Math 343 Midterm I
Fall 2006
sections 002 and 003
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1. Assuming A and B are invertible matrices of the same size, prove that

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{1.1}$$

15 points

Solution

$B^{-1}A^{-1}$ is the inverse of AB if and only if

$$(AB)\left(B^{-1}A^{-1}\right) = \left(B^{-1}A^{-1}\right)(AB) = I, \tag{1.2}$$

to whit we first note that, by the associative property of matrix multiplication,

$$\begin{gathered}(AB)\left(B^{-1}A^{-1}\right) = A\left(BB^{-1}\right)A^{-1} \\ \text{and} \\ \left(B^{-1}A^{-1}\right)(AB) = B^{-1}\left(A^{-1}A\right)B.\end{gathered} \tag{1.3}$$

Then, by the definition of the inverses, in particular that an inverse is both a right and a left inverse, we have, respectively, that

$$\begin{gathered}(AB)\left(B^{-1}A^{-1}\right) = A\left(BB^{-1}\right)A^{-1} = AIA^{-1} = A\left(IA^{-1}\right) \\ \text{and} \\ \left(B^{-1}A^{-1}\right)(AB) = B^{-1}\left(A^{-1}A\right)B = B^{-1}IB = B^{-1}(IB).\end{gathered} \tag{1.4}$$

In (1.4) we have also used the associative property of matrix multiplication again. Using now the fact that the identity matrix is in fact the "multiplicative identity" we get

$$\begin{gathered}(AB)\left(B^{-1}A^{-1}\right) = A\left(IA^{-1}\right) = AA^{-1} \\ \text{and} \\ \left(B^{-1}A^{-1}\right)(AB) = B^{-1}(IB) = B^{-1}B.\end{gathered} \tag{1.5}$$

Finally we use again the definition of the inverses. In particular, using that an inverse is both a right and a left inverse, we have, respectively, that

$$
\begin{gathered}
(AB)\left(B^{-1}A^{-1}\right) = AA^{-1} = I \\
\text{and} \\
\left(B^{-1}A^{-1}\right)(AB) = B^{-1}B = I,
\end{gathered} \tag{1.6}
$$

which is the required (1.2)

2. Use Gaussian elimination, noting along the way the various relevant row operations and their relationship to the (evolving calculation of) the determinant, to show that

$$
\det\begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} = (x-y)(y-z)(z-x). \tag{1.7}
$$

Be sure to note along the way the various relevant row operations to your Gaussian elimination and their accurate relationship to the (evolving calculation of) the determinant.

15 points

Solution

We indicate the various Gaussian elimination row operations in the upper right-hand corner of the matrix (whose determinant is being calculated), making sure to use the correct relationship of such to the evolving determinant calculation:

$$
\det\begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix}^{\substack{R2-R1 \\ R3-R1}} = \det\begin{bmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{bmatrix}^{\substack{R2/(y-x) \\ R3/(z-x)}} = (y-x)(z-x)\det\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{bmatrix}^{R3-R2} =
$$

$$
= (y-x)(z-x)\det\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & z-y \end{bmatrix}^{R3-R2} = (y-x)(z-x)\left(1\cdot 1\cdot (z-y)\right) = (x-y)(y-z)(z-x). \tag{1.8}
$$

In the final steps above we used that a) the determinant of a triangular matrix is the product of its diagonal entries, and b) some obvious algebraic properties of multiplication (of scalars).

3. By using the “permutation definition” of the determinant, prove that if square matrix B is the same as square matrix A, except that one of B’s rows is a scalar multiple k of the corresponding row of A, then

$$\det B = k \det A. \tag{1.9}$$

15 points

__**Solution**__

Let it be the j^{th} row of the matrix B that is the same as the scalar k multiplied by the j^{th} row of the matrix A. Then, by definition of the determinant, and the relationship between the 2 rows (of matrix A and matrix B), we have

$$\begin{aligned} \det B &:= \sum_{p:[n]\to[n]} (-1)^{d(p)} b_{1p(1)} \cdots b_{jp(j)} \cdots b_{np(n)} = \sum_{p:[n]\to[n]} (-1)^{d(p)} a_{1p(1)} \cdots \left(ka_{jp(j)}\right) \cdots a_{np(n)} \\ &= k \sum_{p:[n]\to[n]} (-1)^{d(p)} a_{1p(1)} \cdots a_{jp(j)} \cdots a_{np(n)} =: k \det A. \end{aligned} \tag{1.10}$$

4. Prove that, if the matrix A is invertible, the system $A\mathbf{x} = \mathbf{b}$ has one and only one solution $\mathbf{x}$, namely $\mathbf{x} = A^{-1}\mathbf{b}$. (Warning: there are two things to prove here, namely a) that if the system has a solution, then it can only be $\mathbf{x} = A^{-1}\mathbf{b}$, and that b) $\mathbf{x} = A^{-1}\mathbf{b}$ actually does solve the system. Here then you will have addressed the “one and only one” issues in reverse order: you first show that a) there is at most one solution, and b) that there is in fact one solution (rather than none). In parts a) and b) you will use that A^{-1} is A’s left and right inverse, respectively.)

15 points

__**Solution**__

If the system has a solution $\mathbf{x}$, then, for any such $\mathbf{x}$,we may write

$$A\mathbf{x} = \mathbf{b} \tag{1.11}$$

(without implicitly lying), and then, by left application of A^{-1} to (1.11), as well as by the associative property of matrix multiplication, obtain that

$$A^{-1}\mathbf{b} = A^{-1}\left(A\mathbf{x}\right) = \left(A^{-1}A\right)\mathbf{x} = I\mathbf{x} = \mathbf{x}. \tag{1.12}$$

In (1.12) we also used that A^{-1} is a left inverse of A, as well as the fact that the so-called identity matrix I is in fact a “multiplicative identity”. Here then we have just showed that if (1.11) has a solution, it’s got to be $\mathbf{x} = A^{-1}\mathbf{b}$. Thus we have showed that (1.11) has at most one solution. But our demonstration does not yet preclude their being no solution. To do that, we confirm that, for the only promising candidate $\mathbf{x} = A^{-1}\mathbf{b}$, we get

$$A\mathbf{x} = A\left(A^{-1}\mathbf{b}\right) = \left(AA^{-1}\right)\mathbf{b} = I\mathbf{b} = \mathbf{b}, \tag{1.13}$$

so that our candidate was successful. (Here we have used the associative property of matrix multiplication, the fact that A^{-1} is a right inverse of A, as well as the fact that the so-called identity matrix I is in fact a "multiplicative identity".) Thus we have showed that the system has one and only one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

5. The graph of the function

$$y = ax^2 + bx + c \tag{1.14}$$

contains the points $(1,1)$, $(-1,-1)$ and $(-2,4)$. Find the values of the constants a, b, and c by forming a relevant augmented matrix and performing Gauss-Jordan elimination on it.

15 points

Solution

The stated facts dictate the three equations

$$\begin{aligned} a(1)^2 + b(1) + c &= a + b + c = 1 \\ a(-1)^2 + b(-1) + c &= a - b + c = -1 \\ a(-2)^2 + b(-2) + c &= 4a - 2b + c = 4. \end{aligned} \tag{1.15}$$

We row reduce the associated augmented matrix to reduced row echelon form as follows:

$$\begin{aligned} &\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 4 & -2 & 1 & 4 \end{bmatrix}^{\substack{R2-R1 \\ R3-4R1}} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & -6 & -3 & 0 \end{bmatrix}^{\substack{R2/(-2) \\ R3/(-3)}} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}^{R3-2R2} \\ &\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}^{R1-R3} \sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}^{R1-R2} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}. \end{aligned} \tag{1.16}$$

The latter indicates that $a = 2, b = 1,$ and $c = -2$.

6. Assume that both the matrix B and the matrix C are inverses of the matrix A. Show that B and C are just two aliases for the same matrix, i.e. show that in fact $B = C$.

15 points

Solution

The descriptions of B and C demand that

$$AB = BA = I = AC = CA. \tag{1.17}$$

Using the associative property of matrix multiplication in two different ways on the product BAC we get

$$\begin{gathered} BAC = B\left(AC\right) = BI = B \\ \text{and} \\ BAC = \left(BA\right)C = IC = C, \end{gathered} \tag{1.18}$$

so that indeed

$$\begin{gathered} B = BAC = C \\ \Rightarrow \\ B = C \end{gathered} \tag{1.19}$$

as claimed. Note that in (1.18) we also used that a) C is a right inverse of A, b) B is a left inverse of A, and that c) the identity matrix acts as both a right and left multiplicative identity.

7. The "eigenvalues" of a (square) matrix B turn out to be the zeroes λ of the characteristic polynomial $P_B(\lambda) := \det(\lambda I - B)$, i.e. the eigenvalues of B are the roots λ of the characteristic equation $\det(\lambda I - B) = 0$ (where I denotes the identity matrix of the same size as B). Find the eigenvalues of the matrix

$$B = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix}. \tag{1.20}$$

What is the determinant of this matrix? What is the product of its eigenvalues?

15 points

Solution

The characteristic equation is

$$\begin{aligned} 0 = \det(\lambda I - B) &= \det\begin{bmatrix} \lambda - 2 & -2 \\ -6 & \lambda - 3 \end{bmatrix} = (\lambda - 2)(\lambda - 3) - (-2)(-6) \\ &= \lambda^2 - 5\lambda + 6 - 12 = \lambda^2 - 5\lambda - 6 = (\lambda + 1)(\lambda - 6), \end{aligned} \tag{1.21}$$

so that the eigenvalues are clearly -1 and 6. The determinant of the matrix is $(2)(3) - (2)(6) = 6 - 12 = -6$, while the product of the eigenvalues is $(-1)(6) = -6$ also.

8. Find the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1.22}$$

by row reducing $\left[A | I \right]$ to $\left[I | A^{-1} \right]$. Assume the parameters $a, b, c,$ and d do not take on any special values, nor have a special relationship among them—that is row reduce naively, without worrying about any divisions by hidden zeros.

15 points

Solution

The naïve row reduction mentioned proceeds as follows:

$$\begin{aligned} \left[A | I \right] &= \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]^{aR2 - cR1} \sim \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad - bc & -c & a \end{array}\right]^{(ad-bc)R1 - bR2} \\ &\sim \left[\begin{array}{cc|cc} a(ad - bc) & 0 & ad & -ab \\ 0 & ad - bc & -c & a \end{array}\right]^{R1/a} \sim \left[\begin{array}{cc|cc} ad - bc & 0 & d & -b \\ 0 & ad - bc & -c & a \end{array}\right]^{R1/a} \\ &\sim \left[\begin{array}{cc|c} 1 & 0 & \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ 0 & 1 & \end{array}\right]^{R1/a} = \left[I | A^{-1} \right]. \end{aligned} \tag{1.23}$$

9. When is the product of two symmetric matrices symmetric? Prove this. (Assume that the product of the two matrices makes sense, i.e. assume the two matrices are the same size.)

15 points

Solution

The product AB of two matrices A and B is, by definition, symmetric if and only if $AB = (AB)^T$. On the other hand we have proved that $(AB)^T = B^T A^T$, which, under the present circumstances, is the product BA. Thus, if the matrices A and B are symmetric, their product is symmetric if and only if $AB = BA$, i.e. if and only if A and B commute.

10. As in problem 7, let B be the matrix

$$\begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix}, \tag{1.24}$$

and let $P(X) := (X+I)(X-6I)$, where X is any 2 by 2 matrix, and where I denotes the identity. Calculate $P(B) = (B+I)(B-6I)$ (it's easier to calculate as written—do not expand the polynomial factors unless you want to work harder than necessary). Does the result surprise you (given the outcomes of problem 7)?

15 points

Solution

We have

$$\begin{gathered} B + I = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \\ \text{and} \\ B - 6I = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} + \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix}, \end{gathered} \tag{1.25}$$

so that

$$\begin{aligned} P(B) = (B+I)(B-6I) &= \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix} = \begin{bmatrix} 3(-4)+2(6) & 3(2)+2(-3) \\ 6(-4)+4(6) & 6(2)+4(-3) \end{bmatrix} \\ &= \begin{bmatrix} -12+12 & 6-6 \\ -24+24 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{1.26}$$