Math 343 Midterm II
Fall 2006
sections 002 and 003
Instructor: Scott Glasgow

Please do NOT write on this exam. No credit will be given for such work. Rather write in a blue book, or on your own paper, preferably engineering.

1. Let $\mathbf{u} = (x, -1, 4)$, $\mathbf{v} = (-3, 1, 3x)$, and $\mathbf{w} = (6, -2, -4)$. Find two values of x for which the set $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is NOT a basis for $\mathbb{R}^3$.

15 points

Solution

S is a basis for $\mathbb{R}^3$ iff it spans $\mathbb{R}^3$ and is linearly independent. Both of these properties hold iff the square matrix $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ is nonsingular, which holds iff $\det[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] \neq 0$. Thus to insure that we do not get a basis it is necessary and sufficient to demand that

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$$0 = \det[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \det\begin{bmatrix} x & -3 & 6 \\ -1 & 1 & -2 \\ 4 & 3x & -4 \end{bmatrix} = x\det\begin{bmatrix} 1 & -2 \\ 3x & -4 \end{bmatrix} - (-3)\det\begin{bmatrix} -1 & -2 \\ 4 & -4 \end{bmatrix} + 6\det\begin{bmatrix} -1 & 1 \\ 4 & 3x \end{bmatrix}$$

$$= x(-4+6x) + 3(4+8) + 6(-3x-4) = 6x^2 - 22x + 12 = 2(3x^2 - 11x + 6) = 2(3x-2)(x-3)$$

$\Leftrightarrow$

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$x = 2/3$ or 3.

$$(1.1)$$

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2. Determine the standard matrix for the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ that first rotates a vector counterclockwise about the y axis through an angle θ, then reflects the resulting vector about the xy-plane, and then projects the latter resulting vector orthogonally onto the xz plane. If you do not recall the form of the standard matrices for these three transformations, recall the theorem (mnemonic)

$$[T] = [T\mathbf{e}_1\ T\mathbf{e}_2\ T\mathbf{e}_3], \tag{1.2}$$

where the indicated (boldface column) vectors are the standard basis elements of $\mathbb{R}^3$.

15 points

Solution

The linear transformation T can be expressed as the composition $T_3 \circ T_2 \circ T_1$, where T_1 is the rotation, T_2 is the reflection, and T_3 is the projection. Thus the standard matrix $[T]$ is the following product of the standard matrices of the other transformations: $[T_3][T_2][T_1]$.

By considering the action of these three transformations on the standard basis elements, one finds that

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$$[T_3]=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix},\ [T_2]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix},\ \text{and } [T_1]=\begin{bmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{bmatrix}. \qquad (1.3)$$

Thus

correct order 5

$$[T]=[T_3][T_2][T_1]=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{bmatrix}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}\cos\theta&0&\sin\theta\\0&1&0\\\sin\theta&0&-\cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos\theta&0&\sin\theta\\0&0&0\\\sin\theta&0&-\cos\theta\end{bmatrix}. \qquad (1.4)$$

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3. What is the Wronskian of the functions x^2, x^3, and x^4? Do these functions form a linearly independent set (in the vector space of real-valued functions of a single real variable, with the standard addition and scalar multiplication)?

15 points

Solution

By the definition of the Wronskian we have

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$$W=\det\begin{bmatrix}x^2&x^3&x^4\\2x&3x^2&4x^3\\2&6x&12x^2\end{bmatrix}=x^2\cdot x\det\begin{bmatrix}1&x&x^2\\2&3x&4x^2\\2&6x&12x^2\end{bmatrix}=x^3\cdot x\cdot x^2\det\begin{bmatrix}1&1&1\\2&3&4\\2&6&12\end{bmatrix}=x^6\det\begin{bmatrix}1&1&1\\0&1&2\\0&4&10\end{bmatrix}$$

$$=x^6\det\begin{bmatrix}1&1&1\\0&1&2\\0&0&2\end{bmatrix}=2x^6. \qquad (1.5)$$

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Since the Wronskian is not the zero function, the set indicated is linearly independent.

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4. Let $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ be a basis for a vector space V. Prove that for any vector $\mathbf{v}\in V$ there exists one AND only one coordinate vector $(c_1,c_2,\ldots,c_n)\in\mathbb{R}^n$ such that $\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n$.

about half each

15 points

Solution

-2 "or's pans"

By definition of basis we immediately have that for any $\mathbf{v}\in V$ there exists at least one coordinate vector $(c_1,c_2,\ldots,c_n)\in\mathbb{R}^n$ such that $\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n$. As for uniqueness of this coordinate vector, and by way of contradiction, assume that there is another one, say $(k_1,k_2,\ldots,k_n)\in\mathbb{R}^n$. Then

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$$\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n=k_1\mathbf{v}_1+k_2\mathbf{v}_2+\ldots+k_n\mathbf{v}_n \tag{1.6}$$

so that

$$\begin{aligned}(c_1-k_1)\mathbf{v}_1+(c_2-k_2)\mathbf{v}_2+\ldots+(c_n-k_n)\mathbf{v}_n&=\mathbf{0}\\ &\Leftrightarrow\\ c_1-k_1=c_2-k_2=\ldots=c_n-k_n&=0\\ &\Leftrightarrow\\ c_1=k_1,c_2=k_2,\ldots,c_n&=k_n,\end{aligned} \tag{1.7}$$

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the first equivalence holding since S is independent. The last statement indicates then that $(c_1,c_2,\ldots,c_n)=(k_1,k_2,\ldots,k_n)$, i.e. the "second" coordinate vector was only an alias for the first.

5. Find the ***unique*** reduced row echelon form of the following matrix, identify its rank, and find bases for its row and column space ***by using its unique reduced row echelon form***:

$$A=\begin{bmatrix}1&-3&4&-2&5&4\\2&-6&9&-1&8&2\\2&-6&9&-1&9&7\\-1&3&-4&2&-5&-4\end{bmatrix}. \tag{1.8}$$

15 points

Solution

Since the row space is unchanged by elementary row operations, and since a basis for the row space of a row echelon matrix is obvious (one can always be formed from the rows containing leading ones), we should row reduce A to form such a echelon matrix R and read off the desired basis for the row space of A directly from R. Similarly, but not identically, one can read off a basis for the column space of R, and take note of the location of this special set in R, using then the corresponding column vectors in A as a basis for the column space of A. Thus for both bases we may begin by row reducing A. Continuing to reduced row echelon form the row reduction produces

$$R=\begin{bmatrix} 1 & -3 & 0 & -14 & 0 & -37 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (1.9)$$

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Rank is clearly 3

Thus a basis for the row space of A (not just R) is the set

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$$5 \quad \{(1,-3,0,-14,0,-37),\ (0,0,1,3,0,4),(0,0,0,0,1,5)\}, \qquad (1.10)$$

while the first, third and fifth columns of A form a basis for the column space of A (since the corresponding columns of R form a basis for the column space of R).

6. Find a basis for the null-space of the matrix

$$R=\begin{bmatrix} 1 & -3 & 0 & -14 & 0 & -37 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (1.11)$$

15 points

Solution

Since the matrix is in reduced row echelon form, the desired basis is easily found by considering that the various rows of R are code for certain equations (involving the components of members of the null-space). By straightforward manipulation of those equations one finds that

$$Null(R)=\left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}\middle| 1x_1-3x_2-14x_4-37x_6=0, 1x_3+3x_4+4x_6=0, x_5+5x_6=0; x_2,x_4,x_6\in\mathbb{R}\right\}$$

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$$=\left\{\begin{bmatrix}3x_2+14x_4+37x_6\\x_2\\-3x_4-4x_6\\x_4\\-5x_6\\x_6\end{bmatrix}\middle| x_2,x_4,x_6\in\mathbb{R}\right\}=span\left\{\begin{bmatrix}3\\1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}14\\0\\-3\\1\\0\\0\end{bmatrix},\begin{bmatrix}37\\0\\-4\\0\\-5\\1\end{bmatrix}\right\}, \quad (1.12)$$

so that a basis is apparent in the last representation.

7. Let $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{v}$ denote elements of any vector space. Solve the equation $\mathbf{x}+\mathbf{u}=\mathbf{v}$ for $\mathbf{x}$, stating at each step which vector space axiom or algebraic property or definition you used to arrive at that step.

15 points

Solution

1 point off per incorrect or irrelevant/extra axiom/definition/algebraic property

$$\begin{aligned}
\mathbf{x}+\mathbf{u}&=\mathbf{v} && \\
(\mathbf{x}+\mathbf{u})+(-\mathbf{u})&=\mathbf{v}+(-\mathbf{u}) && \text{Add } -\mathbf{u} \text{ to both sides of the above equation}\\
\mathbf{x}+(\mathbf{u}+(-\mathbf{u}))&=\mathbf{v}+(-\mathbf{u}) && \text{Associative Axiom of Vector Addition}\\
\mathbf{x}+\mathbf{0}&=\mathbf{v}+(-\mathbf{u}) && \text{Additive Inverse Axiom}\\
\mathbf{x}&=\mathbf{v}+(-\mathbf{u}) && \text{Additive Identity Axiom}\\
\mathbf{x}&=\mathbf{v}-\mathbf{u} && \text{Definition of subtraction}
\end{aligned} \quad (1.13)$$

8. For $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$ prove the triangle inequality $\|\mathbf{u}+\mathbf{v}\|\le\|\mathbf{u}\|+\|\mathbf{v}\|$. Note in your calculation the step at which you use the Cauchy-Schwarz inequality.

15 points

Solution

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\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \overset{3}{=} \mathbf{u}\cdot\mathbf{u}+2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} \overset{3}{\le} \|\mathbf{u}\|^2+2|\mathbf{u}\cdot\mathbf{v}|+\|\mathbf{v}\|^2 \\
&\overset{3}{\le} \|\mathbf{u}\|^2+2\|\mathbf{u}\|\|\mathbf{v}\|+\|\mathbf{v}\|^2 \qquad \text{(Cauchy-Schwarz)} \\
&\overset{3}{=} \left(\|\mathbf{u}\|+\|\mathbf{v}\|\right)^2 \\
&\Leftrightarrow \\
\|\mathbf{u}+\mathbf{v}\| &\le \|\mathbf{u}\|+\|\mathbf{v}\| \quad 3
\end{aligned}
\tag{1.14}
$$

(the latter since the two squared quantities are nonnegative).

9. Let $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$ and let those vectors be orthogonal with respect to the Euclidean inner product. Prove the Pythagorean theorem $\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2$. Note in your calculation the step at which you use the vectors' orthogonality.

15 points

Solution

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\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &\overset{3}{=} (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \overset{3}{=} \mathbf{u}\cdot\mathbf{u}+2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} \overset{3}{=} \|\mathbf{u}\|^2+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2 \\
&\overset{3}{=} \|\mathbf{u}\|^2+2\cdot 0+\|\mathbf{v}\|^2 \qquad \text{(Orthogonality)} \\
&\overset{3}{=} \|\mathbf{u}\|^2+\|\mathbf{v}\|^2
\end{aligned}
\tag{1.15}
$$

10. Given

$$
A=\begin{bmatrix} 2 & 1 & 8 \\ 7 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix},\ \mathbf{u}=\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix},\ \text{and } \mathbf{v}=\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix},
\tag{1.16}
$$

compute both $(A\mathbf{u})\cdot\mathbf{v}$ and $\mathbf{u}\cdot\left(A^T\mathbf{v}\right)$.

15 points

Solution

We have

$$\left\{\begin{aligned}
(A\mathbf{u})\cdot\mathbf{v} &= \left(\begin{bmatrix}2&1&8\\7&1&2\\0&1&3\end{bmatrix}\begin{bmatrix}2\\1\\3\end{bmatrix}\right)\cdot\begin{bmatrix}4\\2\\1\end{bmatrix}=\begin{bmatrix}29\\21\\10\end{bmatrix}\cdot\begin{bmatrix}4\\2\\1\end{bmatrix}=29\cdot4+21\cdot2+10\cdot1=168,\\
\mathbf{u}\cdot\left(A^T\mathbf{v}\right) &= \begin{bmatrix}2\\1\\3\end{bmatrix}\cdot\left(\begin{bmatrix}2&7&0\\1&1&1\\8&2&3\end{bmatrix}\begin{bmatrix}4\\2\\1\end{bmatrix}\right)=\begin{bmatrix}2\\1\\3\end{bmatrix}\cdot\begin{bmatrix}22\\7\\39\end{bmatrix}=2\cdot22+1\cdot7+3\cdot39=168.
\end{aligned}\right. \tag{1.17}$$