Linear Algebra

Darrin Doud

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Contents

Chapter 1

Solving Linear Systems

Chapter 2

Determinants

2.1 Defining determinants

In the case of two-by-two matrices, we have seen that to each matrix there is an associated number, which entirely determines whether or not a matrix is invertible. We review this result:

Theorem 2.1. *Let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a 2×2 *matrix with real entries. Then* A *is invertible if and only if* $ad - bc \neq 0$. *When* A *is invertible, the inverse of* A *is given by the formula*

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof. If $ad - bc$ is nonzero, we may show that the given matrix is really A^{-1} (and hence that A is actually invertible) by multiplying them together.

Suppose then that A is invertible. We wish to show that $ad - bc \neq 0$. Clearly either a or b must be nonzero, since we have seen that a matrix with a row of zeros can not be invertible. We will deal with two cases:

1. Suppose that $a \neq 0$. We row reduce A as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{a} R_1} \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \xrightarrow{R_2 \to R_2 - cR_1} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{cb}{a} \end{bmatrix}$$

   If the entry $d - \frac{cb}{a}$ were equal to zero, then the matrix would be in reduced row echelon form, and we would see that A is not invertible. Since we are assuming that A is invertible, we see that this can not happen, so that $d - \frac{cb}{a} \neq 0$, or in other words, $ad - bc \neq 0$.

2. Suppose that $a = 0$. Then we know that $b \neq 0$. Note also that since A is invertible, we know that $c \neq 0$ (since otherwise A would contain a column of zeros, and would fail to be invertible). Then $ad - bc = 0d - bc = bc \neq 0$.

□

We will now define a function det from the set of square matrices to the set of real numbers such that a matrix A is invertible exactly when $\det(A)$ is nonzero. The definition will take up the rest of this section, and determining properties of det will be postponed until later.

2.1.1 Permutations

In order to define the determinant function, we will need to understand some simple facts about permutations. We begin with some definitions and very simple properties.

Definition 1. Let $S = \{1, \ldots, n\}$. A *permutation* of S is a function $\sigma : S \to S$ such that the following two properties hold:

1. If $i \in S$ and $j \in S$ are different, then $\sigma(i) \neq \sigma(j)$.
2. If $i \in S$ then $i = \sigma(j)$ for some $j \in S$.

One way to write a permutation is just to list what $\sigma(1), \ldots, \sigma(n)$ are. The result will be a rearrangement of the numbers from 1 through n, with every number appearing at least once (by property 2 of Definition 1), and no number appearing more than once (by property 1). For instance, for the permutation of $\{1, 2, 3\}$ with $\sigma(1) = 2$, $\sigma(2) = 3$ and $\sigma(3) = 1$, we might talk about the permutation $(2, 3, 1)$, or even just the permutation 231 (note that this last way of writing things breaks down when $n \geq 10$). When n is small, we can easily list all permutations of $\{1, \ldots, n\}$. For instance for $n = 3$, there are 6 permutations:

$$\begin{array}{ll} (1,2,3), & (1,3,2) \\ (2,1,3), & (2,3,1) \\ (3,1,2), & (3,2,1). \end{array}$$

Definition 2. S_n is the set of all permutations of the numbers $\{1, \ldots, n\}$.

Definition 3. Let $S = \{1, \ldots, n\}$, and let $\sigma : S \to S$ be a permutation. An **inversion** in σ is a pair $(\sigma(i), \sigma(j))$ with $i \in S$, $j \in S$, and $i < j$, such that $\sigma(i) > \sigma(j)$.

If we write our permutation as a rearrangement, we can easily count the number of inversions, by counting how often a larger number precedes a smaller one. We count the number of inversions in several permutations:

$n = 5$	32451	5 inversions (3,2), (3,1), (2,1), (4,1), (5,1).
$n = 5$	13245	1 inversion: (3,2).
$n = 5$	54123	7 inversions: (5,4), (5,3), (5,2), (5,1), (4,3), (4,2), (4,1).
$n = 4$	4321	6 inversions: (4,3), (4,2), (4,1), (3,2), (3,1), (2,1).

Note how organizing the counting makes it easier to keep track of which inversions have already been counted–we first count the number of inversions beginning with $\sigma(1)$ (the first number), then those beginning with $\sigma(2)$, etc.

Definition 4. A permutation is called *even* if it contains an even number of inversions. It is called *odd* if it contains an odd number of inversions. If σ is a permutation, we will define $\operatorname{sign}(\sigma)$ by the following rule:

$$\operatorname{sign}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

We are now prepared to define the determinant.

2.1.2 Definition of the determinant

Throughout this section, let A be an $n \times n$ matrix, and denote its (i, j) entry by a_{ij}.

Definition 5. Let $\sigma \in S_n$. The *elementary product* associated to σ is the product

$$a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

The *signed elementary product* associated to σ is the product

$$\operatorname{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

Possibly the most important fact about elementary products of A is that each elementary product contains exactly one element from every row of A, and exactly one element from every column of A. This fact will play an important role in many proofs of simple facts about determinants.

Example 2.1.1. Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and let $\sigma(1) = 3$, $\sigma(2) = 2$, and $\sigma(3) = 1$. Then the elementary product associated to σ is

$$a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)} = a_{13}a_{22}a_{31} = ceg$$

and since σ is odd, the signed elementary product associated to σ is $-ceg$.

Definition 6. Let A be an $n \times n$ matrix. The *determinant* of A is the sum of all the signed elementary products of A (as σ runs through all possible permutations). In symbols,

$$\det(A) = \sum_{\sigma \text{ in } S_n} \operatorname{sign}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

2.1.3 Two-by-two determinant

Let A be the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then $n = 2$. There are only two permutations of $\{1, 2\}$, namely 12 and 21. The permutation 12 is even and 21 is odd. We then obtain the formula

$$\det(A) = a_{11}a_{22} + (-1)a_{12}a_{21} = ad - bc$$

2.1.4 Three-by-three determinants

Let A be the matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

There are six permutations in S_n, giving the following signed elementary products:

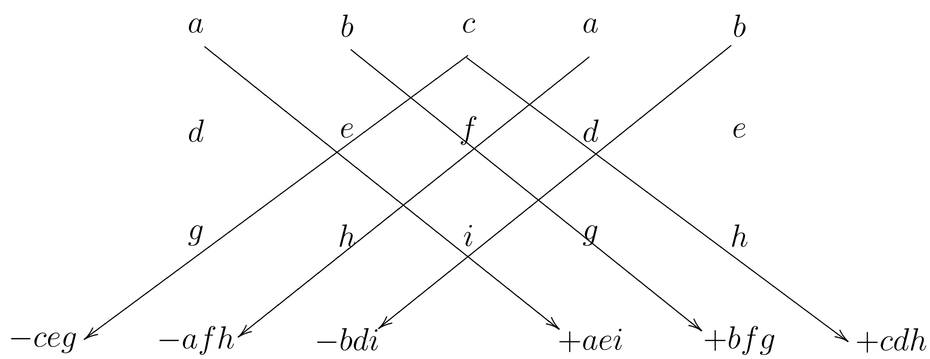
σ	sign(σ)	el. prod.	signed el. prod.
123	1	$a_{11}a_{22}a_{33} = aei$	aei
132	-1	$a_{11}a_{23}a_{32} = afh$	$-afh$
213	-1	$a_{12}a_{21}a_{33} = bdi$	$-bdi$
231	1	$a_{12}a_{23}a_{31} = bfg$	bfg
312	1	$a_{13}a_{21}a_{32} = cdh$	cdh
321	-1	$a_{13}a_{22}a_{31} = ceg$	$-ceg$

We see that that

$$\det(A) = aei - afh + bfg - bdi + cdh - ceg.$$

2.1.5 Shortcuts for computing small determinants

The formula for a 2×2 determinant is short and easy to remember. The formula for a 3×3 determinant is longer, involving six terms, each of which has three variables. In order to remember this formula, the following technique is useful. Write the matrix, and copy the first two columns to the right of the matrix, as below. Examine the diagonals indicated, and for the diagonals which go to the right, multiply the terms and add. For the diagonals which go to the left, multiply the terms and subtract.



It is very important to note that similar rules do not hold for 4×4 or larger matrices! You will get incorrect answers!

2.1.6 Larger Determinants

Note that there are $n! = n(n-1)(n-2)\cdots(2)(1)$ permutations in S_n. Hence, to compute the determinant of a four-by-four matrix would require adding together $4! = 24$ elementary products, and to compute the determinant of a 10×10 matrix would require adding together $10! = 3,628,800$ elementary products. Clearly, the definition is not a convenient way to compute large determinants! We will learn later how to compute determinant of large matrices efficiently.

2.1.7 Upper triangular matrices

Theorem 2.2. *Let A be an upper triangular matrix. Then* $\det(A)$ *is the product of the diagonal entries of A.*

Proof. Denote the (i, j) entry of A by a_{ij}, and note that if $j < i$ then $a_{ij} = 0$ (this is just the definition of upper triangular). Let σ be a permutation, and suppose that the elementary product

$$a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$$

is nonzero. Then each value $a_{i\sigma(i)}$ must be nonzero, so $\sigma(i) \geq i$ for every i from 1 to n. Since $\sigma(n) \geq n$, we must have $\sigma(n) = n$. Then since $\sigma(n-1) \geq n-1$, $\sigma(n-1)$ must equal either $n-1$ or n. Since $\sigma(n) = n$, $\sigma(n-1)$ cannot equal n, so $\sigma(n-1) = n-1$. Similarly, $\sigma(n-2) = n-2$, and in general, $\sigma(i) = i$. Hence, we see that the only permutation which yields a nonzero elementary product is $(1, 2, \ldots, n)$. This is an even permutation, so when we add all the signed elementary products together, we get

$$\det(A) = a_{11}a_{22}\cdots a_{nn}$$

which is just the product of the diagonal entries. □

2.1.8 Exercises

Exercise 2.1.1. Compute the following determinants:

1. $\det\left(\begin{pmatrix} 4 & 5 \\ 3 & 2 \end{pmatrix}\right)$

2. $\det\left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}\right)$

Exercise 2.1.2. Prove that a matrix with a row of zeros has determinant equal to zero.

Solution: Suppose that row i consists completely of zeros. every elementary product will contain a factor of the form $a_{i,\sigma(i)}$, which will be zero (since it comes from row i). Hence, every elementary product will be zero, so the sum of the signed elementary products will be zero. Thus, $\det(A) = 0$.

Exercise 2.1.3. Prove that the determinant of a lower triangular matrix is the product of the diagonal entries.

Exercise 2.1.4. Prove that if the entries of a matrix are all integers, then the detrminant is also an integer.

Exercise 2.1.5. Prove that if the determinant of a matrix A is d, and B is a matrix obtained from A by multiplying all the elements in one row by a number k, then $\det(B) = k\det(A)$.

Exercise 2.1.6. If A is an $n \times n$ matrix, prove that $\det(kA) = k^n \det(A)$.

Exercise 2.1.7. Compute the determinant of the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 1 & 1 \end{pmatrix}.$$

2.2 Cofactor Expansions

One way to think of elementary products is as products of entries of the matrix, with exactly one from every row and exactly one from every column. We will use this idea to write the determinant of an $n \times n$ matrix in terms of smaller determinants, which will be easier to compute.

Definition 7. Let A be an $n \times n$ matrix. We define the (i, j) *minor* M_{ij} of A to be the determinant of the matrix obtained from A by deleting the ith row and the jth column.

For instance, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

the minors of A are

$$M_{11} = \det\left(\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}\right) = -3, \quad M_{12} = \det\left(\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}\right) = -6, \quad M_{13} = \det\left(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}\right) = -3$$
$$M_{21} = \det\left(\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}\right) = -6, \quad M_{22} = \det\left(\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}\right) = -12, \quad M_{23} = \det\left(\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}\right) = -6$$
$$M_{31} = \det\left(\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}\right) = -3, \quad M_{32} = \det\left(\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}\right) = -6, \quad M_{33} = \det\left(\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\right) = -3$$

Definition 8. Let A be an $n \times n$ matrix. We define the (i, j) *cofactor* C_{ij} of A to be $(-1)^{i+j} M_{ij}$.

For the matrix A above, we have the cofactors:

$$\begin{array}{lll} C_{11} = -3, & C_{12} = 6, & C_{13} = -3 \\ C_{21} = 6, & C_{22} = -12, & C_{23} = 6 \\ C_{31} = -3, & C_{32} = 6, & C_{33} = -3 \end{array}$$

Notice that if we take a_{ij} and multiply it by C_{ij}, we will get a sum that includes all the elementary products of A that include a_{ij}. This is easy to see, because any such elementary product must be a_{ij} times an elementary product of M_{ij}. It is not quite so easy to see (and the proof will be delayed until later–see Theorem 2.8) that to get the signs right on all these elementary products, we need to multiply by $(-1)^{i+j}$. We then get the following theorem:

Theorem 2.3. *Let A be an $n \times n$ matrix. Choose an integer i between 1 and n. Then*

$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}M_{ij}$$

Theorem 2.4. *Let A be an $n \times n$ matrix. Choose an integer j between 1 and n. Then*

$$\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij} = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}M_{ij}$$

Proof. The sums certainly include all the elementary products, and as mentioned above (and proven later in Theorem 2.8), they include them with the correct signs. Hence, the sums are in fact equal to the determinant of A. □

Definition 9. If we compute $\det(A)$ using Theorem 2.3, we say that we have *expanded the determinant on row i*. Similarly, if we compute $\det(A)$ using Theorem 2.4, we say that we have *expanded the determinant on column j*.

To keep track of the signs in the sums it is convenient to fill in a matrix with a checkerboard pattern of pluses and minuses:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and then the sign in front of $a_{ij}M_{ij}$ is just the sign in the (i, j) position of the matrix above.

Example 2.2.1. We compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

in several different ways. We first choose to expand the determinant on row number 2. Then we have that

$$\begin{aligned} \det(A) &= (-1)(4)\det\left(\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}\right) + (1)(5)\det\left(\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}\right) + (-1)(6)\det\left(\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}\right) \\ &= (-4)(-6) + 5(-12) + (-6)(-6) \\ &= 24 - 60 + 36 \\ &= 0 \end{aligned}$$

We may also choose to expand the determinant on column 3, obtaining

$$\begin{aligned}\det(A) &= (1)(3)\det\left(\begin{bmatrix}4 & 5\\ 7 & 8\end{bmatrix}\right) + (-1)(6)\det\left(\begin{bmatrix}1 & 2\\ 7 & 8\end{bmatrix}\right) + (1)(9)\det\left(\begin{bmatrix}1 & 2\\ 4 & 5\end{bmatrix}\right)\\ &= (3)(-3) + (-6)(-6) + (9)(-3)\\ &= -9 + 36 - 27\\ &= 0\end{aligned}$$

One benefit of using cofactor expansions to compute determinants is that we may often choose a row or column that has many zeros, thus reducing the amount of work that we have to do.

Example 2.2.2. We compute the determinant of

$$A = \begin{bmatrix}1 & 2 & 3 & 4 & 0\\ 0 & 1 & 0 & 0 & 0\\ 2 & 7 & 1 & 2 & 3\\ 1 & 1 & 0 & 2 & 0\\ 2 & 3 & 0 & 2 & 0\end{bmatrix}$$

by expanding on the second row:

$$\begin{aligned}\det(A) &= 1\det\left(\begin{bmatrix}1 & 3 & 4 & 0\\ 2 & 1 & 2 & 3\\ 1 & 0 & 2 & 0\\ 2 & 0 & 2 & 0\end{bmatrix}\right)\\ &= (1)(3)\det\left(\begin{bmatrix}1 & 3 & 4\\ 1 & 0 & 2\\ 2 & 0 & 2\end{bmatrix}\right)\\ &= (1)(3)(-3)\det\left(\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}\right)\\ &= (1)(3)(-3)(2-4)\\ &= 18.\end{aligned}$$

We will now use cofactor expansion to prove a useful theorem about determinants, which will then be used to compute inverse matrices.

Theorem 2.5. *Let A be an $n \times n$ matrix, with two identical rows or columns. Then* $\det(A) = 0$.

Proof. We prove this theorem for matrices with two identical rows by induction. First, we note that it is obviously true for 2×2 matrices, since

$$\det\left(\begin{bmatrix}a & b\\ a & b\end{bmatrix}\right) = ab - ba = 0.$$

Now we will assume that we have already proven the theorem for $n \times n$ matrices, and we will prove it for $(n+1) \times (n+1)$ matrices.

Suppose that A is $(n+1) \times (n+1)$ and has two identical rows. Expand the determinant along a row which is not one of the two identical rows. Each minor will then be a determinant of an $n \times n$ submatrix, and each of these submatrices will have two identical rows. Hence, by induction, each minor in the expansion of the determinant will be zero, so the determinant will be zero. □

Definition 10. Let A be an $n \times n$ matrix. The *adjoint* of A, denoted $\operatorname{adj}(A)$ is the $n \times n$ matrix whose (i,j) entry is C_{ji}.

Note that if we compute all the cofactors, C_{ij} and make a matrix whose (i,j) entry is C_{ij}, then the adjoint is just the transpose of this matrix.

The adjoint of A has the following special property.

Theorem 2.6. *Let A be an $n \times n$ matrix. Then the product $A \operatorname{adj}(A) = \det(A)I$.*

Proof. The (i,j) entry of $A \operatorname{adj}(A)$ is the dot product of the ith row of A, or $(a_{i1}, a_{i2}, \ldots, a_{in})$ with the jth column of $\operatorname{adj}(A)$, or

$$\begin{pmatrix} C_{j1} \\ C_{j2} \\ \vdots \\ C_{jn} \end{pmatrix}$$

This product is just

$$a_{i1}C_{j1} + \cdots + a_{in}C_{jn} = \sum_{k=1}^{n} a_{ik}C_{jk}.$$

Note that if $i = j$, then this is just the cofactor expansion of $\det(A)$ on the ith row. If $i \neq j$, however, then we can see that this is the cofactor expansion (on the jth row) of the determinant of a different matrix B_{ij}, which is obtained by replacing the jth row of A with the ith row. Since B_{ij} has two identical rows, $\det(B_{ij}) = 0$. Hence, we have that $A \operatorname{adj}(A) = \det(A)I$. □

Theorem 2.7. *Let A be an $n \times n$ matrix. Then if $\det(A) \neq 0$, A is invertible, and $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$.*

Proof. This follows immediately from Theorem 2.6. □

Note that it is also true, and will be proved later, that if A is invertible, then $\det(A) \neq 0$.

2.2.1 Elementary products and cofactors–getting the signs right

In this subsection, we do a rather subtle bit of work to show that the signs on the elementary products in the cofactor expansion are the same as the signs on the signed elementary products in the definition of the determinant. This proof is more involved than most of the rest of the book, so we put it in this optional subsection.

Theorem 2.8. *Let A be an $n \times n$ matrix, and let A_{ij} be the $(n-1) \times (n-1)$ submatrix obtained by deleting the ith row and jth column. The signed elementary products arising in the product*

$$(-1)^{i+j} a_{ij} \det(A_{ij})$$

all have the same sign as they do in the definition of $\det(A)$*.*

Proof. We begin by choosing a row i and a column j of A. Deleting this row and column leaves an $(n-1) \times (n-1)$ matrix A_{ij}. Let $\sigma \in S_{n-1}$, so that σ defines an elementary product of A_{ij}. We define a representation $\tau \in S_n$ by

$$\tau(k) = \begin{cases} \sigma(k) & \text{if } k < i \text{ and } \sigma(k) < j \\ \sigma(k+1) & \text{if } k > i \text{ and } \sigma(k+1) < j \\ \sigma(k)+1 & \text{if } k < i \text{ and } \sigma(k) > j \\ \sigma(k+1)+1 & \text{if } k > i \text{ and } \sigma(k) > i. \end{cases}$$

Note that although the definition of τ looks quite complicated, it is really just a permutation in S_n, which gives the elementary product of A resulting from multiplying a_{ij} by the elementary product of A_{ij} coming from σ. Note that every inversion of σ will be an inversion of τ, and that every inversion of τ not involving j will correspond to an inversion of σ. So, to compare the signs of σ and τ, we just need to count how many inversions of τ involve j.

Note that there are $n-j$ elements of $\{1, \ldots, n\}$ larger than j, and there are $i-1$ spots where they could be put before j. Let k be the number of elements greater than j which come before j in the permutation θ. Then $n-j-k$ elements greater than j follow j, but there are $n-i$ spots following j, so $(n-i)-(n-j-k)$ elements smaller than j must follow j. Hence θ has

$$k + (n-i) - (n-j-k) = j - i + 2k = i + j + 2(k-i)$$

inversions more than τ does. Hence, when we multiply a_{ij} together with a signed elementary product of A_{ij}, we need to multiply by a factor of

$$(-1)^{i+j+2(k-i)} = (-1)^{i+j}$$

to get a signed elementary product of A. This is exactly the factor indicated. □

2.3 Row operations and determinants

In this section, we will derive a new technique for computing determinants which works quite efficiently. The basic idea behind the technique is to use row operations to put a matrix into a form in which the determinant is easily calculated, keeping track of the row operations used, and how they affect the determinant, we can backtrack, and determine what the original determinant was.

Theorem 2.9. *Let A be an $n \times n$ matrix. Then*

1. *if B is obtained from A by multiplying one row or column by k, then* $\det(B) = k\det(A)$*;*
2. *if B is obtained from A by swapping two rows, then* $\det(B) = -\det(A)$*.*
3. *if B is obtained from A by adding a multiple of one row to another row, then* $\det(B) = \det(A)$*.*

Proof. We note that we have already proved part 1 of the theorem in Exercise 2.1.5. We will prove the remaining parts by induction on the size of A.

To begin suppose that A is a 2×2 matrix, say

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Letting B be the matrix which results from swapping the two rows, we have that

$$B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

and $-\det(A) = -(ad - bc) = cb - da = \det(B)$. A similar argument proves that swapping two columns changes the sign of the determinant.

Now, letting B be obtained by adding k times row 1 to row 2, we have that

$$B = \begin{bmatrix} a & b \\ c + ka & d + kb \end{bmatrix},$$

and we see that

$$\det(B) = a(d + kb) - b(c + ka) = ad + kab - bc - kab = ad - bc = \det(A).$$

A similar argument works for adding a multiple of row two to row one, or adding a multiple of one column to another.

Having proved the theorem for 2×2 matrices, we proceed by induction. Assume the theorem is true for all matrices smaller than $n \times n$, and let A be an $n \times n$ matrix. Let B be a matrix obtained from A by swapping two rows. Expanding $\det(B)$ on a row that is not one of the two that were swapped, we find that each minor that we compute is the same as the corresponding minor of A, with two rows swapped. Hence, each cofactor used in calculating $\det(B)$ has the opposite sign from the cofactor used in determining $\det(A)$. Factoring these sign changes out, we see that $\det(B) = -\det(A)$.

Similarly, if B is obtained by adding a multiple of one row to another, we expand the determinant on a third row, not involved in the row operation. We find, then that each minor of B is obtained from the corresponding minor of A by adding a multiple of one row to another. Hence, each cofactor used to calculate $\det(B)$ is the same as the corresponding cofactor used to calculate $\det(A)$. It follows that $\det(B) = \det(A)$.

A similar argument holds for a matrix B obtained from A by a column operation. □

We note the important fact that row operations can only multiply a determinant by a nonzero number, so that they can never change whether or not a determinant is zero. This leads immediately to the following theorem.

Theorem 2.10. *Let A be an $n \times n$ matrix. If A is invertible, then* $\det(A) \neq 0$.

Proof. Let A be an invertible matrix. We have seen that since A is invertible, its reduced row echelon form is equal to the identity. The identity is upper triangular, and so has determinant equal to 1 (the product of its diagonal entries. Since row operations can be used to change A into a matrix with nonzero determinant, and row operations cannot change a zero determinant into a nonzero one, we see that $\det(A)$ must have been nonzero. □

For completeness, we combine Theorems 2.7 and 2.10

Theorem 2.11. *An $n \times n$ matrix A is invertible if and only if* $\det(A) \neq 0$.

Proof. This follows immediately from Theorems 2.7 and 2.10. □

2.3.1 Calculating determinants using row operations

We now give a procedure to calculate determinants using row operations.

Let A be an $n \times n$ matrix. We perform row operations on A to reduce it to upper triangular form. For each row operation, we write down the following information:

1. If we swap two rows, we write down a -1.
2. If we multiply a row by a nonzero constant k, we write down k.

3. If we add a multiple of one row to another, we write down a 1.

After performing row operations on A, we will obtain a new matrix B, which will be upper triangular. We can easily calculate the determinant of B. Let C be the product of all the numbers that we wrote down while performing row operations. Then, $\det(A)C = \det(B)$, so that

$$\det(A) = \frac{\det(B)}{C}.$$

We illustrate with an example.

Example 2.3.1. Calculate the determinant of the matrix

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix}$$

.

We perform the following row operations, writing down the appropriate numbers in the column on the right:

$$\begin{bmatrix} 2 & 4 & 6 \\ 2 & 4 & 3 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2} R1} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \\ 2 & 3 & 1 \end{bmatrix} \qquad \frac{1}{2}$$

$$\xrightarrow{R_2 = R_2 + (-2)R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -3 \\ 2 & 3 & 1 \end{bmatrix} \qquad 1$$

$$\xrightarrow{R_3 = R_3 + (-2)R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -3 \\ 0 & -1 & -5 \end{bmatrix} \qquad 1$$

$$\xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & -3 \end{bmatrix} \qquad -1$$

The matrix that we end up with has determinant $(1)(-1)(-3) = 3$, and the numbers that we wrote down to the right have product $(\frac{1}{2})(1)(1)(-1) = -\frac{1}{2}$, so we see that $\det(A) = 3/(-\frac{1}{2}) = -6$.

2.3.2 Determinants of elementary matrices

If E is an elementary matrix, then it is obtained from the identity matrix by performing a single row operation. Since we know the determinant of I, and we know how row operations affect the determinant, we know the determinant of E.

Theorem 2.12. *Let E be an $n \times n$ elementary matrix.*

1. *If E is obtained by multiplying a row of I by k, then* $\det(E) = k$.
2. *If E is obtained by swapping two rows of I, then* $\det(E) = -1$.
3. *If E is obtained by adding a multiple of one row to another row, then* $\det(E) = 1$.

Proof. This follows immediately from Theorem 2.9. □

2.4 Determinants, matrix multiplication, and transposes

Lemma 2.13. *If E is an $n \times n$ elementary matrix, and B is any $n \times n$ matrix, then* $\det(EB) = \det(E)\det(B)$.

Proof. This follows from Theorem 2.12, since multiplying E by B performs the row operation corresponding to E on B, and performing that row operation multiplies the determinant of B by $\det(E)$. □

Lemma 2.14. *If $E_1, \ldots, E_k$ are $n \times n$ elementary matrices, and B is an $n \times n$ matrix, then*

$$\det(E_k \ldots E_1 B) = \det(E_k) \cdots \det(E_1)\det(B).$$

Proof. If $k = 1$ then this is just Lemma 2.13.

Suppose now that the theorem is true for fewer than k elementary matrices. We note that

$$\begin{aligned}\det(E_k E_{k-1} \cdots E_1 B) &= \det(E_k(E_{k-1} \cdots E_1 B)) \\ &= \det(E_k)\det(E_{k-1} \cdots E_1 B) \\ &= \det(E_k)\det(E_{k-1}) \cdots \det(E_1)\det(B),\end{aligned}$$

exactly as desired. Hence, the theorem is true for k elementary matrices. By induction, then, the theorem is true. □

Theorem 2.15. *Let A and B be $n \times n$ matrices. Then* $\det(AB) = \det(A)\det(B)$.

Proof. Suppose that $\det(A) = 0$. Then A is not invertible. If AB were invertible, so that for some matrix C, $(AB)C = I$, then $A(BC) = I$, and A would be invertible. Hence, we see that AB is not invertible, so that by Theorem 2.11,

$$\det(AB) = 0 = 0\det(B) = \det(A)\det(B).$$

We now suppose that $\det(A) \neq 0$. Then A is invertible by Theorem 2.11, and hence A is a product of elementary matrices, say $A = E_1 E_2 \ldots E_k$.

Hence, by Lemma 2.14, we see that

$$\begin{aligned}\det(AB) &= \det(E_1 \cdots E_k B) \\ &= \det(E_1) \cdots \det(E_k) \det(B) \\ &= \det(E_1 \cdots E_k) \det(B) \\ &= \det(A) \det(B)\end{aligned}$$

□

2.4.1 The determinant of the inverse

Corollary 2.16. *If A is an $n \times n$ invertible matrix, then* $\det(A^{-1}) = 1/\det(A)$.

Proof. Since $AA^{-1} = I$,

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}),$$

so $\det(A^{-1}) = 1/\det(A)$. □

2.4.2 The determinant of a transpose

Theorem 2.17. *Let A be a square matrix. Then* $\det(A) = \det(A^T)$.

Proof. We proceed by induction. The theorem is easily seen to be true if A is a 2×2 matrix. Assume that the theorem is true for all matrices smaller than $n \times n$, and we will prove that the theorem is true for $n \times n$ matrices.

Let $B = A^T$. Denote the (i, j) entries of A and B by a_{ij} and b_{ij}, respectively, and denote the matrices obtained by deleting the ith row and jth column of A and B by A_{ij} and B_{ij}. Then we see easily that $a_{ij} = b_{ji}$ and $A_{ij}^T = B_{ji}$. Since A_{ij} is smaller than A, we know that $\det(A_{ij}) = \det(A_{ij}^T = \det(B_{ji})$.

Expanding the determinant of A on row 1 and the determinant of B on column 1, we then obtain

$$\begin{aligned}\det(A) &= \sum_{k=1}^{n} a_{1k}(-1)^{1+k} \det(A_{1k}) \\ &= \sum_{k=1}^{n} b_{k1}(-1)^{1+k} \det(B_{k1}) \\ &= \det(B)\end{aligned}$$

□

2.4.3 The characteristic polynomial

We now define a special polynomial which will be important in our future study of matrices.

Definition 11. Let A be an $n \times n$ matrix, and let x be a variable. The *characteristic polynomial* of A is the polynomial

$$\det(xI - A).$$

Example 2.4.1. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

The characteristic polynomial of A is

$$\begin{aligned} \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) &= \det\left(\begin{bmatrix} x-1 & -2 \\ -3 & x-4 \end{bmatrix}\right) \\ &= (x-1)(x-4) - (-2)(-3) \\ &= x^2 - 5x - 2 \end{aligned}$$

Note that we can use any technique available to evaluate the determinant used to compute the characteristic polynomial.

Example 2.4.2. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

We may compute the characteristic polynomial using the formula derived for 3×3 determinants:

$$xI - A = \begin{bmatrix} x-1 & -2 & -3 \\ -1 & x-1 & -2 \\ -1 & -1 & x-1 \end{bmatrix}$$

so

$$\begin{aligned} \det(xI - A) &= (x-1)^3 + (-2)(-2)(-1) + (-3)(-1)(-1) \\ &\qquad - (-3)(x-1)(-1) - (-2)(-1)(x-1) - (x-1)(-2)(-1) \\ &= (x^3 - 3x^2 + 3x - 1) - 4 - 3 - 7(x-1) \\ &= x^3 - 3x^2 - 4x - 1. \end{aligned}$$

Alternatively, we write down $xI - A$ and perform row operations

$$\begin{bmatrix} x-1 & -2 & -3 \\ -1 & x-1 & -2 \\ -1 & -1 & x-1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1 a} \begin{bmatrix} -1 & -1 & x-1 \\ -1 & x-1 & -2 \\ x-1 & -2 & -3 \end{bmatrix} \qquad -1$$

$$\xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} -1 & -1 & x-1 \\ 0 & x & -1-x \\ x-1 & -2 & -3 \end{bmatrix} \qquad 1$$

$$\xrightarrow{R_3 = R_3 + (x-1)R_1} \begin{bmatrix} -1 & -1 & x-1 \\ 0 & x & -1-x \\ 0 & -1-x & -3+(x-1)^2 \end{bmatrix} \qquad 1$$

$$\xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} -1 & -1 & x-1 \\ 0 & x & -1-x \\ 0 & -1 & x^2-3x-3 \end{bmatrix} \qquad 1$$

$$\xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} -1 & -1 & x-1 \\ 0 & -1 & x^2-3x-3 \\ 0 & x & -1-x \end{bmatrix} \qquad -1$$

$$\xrightarrow{R_3 = R_3 + xR_2} \begin{bmatrix} -1 & -1 & x-1 \\ 0 & -1 & x^2-3x-3 \\ 0 & 0 & x^3-3x^2-4x-1 \end{bmatrix} \qquad 1$$

Now the final matrix has determinant $x^3 - 3x^2 - 4x - 1$, and we divide it by $(1)(-1)(1)(1)(1)(-1) = 1$ to find that $\det(xI - A) = x^3 - 3x^2 - 4x - 1$.

Definition 12. A real number λ is an eigenvalue of a matrix A if the equation

$$Ax = \lambda x$$

has nontrivial solutions. A nontrivial solution to this equation is called an eigenvector of A with eigenvalue λ.

Theorem 2.18. *The real number λ is an eigenvalue of A if and only if* $\det(\lambda I - A) = 0$

Proof. The equation

$$Ax = \lambda x$$

or

$$Ax = \lambda Ix$$

is the same as the equation

$$(\lambda I - A)x = 0.$$

Hence, there is a nontrivial solution (so that λ is an eigenvalue) exactly when $\lambda I - A$ is singular, or in other words, exactly when $\det(\lambda I - A) = 0$. □

Corollary 2.19. *The eigenvalues of A are exactly the roots of the characteristic polynomial of A.*

2.4.4 Exercises

Exercise 2.4.1. Suppose that A is an $n \times n$ invertible matrix. What is the determinant of $\operatorname{adj}(A)$?

Exercise 2.4.2. Prove by induction that $\det(A^k) = \det(A)^k$. (Take care to prove the theorem for both positive and negative values of k.)

Exercise 2.4.3. An *orthogonal* matrix is a matrix such that $A^T = A^{-1}$. If A is an orthogonal matrix, determine the possible values of $\det(A)$. (Hint: there are only two possible values.)

Exercise 2.4.4. Prove or give a counterexample: If A and B are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$.

Exercise 2.4.5. Prove that if A is a 2×2 matrix, then $\det(A) = \det(A^T)$.

Exercise 2.4.6. Prove that A and A^T have the same eigenvalues.

Exercise 2.4.7. Show that the eigenvalues of an upper triangular matrix A are the diagonal elements of A.