Math 343 Midterm III KEY
Fall 2006
sections 002 and 003
Instructor: Scott Glasgow

Please do NOT write on this exam. No credit will be given for such work. Rather write in a blue book, or on your own paper, preferably engineering.

1. Find the orthogonal projection of $\mathbf{u}=(6,1,5,2)$ onto the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1=(1,2,1,2)$, $\mathbf{v}_2=(2,3,2,1)$ and $\mathbf{v}_3=(1,3,3,1)$.

15 points

<u>Solution</u>

The projection $\mathbf{u}_{\|}$ is $\mathbf{u}=(6,1,5,2)$ iff $\mathbf{u}\in Span\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, otherwise it is the vector $\mathbf{u}_{\|}\in Span\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ nearest $\mathbf{u}$. Actually, in either case, we have

$$\mathbf{u}_{\|}=A\mathbf{x} \tag{1.1}$$

where $\mathbf{x}$ is *any* solution of

$$A^T A\mathbf{x}=A^T\mathbf{u}\,,^1 \tag{1.2}$$

and where

$$A=[\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3]=\begin{bmatrix}1&2&1\\2&3&3\\1&2&3\\2&1&1\end{bmatrix}. \tag{1.3}$$

We solve (1.2) by row reducing $\left[A^T A\middle| A^T\mathbf{u}\right]$ after calculating $A^T A$ and $A^T\mathbf{u}$:

---

[1] The vector $\mathbf{u}_{\|}=A\mathbf{x}$ is unique even if $\mathbf{x}$ solving (1.2) isn't: Let $\mathbf{x}_1$ and $\mathbf{x}_2$ both solve (1.2). Then $\mathbf{d}=\mathbf{x}_1-\mathbf{x}_2$ satisfies $A^T A\mathbf{d}=\mathbf{0}$, so that either $\mathbf{d}\in Nul(A)$ or $A\mathbf{d}\in Nul(A^T)=(ColA)^{\perp}$. In the first case $\mathbf{u}_{\|1}-\mathbf{u}_{\|2}=A\mathbf{x}_1-A\mathbf{x}_2=A\mathbf{d}=\mathbf{0}$ and we are done. In the second case $A\mathbf{d}=\mathbf{0}\in(ColA)^{\perp}$ also since otherwise

$$\|\mathbf{u}_{\perp 1}\|^2=\|\mathbf{u}-\mathbf{u}_{\|1}\|^2=\left\|(\mathbf{u}-\mathbf{u}_{\|2})+(\mathbf{u}_{\|2}-\mathbf{u}_{\|1})\right\|^2=\left\|\mathbf{u}_{\perp 2}+(\mathbf{u}_{\|2}-\mathbf{u}_{\|1})\right\|^2=\|\mathbf{u}_{\perp 2}\|^2+\|\mathbf{u}_{\|2}-\mathbf{u}_{\|1}\|^2$$
$$\|\mathbf{u}_{\perp 2}\|^2+\|A\mathbf{d}\|^2>\|\mathbf{u}_{\perp 2}\|^2,$$

so that $\|\mathbf{u}_{\perp 1}\|^2$ and $\|\mathbf{u}_{\perp 2}\|^2$ are not both least.

$$A^T A=\begin{bmatrix}1&2&1&2\\2&3&2&1\\1&3&3&1\end{bmatrix}\begin{bmatrix}1&2&1\\2&3&3\\1&2&3\\2&1&1\end{bmatrix}=\begin{bmatrix}10&12&12\\12&18&18\\12&18&20\end{bmatrix},\ A^T\mathbf{u}=\begin{bmatrix}1&2&1&2\\2&3&2&1\\1&3&3&1\end{bmatrix}\begin{bmatrix}6\\1\\5\\2\end{bmatrix}=\begin{bmatrix}17\\27\\26\end{bmatrix},$$

$$\left[A^T A\middle|A^T\mathbf{u}\right]=\left[\begin{array}{ccc|c}10&12&12&17\\12&18&18&27\\12&18&20&26\end{array}\right]\sim\left[\begin{array}{ccc|c}10&12&12&17\\2&6&6&10\\2&6&8&9\end{array}\right]\sim\left[\begin{array}{ccc|c}1&3&3&5\\10&12&12&17\\2&6&8&9\end{array}\right]\sim\left[\begin{array}{ccc|c}1&3&3&5\\0&18&18&33\\0&0&2&-1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}1&3&3&5\\0&6&6&11\\0&0&2&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}1&3&3&5\\0&6&0&14\\0&0&2&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}2&6&6&10\\0&3&0&7\\0&0&2&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}2&6&0&13\\0&3&0&7\\0&0&2&-1\end{array}\right]\sim\left[\begin{array}{ccc|c}2&0&0&-1\\0&3&0&7\\0&0&2&-1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}6&0&0&-3\\0&6&0&14\\0&0&6&-3\end{array}\right].$$

(1.4)

Thus

$$\mathbf{u}_{\parallel}=A\mathbf{x}=\frac{1}{6}\begin{bmatrix}1&2&1\\2&3&3\\1&2&3\\2&1&1\end{bmatrix}\begin{bmatrix}-3\\14\\-3\end{bmatrix}=\frac{1}{6}\begin{bmatrix}22\\27\\16\\5\end{bmatrix}=\begin{bmatrix}11/3\\9/2\\8/3\\5/6\end{bmatrix}. \tag{1.5}$$

2. Use the Gram-Schmidt process, etc., to transform the following members of the Euclidean space $\mathbb{R}^3$ into an orthonormal set in that innerproduct space:

$$\mathbf{u}_1=(1,1,1),\mathbf{u}_2=(0,2,2),\text{ and }\mathbf{u}_3=(0,0,3). \tag{1.6}$$

15 points

<u>Solution</u>

After the required normalization, only the directions, not sizes, of the original vectors enters in. Thus, for simplicity of calculation, and without change of notation, we immediately exchange the set (1.6) for the following:

$$\mathbf{u}_1=(1,1,1),\mathbf{u}_2=(0,1,1),\text{ and }\mathbf{u}_3=(0,0,1). \tag{1.7}$$

Then we produce an orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ via (a modified version of) Gram-Schmidt:

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 = (1,1,1) \\
\mathbf{v}_2 &\propto \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (0,1,1) - \frac{2}{3}(1,1,1) = \frac{1}{3}(-2,1,1) \propto (-2,1,1) = \mathbf{v}_2 \\
\mathbf{v}_3 &\propto \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{6}(-2,1,1) \\
&= \frac{1}{6}(0,-3,3) = (0,-1,1) = \mathbf{v}_3.
\end{aligned}
\tag{1.8}
$$

Finally we normalize each vector to obtain the required orthonormal set $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$:

$$
\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{3}}(1,1,1),\ \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{6}}(-2,1,1),\ \mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{2}}(0,-1,1). \tag{1.9}
$$

3. Find the transition matrix $P_{B'B}$ from the $\mathbb{R}^3$ basis $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ to the $\mathbb{R}^3$ basis $B' = \{(1,2,4),(2,5,8),(3,6,13)\}$.

15 points

<u>Solution</u>

Let $(\mathbf{v})_B$ and $(\mathbf{v})_{B'}$ be coordinate vectors with respect to the bases B and B' for the vector $\mathbf{v}$. Then, by definition of coordinate vector,

$$
B(\mathbf{v})_B = B'(\mathbf{v})_{B'} = \mathbf{v}, \tag{1.10}
$$

where, abusing notation, B and B' here represent matrices containing the vectors of the bases B and B' as columns. Thus B in (1.10) is the identity matrix. According to (1.10) then,

$$
(\mathbf{v})_{B'} = B'^{-1}(\mathbf{v})_B =: P_{B'B}(\mathbf{v})_B. \tag{1.11}
$$

We calculate the inverse required in (1.11) by row reducing $\left[B' \middle| I\right]$ to $\left[I \middle| B'^{-1}\right] = \left[I \middle| P_{B'B}\right]$:

$$\left[B'\middle|I\right]=\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\2&5&6&0&1&0\\4&8&13&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&0&-2&1&0\\0&0&1&-4&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&2&0&13&0&-3\\0&1&0&-2&1&0\\0&0&1&-4&0&1\end{array}\right]$$
$$\sim\left[\begin{array}{ccc|ccc}1&0&0&17&-2&-3\\0&1&0&-2&1&0\\0&0&1&-4&0&1\end{array}\right]=\left[I\middle|P_{B'B}\right]. \tag{1.12}$$

4. Prove that if $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ is an orthonormal basis for an innerproduct space V, then, for any vector $\mathbf{v}\in V$,

$$\mathbf{v}=\langle\mathbf{v},\mathbf{v}_1\rangle\mathbf{v}_1+\langle\mathbf{v},\mathbf{v}_2\rangle\mathbf{v}_2+\ldots+\langle\mathbf{v},\mathbf{v}_n\rangle\mathbf{v}_n. \tag{1.13}$$

15 points

Solution

By previous theorem, since $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ is a basis for V, we immediately have that for any $\mathbf{v}\in V$ there exists one and only one coordinate vector $(c_1,c_2,\ldots,c_n)\in\mathbb{R}^n$ such that

$$\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n. \tag{1.14}$$

Taking the innerproduct of both sides of (1.14) with $\mathbf{v}_j=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$, we get

$$\begin{aligned}\langle\mathbf{v},\mathbf{v}_j\rangle&=\langle\mathbf{v},c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n\rangle=c_1\langle\mathbf{v}_1,\mathbf{v}_j\rangle+c_2\langle\mathbf{v}_2,\mathbf{v}_j\rangle+\ldots+c_n\langle\mathbf{v}_n,\mathbf{v}_j\rangle\\&=0+0+\ldots0+c_j\langle\mathbf{v}_j,\mathbf{v}_j\rangle+0+\ldots+0=c_j,\end{aligned} \tag{1.15}$$

so that (1.13) follows.

5. Find the angle $\theta\in[0,\pi]$ between $\mathbf{u}=(1,-3)$ and $\mathbf{v}=(2,4)$ in the Euclidean innerproduct space $\mathbb{R}^2$.

15 points

Solution

By (good) definition,

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) = \cos^{-1}\left(\frac{-10}{\sqrt{20}\sqrt{10}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3}{4}\pi. \tag{1.16}$$

6. If possible, find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}. \tag{1.17}$$

If such a matrix P exists, what is the associated diagonalized form of A?

15 points

Solution

We seek an invertible matrix P with eigenvectors of A as its columns. First we calculate A's eigenvalues:

$$\begin{aligned}
&\lambda \text{ is an eigenvalue of } A \text{ iff} \\
&0 = \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} = -\lambda(2-\lambda)(3-\lambda) - 2\big(0 - 1(2-\lambda)\big) \\
&= (2-\lambda)\big(-\lambda(3-\lambda)+2\big) = (2-\lambda)\big(\lambda^2 - 3\lambda + 2\big) = (2-\lambda)(\lambda-2)(\lambda-1) \\
&\Leftrightarrow \\
&\lambda = 1, 2, 2.
\end{aligned} \tag{1.18}$$

The associated eigenvectors are elements of the nontrivial nullspaces of $A - \lambda I$:

$$\mathbf{0} \neq \mathbf{x}_{\lambda=1} \in Nul\left(A-1\cdot I\right) = Nul\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = Nul\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = Nul\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= Span\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}, \tag{1.19}$$

$$\mathbf{0} \neq \mathbf{x}_{\lambda=2} \in Nul\left(A-2\cdot I\right) = Nul\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = Nul\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = Span\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

Thus

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \tag{1.20}$$

diagonalizes A to

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.21}$$

7. Find a basis of the orthogonal complement of the column space of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}. \tag{1.22}$$

15 points

<u>Solution</u>

By theorem

$$\left(ColA\right)^{\perp} = NulA^T = Nul\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = Nul\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} = Nul\begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= Nul\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = Span\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}. \tag{1.23}$$

8. Let A be an nxn matrix. Prove that if

$$A\mathbf{x}\cdot A\mathbf{y} = \mathbf{x}\cdot\mathbf{y} \tag{1.24}$$

holds for every $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, then A is orthogonal.

15 points

Solution

$$\mathbf{x}^T I\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x}\cdot\mathbf{y} = A\mathbf{x}\cdot A\mathbf{y} = \left(A\mathbf{x}\right)^T A\mathbf{y} = \mathbf{x}^T A^T A\mathbf{y}$$
$$\Leftrightarrow \tag{1.25}$$
$$0 = \mathbf{x}^T A^T A\mathbf{y} - \mathbf{x}^T I\mathbf{y} = \mathbf{x}^T\left(A^T A - I\right)\mathbf{y}.$$

Since (1.25) holds for $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, it holds for $\mathbf{x} = \left(A^T A - I\right)\mathbf{y}$, so that (1.25) becomes

$$0 = \left(\left(A^T A - I\right)\mathbf{y}\right)^T\left(A^T A - I\right)\mathbf{y} = \left\|\left(A^T A - I\right)\mathbf{y}\right\|^2$$
$$\Leftrightarrow \tag{1.26}$$
$$\mathbf{0} = \left(A^T A - I\right)\mathbf{y}$$

for every $\mathbf{y}\in\mathbb{R}^n$. Thus (1.26) holds for $\mathbf{y}$ chosen as each of the rows of the matrix $A^T A - I$, showing that each of these rows is the zero vector and, so, showing that $A^T A - I$ is the zero matrix, i.e. $A^T A = I$, which, together with A square, means that A is an orthogonal matrix.

9. Prove the Best Approximation Theorem: If W is a finite-dimensional subspace of an innerproduct space V, and if $\mathbf{u}\in V$, then the orthogonal projection of $\mathbf{u}$ onto W, denoted $proj_W\mathbf{u}$ or, say, $\mathbf{u}_{\parallel}$, is the best approximation to $\mathbf{u}$ in the sense that

$$\|\mathbf{u}-\mathbf{u}_{||}\| < \|\mathbf{u}-\mathbf{w}\| \tag{1.27}$$

for every vector $\mathbf{w} \in W$ that is different from $\mathbf{u}_{||}$. (Hint: Recall $\mathbf{u}_{||}$ can be defined by demanding that $\mathbf{u}-\mathbf{u}_{||}$ is in the orthogonal complement of W.)

15 points

Solution

For every $\mathbf{w} \in W$, $\mathbf{u}-\mathbf{u}_{||}$ is, by definition of $\mathbf{u}_{||}$, orthogonal to $\mathbf{w}$, and, since $\mathbf{u}_{||} \in W$, $\mathbf{u}-\mathbf{u}_{||}$ is also orthogonal to $\mathbf{u}_{||}-\mathbf{w} \in W$. Thus, by Pythagoras,

$$\begin{aligned} \|\mathbf{u}-\mathbf{w}\|^2 &= \left\|\left(\mathbf{u}-\mathbf{u}_{||}\right)+\left(\mathbf{u}_{||}-\mathbf{w}\right)\right\|^2 = \|\mathbf{u}-\mathbf{u}_{||}\|^2 + \|\mathbf{u}_{||}-\mathbf{w}\|^2 \\ &> \|\mathbf{u}-\mathbf{u}_{||}\|^2 \\ &\Leftrightarrow \\ \|\mathbf{u}-\mathbf{w}\| &> \|\mathbf{u}-\mathbf{u}_{||}\|, \end{aligned} \tag{1.28}$$

where we used that $\mathbf{w}$ is different than $\mathbf{u}_{||}$.

10. Prove that if A is an orthogonal $n\text{x}n$ matrix, then

$$\|A\mathbf{x}\| = \|\mathbf{x}\| \tag{1.29}$$

for every $\mathbf{x} \in \mathbb{R}^n$.

15 points

Solution

For every $\mathbf{x} \in \mathbb{R}^n$ we have

$$\begin{aligned} \|A\mathbf{x}\|^2 &= \left(A\mathbf{x}\right)^T A\mathbf{x} = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T I\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2 \\ &\Leftrightarrow \\ \|A\mathbf{x}\| &= \|\mathbf{x}\|. \end{aligned} \tag{1.30}$$