA and $A^T\mathbf{u}$:

$$A^TA=\begin{bmatrix}1&2&1&2\\2&3&2&1\\1&3&3&1\end{bmatrix}\begin{bmatrix}1&2&1\\2&3&3\\1&2&3\\2&1&1\end{bmatrix}=\begin{bmatrix}10&12&12\\12&18&18\\12&18&20\end{bmatrix},\ A^T\mathbf{u}=\begin{bmatrix}1&2&1&2\\2&3&2&1\\1&3&3&1\end{bmatrix}\begin{bmatrix}6\\1\\5\\2\end{bmatrix}=\begin{bmatrix}17\\27\\26\end{bmatrix},$$

$$\left[A^TA\middle|A^T\mathbf{u}\right]=\left[\begin{array}{ccc|c}10&12&12&17\\12&18&18&27\\12&18&20&26\end{array}\right]\sim\left[\begin{array}{ccc|c}10&12&12&17\\2&6&6&10\\2&6&8&9\end{array}\right]\sim\left[\begin{array}{ccc|c}1&3&3&5\\10&12&12&17\\2&6&8&9\end{array}\right]\sim\left[\begin{array}{ccc|c}1&3&3&5\\0&18&18&33\\0&0&2&-1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}1&3&3&5\\0&6&6&11\\0&0&2&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}1&3&3&5\\0&6&0&14\\0&0&2&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}2&6&6&10\\0&3&0&7\\0&0&2&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}2&6&0&13\\0&3&0&7\\0&0&2&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}2&0&0&-1\\0&3&0&7\\0&0&2&-1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}6&0&0&-3\\0&6&0&14\\0&0&6&-3\end{array}\right].$$

(1.22)

Thus

$$\mathbf{u}_{\|}=A\mathbf{x}=\frac{1}{6}\begin{bmatrix}1&2&1\\2&3&3\\1&2&3\\2&1&1\end{bmatrix}\begin{bmatrix}-3\\14\\-3\end{bmatrix}=\frac{1}{6}\begin{bmatrix}22\\27\\16\\5\end{bmatrix}=\begin{bmatrix}11/3\\9/2\\8/3\\5/6\end{bmatrix}. \tag{1.23}$$

8. Find the transition matrix $P_{B'B}$ from the $\mathbb{R}^3$ basis $B=\{(1,0,0),(0,1,0),(0,0,1)\}$ to the $\mathbb{R}^3$ basis $B'=\{(1,2,4),(2,5,8),(3,6,13)\}$.

25 points

<u>Solution</u>

Let $(\mathbf{v})_B$ and $(\mathbf{v})_{B'}$ be coordinate vectors with respect to the bases B and B' for the vector $\mathbf{v}$. Then, by definition of coordinate vector,

$$B(\mathbf{v})_B = B'(\mathbf{v})_{B'} = \mathbf{v}, \tag{1.24}$$

where, abusing notation, B and B' here represent matrices containing the vectors of the bases B and B' as columns. Thus B in (1.24) is the identity matrix. According to (1.24) then,

$$(\mathbf{v})_{B'} = B'^{-1}(\mathbf{v})_B =: P_{B'B}(\mathbf{v})_B. \tag{1.25}$$

We calculate the inverse required in (1.25) by row reducing $\left[B' \middle| I\right]$ to $\left[I \middle| B'^{-1}\right] = \left[I \middle| P_{B'B}\right]$:

$$\begin{aligned}\left[B' \middle| I\right] &= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 4 & 8 & 13 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 13 & 0 & -3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 0 & 1 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 17 & -2 & -3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 0 & 1 \end{array}\right] = \left[I \middle| P_{B'B}\right].\end{aligned} \tag{1.26}$$

9. Let A be an nxn matrix. Prove that if

$$A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y} \tag{1.27}$$

holds for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then A is orthogonal.

25 points

<u>Solution</u>

$$\begin{aligned} &\mathbf{x}^T I\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y} = A\mathbf{x} \cdot A\mathbf{y} = (A\mathbf{x})^T A\mathbf{y} = \mathbf{x}^T A^T A\mathbf{y} \\ &\quad\Leftrightarrow \\ &0 = \mathbf{x}^T A^T A\mathbf{y} - \mathbf{x}^T I\mathbf{y} = \mathbf{x}^T\left(A^T A - I\right)\mathbf{y}. \end{aligned} \tag{1.28}$$

Since (1.28) holds for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, it holds for $\mathbf{x} = \left(A^T A - I\right)\mathbf{y}$, so that (1.28) becomes

$$0 = \left(\left(A^T A - I\right)\mathbf{y}\right)^T \left(A^T A - I\right)\mathbf{y} = \left\|\left(A^T A - I\right)\mathbf{y}\right\|^2$$
$$\Leftrightarrow \qquad (1.29)$$
$$\mathbf{0} = \left(A^T A - I\right)\mathbf{y}$$

for every $\mathbf{y} \in \mathbb{R}^n$. Thus (1.29) holds for $\mathbf{y}$ chosen as each of the rows of the matrix $A^T A - I$, showing that each of these rows is the zero vector and, so, showing that $A^T A - I$ is the zero matrix, i.e. $A^T A = I$, which, together with A square, means that A is an orthogonal matrix.

10. Let $T : P_2[0,1] \to P_2[0,1]$ be defined by

$$T[\mathbf{f}](x) = \int_0^1 \left(4 - 6x + (12x - 6)s\right)\mathbf{f}(s)\,ds. \qquad (1.30)$$

($P_2[0,1]$ is the vector space of polynomials with ordinary addition and multiplication as the vector addition and scalar multiplication.) Find a basis B of $P_2[0,1]$ in which $[T]_B$ is diagonal.

25 points

<u>Solution</u>

With respect to the (standard) basis $S = \{1, x, x^2\}$ of $P_2[0,1]$, we have

$$[T]_s = \left[\left(T(1)\right)_S \middle| \left(T(x)\right)_S \middle| \left(T(x^2)\right)_S\right] = \left[(1)_S \middle| (x)_S \middle| \left(x - \frac{1}{6}\right)_S\right] = \begin{bmatrix} 1 & 0 & -\frac{1}{6} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad (1.31)$$

which is not diagonal. $P^{-1}[T]_s P$ will be diagonal provided P is an invertible matrix of eigenvectors of $[T]_s$. This matrix's columns will be the coordinate vectors of elements of the desired basis B with respect to the basis $S = \{1, x, x^2\}$, which will then readily specify such a basis B. Now the eigenvalues of $[T]_s$ are $1,1,0$, since $[T]_s$ is upper triangular, and since these values appear on the diagonal. The associated eigenspaces are then computed as follows:

$$E_{\lambda=1} = Nul\left(\left[T\right]_s - 1\cdot I\right) = Nul\begin{bmatrix} 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = Nul\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = Span\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\},$$

$$E_{\lambda=0} = Nul\left(\left[T\right]_s - 0\cdot I\right) = Nul\begin{bmatrix} 1 & 0 & -\frac{1}{6} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = Span\left\{\begin{pmatrix} 1/6 \\ -1 \\ 1 \end{pmatrix}\right\} = Span\left\{\begin{pmatrix} 1 \\ -6 \\ 6 \end{pmatrix}\right\}. \tag{1.32}$$

Thus one of the desired bases B is, using the "abusive notation", given by

$$B = \left\{\left[1 | x | x^2\right]\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \left[1 | x | x^2\right]\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \left[1 | x | x^2\right]\begin{pmatrix} 1 \\ -6 \\ 6 \end{pmatrix}\right\} = \left\{1, x, 1-6x+6x^2\right\}. \tag{1.33}$$