REPRESENTATIONS OF BRAID GROUPS VIA DETERMINANTAL RINGS

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Abstract. We construct representations for braid groups B_n via actions of B_n on a determinantal ring, thus mirroring the setting of the classical representation theory for GL_n. The representations that we construct fix a certain unitary form.

§1. Introduction

Let C be a commutative ring with identity. In this paper we attempt to do for the braid groups B_n what has been done for $GL_n(C)$ relative to their representation theory. The point of view will be the following classical way of understanding the representation theory of $GL_n(C)$: Let $X = (a_{ij})$ be a generic $n \times m$ matrix (where $m \geq n \geq 1$) with indeterminate entries and let $R_n = C[a_{ij}, 1 \leq i, j \leq n]$ be the corresponding coordinate ring. Let $G = GL_n(C) \times GL_m(C)$. Then G acts on R as follows:

$$(A, B)(X) = AXB^{-1} \quad \text{for} \quad (A, B) \in G.$$

The representation theory of G is described using Young diagrams. Recall that a *Young diagram* is a finite subset σ of $\mathbb{N} \times \mathbb{N}$ such that $(i, j) \in \sigma$ and $1 \leq i' \leq i, 1 \leq j' \leq j$ implies that $(i', j') \in \sigma$. The kth row of σ will be denoted by σ_k. There is a natural partial ordering of the σ. These diagrams correspond to irreducible representations of $GL_n(C)$ as follows.

Given such a σ a *tableau* of shape σ is a function $S : \sigma \to \{1, \dots, n\}$. The kth row of S will be denoted S_k. One writes $\sigma = |S|$ and there is a natural partial ordering of tableau extending the above order of diagrams. A *bitableau* is a pair $(S|T)$ of tableau of shape σ. Now to each bitableau we can associate a product of minors of the matrix X: for each row σ_k of $\sigma = |S| = |T|$ we get the minor $\mu(S_k|T_k)$ corresponding to the entries a_{ij} where $i \in S_k, j \in T_k$. Then $\mu(S|T)$ is the product $\mu(S_1|T_1)\mu(S_2|T_2)\dots\mu(S_r|T_r)$.

For each σ we let A_σ denote the subspace of R spanned by all tableau τ with $\tau \geq \sigma$ and let A'_σ denote the subspace of R spanned by all tableau τ with $\tau > \sigma$. Then A_σ/A'_σ is an irreducible $G-module$ and this construction gives all of the irreducible representations of G. A fundamental role in this is given to the Plücker relations, these giving all relations among products of the monomials and a partial ordering to the minors. Details of this can be found in [BV, DEP1, DEP2, Gr]. This approach also allows the calculation of the ring of invariants and many other important properties, even in arbitrary characteristic [op. cit.].

We now describe an analogous setup for B_n, noting the following papers which contain results on the general representation theory of the braid groups [A2, At, B, BLM, BW, F, Iv, J, La, Le, Li]. However we should also note that the representation theory for braid groups is more involved than that for linear groups since, for example, the braid groups are not rigid (see Theorem 1.9 below).

Let B_n denote the (algebraic) braid group, with standard generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and with relations

$$\sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1}, \quad \sigma_j\sigma_i = \sigma_i\sigma_j \tag{1.1}$$

for $1 \le i, j < n-1$ and $|i-j| > 1$. It is well known that B_n has a faithful representation in $Aut(F_n)$, where $F_n =< T_1, \dots, T_n >$ is a free group on n generators [Bi]. This comes from an action of B_n on the disc D_n with n punctures $\pi_1, \dots, \pi_n$ as isotopy classes of diffeomorphisms of D_n each fixing the boundary of the disc, so that the action of B_n on the fundamental group $\pi_1(D_n, p)$ for p on the boundary of D_n gives the monomorphism $\phi_n : B_n \to Aut(F_n)$. The action of a generator $\sigma_i, i < n$ is as follows:

$$\phi_n(\sigma_i)(T_i) = T_iT_{i+1}T_i^{-1}; \quad \phi_n(\sigma_i)(T_{i+1}) = T_i; \quad \phi_n(\sigma_i)(T_j) = T_j \text{ for } j \neq i, i+1.$$

Artin characterised the image of ϕ_n: each $\psi \in Aut(F_n)$ such that
(i) $\psi(T_j)$ is a conjugate of some T_k; and
(ii) $\psi(T_1)\psi(T_2)\dots\psi(T_n) = T_1T_2\dots T_n$ [Bi].
We will also need to note that there is an epimorphism $\Pi_n : B_n \to S_n$, given by the permutation action of a braid on the punctures $\{\pi_1, \dots, \pi_n\}$, whose kernel is the *pure braid group* P_n.

We obtain an action of B_n on a finitely generated polynomial algebra as follows; this will be defined by representing the generators T_j of F_n as transvections, specifically we let

$$T_i = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & 1 & \dots & a_{in-1} & a_{in} \\ \vdots & \vdots & \dots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 & 1 \end{pmatrix},$$

where the non-zero off-diagonal entries occur in the ith row. Here a matrix M is a *transvection* [A1] if $M = I_n + A$ where I_n is the identity matrix, $det(M) = 1$, $rank(A) = 1$ and $A^2 = 0$. In particular, conjugates of transvections are transvections. We let $R_n^{(0)} = C[a_{ij}, 1 \le i \neq j \le n]$ be the corresponding ring, so that $T_i \in SL_n(R_n^{(0)})$, and in this context it will be convenient to put $a_{ii} = 0$ for $i = 1, \dots, n$.

The fact that the group $< T_1, T_2, \dots, T_n >$ generated by these transvections is a free group of rank n was noted in [Hu2, Lemma 2.5]. The action of B_n on $R_n^{(0)}$ comes from the action of B_n on the trace algebra associated to the matrix group F_n: note that the element $T_iT_j, i \neq j$, represents the conjugacy class of the simple closed curve containing the punctures π_i, π_j in its interior. Now

$$trace(T_iT_j) = a_{ij}a_{ji} + n,$$

and, if $A, B \in F_n$, then, since AT_rA^{-1}, BT_sB^{-1} are transvections, it similarly follows that

$$trace(AT_rA^{-1}BT_sB^{-1}) = b_{rs}b_{sr} + n,$$

for some $b_{rs}, b_{sr} \in R_n^{(0)}$. For $\alpha \in B_n$ Artin's characterisation of braids given above shows that $\alpha(T_i) = AT_rA^{-1}, \alpha(T_j) = BT_sB^{-1}$ for some $A, B \in F_n$, where $\Pi_n(i) = r, \Pi_n(j) = s$, and so if $trace(AT_rA^{-1}BT_sB^{-1}) = b_{rs}b_{sr} + n$, then [Hu1, Hu2, Hu4] we may choose b_{rs}, b_{sr} such that

$$b_{rs} = a_{rs} + \text{terms of higher order}, \quad b_{sr} = a_{sr} + \text{terms of higher order}.$$

The action of B_n on $R_n^{(0)}$ is then given by $\alpha(a_{ij}) = b_{rs}$.

This action is non-linear; on generators it is given as follows:

$$\begin{aligned}
&\sigma_i(a_{i\,i+1}) = a_{i+1\,i}, \quad \sigma_i(a_{i+1\,i}) = a_{i\,i+1}, \quad \sigma_i(a_{h\,i+1}) = a_{h\,i},\\
&\sigma_i(a_{h\,i}) = a_{h\,i+1} + a_{h\,i}a_{i,i+1}, \quad \sigma_i(a_{i+1\,j}) = a_{i\,j},\\
&\sigma_i(a_{i\,j}) = a_{i+1\,j} - a_{i+1\,i}a_{i\,j},
\end{aligned} \tag{1.2}$$

where $h, j \neq i,\ i+1$.

A quotient of the above representation (1.2) was found by Magnus [Ma] when he was looking at the action of $Aut(F_n)$ on the character variety of 2×2 complex matrices. This character variety is essentially a polynomial ring with some quadratic relations; however Magnus noted ("somewhat surprisingly" [Ma p.100]) that, relative to a certain set of generators, the action of $B_n \subset Aut(F_n)$ on this character variety lifted to an action on a polynomial algebra with $n(n-1)/2$ generators. In [Hu1] we gave the above explanation for the existence of a lift of Magnus's representation, extending it to act on the $n(n-1)$ indeterminates a_{ij}.

Let $c_{ijk\ldots rs}$ represent the *cycle* $a_{ij}a_{jk}\ldots a_{rs}a_{si} \in R_n^{(0)}$. Then the cycles generate a subalgebra of $R_n^{(0)}$ denoted $Y_n^{(0)}$. Then $Y_n^{(0)}$ is the trace ring for the matrix group F_n. A cycle $c_{ijk\ldots rs}$ will be called *simple* if $i, j, k, \ldots, r, s$ are all distinct. It is easy to see that every cycle in $R_n^{(0)}$ is a product of simple cycles. It is also clear from the above action (1.2) of B_n on $R_n^{(0)}$ that if c_I is a cycle in $R_n^{(0)}$ and $\alpha \in B_n$, then $\alpha(c_I)$ is a sum of integral multiples of monomials, each of which is a cycle. Thus $Y_n^{(0)}$ is a B_n-invariant subring of $R_n^{(0)}$.

It follows from [Hu1, Theorem 2.5 and Theorem 6.2] that the kernel of the action of B_n on $R_n^{(0)}$ is the centre of B_n and that if B_n and $R_n^{(0)}$ are thought of as sub-objects of B_{n+1} and $R_{n+1}^{(0)}$ (respectively), then the action of B_n on $R_{n+1}^{(0)}$ is faithful.

We note as in [Hu2] that there is a natural ring involution $*$ on $R_n^{(0)}$ which commutes with the action of B_n: $\alpha(w^*) = \alpha(w)^*$ for all $\alpha \in B_n$ and all $w \in R_n^{(0)}$. This involution is determined by its action on the generators a_{ij} of $R_n^{(0)}$ which is as follows: $a_{ij}^* = -a_{ji}$. This involution has the following property:

$$trace(A^{-1}) = trace(A)^*,$$

for all $A \in F_n$. The action that Magnus discovered in [Ma], and that we referred to above, was the action on the $n(n-1)/2$ symbols $a_{ij} + a_{ij}^*$.

It is clear from the above presentation of B_n that, for $r < n$, the subgroup

$$B_{r,n} =< \sigma_r, \sigma_{r+1}, \ldots, \sigma_{n-1} >$$

of B_n is isomorphic to B_{n-r+1} with $B_{1,n} = B_n$. Now given $n_1, n_2, \ldots, n_s \geq 1$ we let

$$\begin{aligned}
G = G_{n_1,n_2,\ldots,n_s} &= B_{1,n_1} \times B_{n_1+1,n_1+n_2} \times B_{n_1+n_2+1,n_1+n_2+n_3} \times \ldots\\
&\qquad\qquad\qquad\qquad \times B_{n_1+\cdots+n_{s-1}+1,n_1+\cdots+n_s}\\
&\cong B_{n_1} \times B_{n_2} \times \cdots \times B_{n_s}.
\end{aligned}$$

Then by the above there is an action of G on $R^{(0)}_{n_1+\cdots+n_s+r}$ and on $Y^{(0)}_{n_1+\cdots+n_s+r}$ for any $r \geq 0$. Let $n = n_1+n_2+\cdots+n_s+r$ and $M_n^{(0)} = (a_{ij})$ where we have $a_{ii} = 0$ for all $i = 1,\ldots,n$. Then this action respects the minors of $M_n^{(0)}$ as follows: for any subsequences S, T of $\{1, 2, \ldots, n\}$ of the same length (thought of as bitableau with a single row) we let $(S|T)^{(0)}$ denote the minor of $M_n^{(0)}$ having rows taken from S and columns taken from T. (If either of S or T is the empty sequence, then the corresponding determinant will be taken to be 0). The action of B_n (or the subgroup G) on $R_n^{(0)}$ induces an action of B_n on the $(S|T)^{(0)}$ which, for S, T with one row, is given on generators as follows:

$$\begin{aligned}\sigma_r(S|T)^{(0)} = t_r[(S|T)^{(0)} + (S|S_r^{r+1}T)^{(0)}a_{r+1r} - a_{rr+1}(S_r^{r+1}S|T)^{(0)} \\ - a_{rr+1}(S_r^{r+1}S|S_r^{r+1}T)^{(0)}a_{r+1r}].\end{aligned} \tag{1.3}$$

Here t_r is the transposition $(r, r+1) \in S_n$ acting on the indices of the a_{ij} and S_r^{r+1} acts on the sequences S, T as follows: $S_r^{r+1}T$ is the empty sequence unless r is in T, while if r is in T, then $S_r^{r+1}T$ is T with r replaced by $r+1$. We shall sometimes write (1.3) as

$$\sigma_r(S|T)^{(0)} = t_r\left[(S - a_{rr+1}S_r^{r+1}(S)|T + S_r^{r+1}(T)a_{r+1r})^{(0)}\right], \tag{1.4}$$

where linearity in the two entries is understood (we will later give a better account of the context in which this action occurs).

The above shows that there is an action of B_n on the determinantal ideals generated by the minors of $M_n^{(0)}$, the complication being that if $\alpha \in B_n$, then $\alpha(S|T)^{(0)}$ is a sum of terms of the form $w(S'|T')^{(0)}$, where $w \in R_n^{(0)}$, and so this does not result in a finite-dimensional representation over C. We will indicate below how this situation can be modified so as to produce a finite-dimensional representation.

The action of B_n on the $(S|T)^{(0)}$ can be extended to an action on products

$$(S_1|T_1)^{(0)} \times \cdots \times (S_r|T_r)^{(0)}$$

in the obvious way. These products can then be represented using bitableau, as in the GL_n case. The set of all minors corresponding to such bitableau of a given shape, generate an ideal of $R_n^{(0)}$ which, by (1.3), is B_n-invariant. Thus we now have a way of assigning to each Young diagram σ a B_n-invariant ideal $A_\sigma^{(0)}$ of $R_n^{(0)}$. Again the problem is that, since the action of B_n on the a_{ij} is non-linear, this ideal is not finitely generated as a C-module.

The first modification will be to show (in Proposition 3.1) that the action of B_n on the $(S|T)^{(0)}$ lifts to an action on the $R_n^{(0)}$-module freely generated by products of abstract symbols $(S|T)'$ where S, T are tableau of shape σ taking values in $\{1, 2, \ldots, n\}$, having the same length, the action on such symbols being given by the analogue of (1.3). We will also impose the conditions:
(i) $(S|T)' = 0$ if either S or T contains repeated entries;
(ii) if S' is S with two entries interchanged, then $(S'|T)' = -(S|T)'$ (with a similar condition for T);
(iii) if $S = (S_1, \ldots, S_k)$ and $T = (T_1, \ldots, T_k)$, then $(S|T)' = (S_1|T_1)'(S_2|T_2)' \ldots (S_k|T_k')$.

The second modification is that we think of the $(S|T)'$ as generating an $R_n^{(0)}$-algebra and then reduce the elements of $R_n^{(0)}$ modulo a certain ideal that we now describe. Let u be another indeterminate and replace C by the field of rational functions $C(u)$ (so that we will now have to assume that C is an integral domain). For $S \subseteq \{1, 2, \dots, n\}$ we let $I(S)$ be the ideal of $R_n^{(0)}$ generated by the elements

$$\begin{aligned} &a_{ij}a_{ji} - \frac{1}{u(u+1)}, \quad \text{for } i, j \in S,\ i \neq j; \\ &a_{ij}a_{jk} - \frac{1}{u}a_{ik}, \qquad \text{if } i, j, k \in S \text{ and } (j-i)(k-i)(k-j) > 0; \\ &a_{ij}a_{jk} - \frac{1}{u+1}a_{ik} \quad \text{if } i, j, k \in S \text{ and } (j-i)(k-i)(k-j) < 0. \end{aligned} \tag{1.5}$$

For disjoint subsets $S_1, S_2, \dots, S_r \subseteq \{1, 2, \dots, n\}$ we let

$$I(S_1, S_2, \dots, S_r) =< I(S_1), I(S_2), \dots, I(S_r) > .$$

Then for example we see that the invariance of $I(\{1, 2, \dots, m\})$ under the action of B_m implies the invariance of $I(\{1, 2, \dots, n_1\}, \{n_1 + 1, n_1 + 2, \dots, n_1 + n_2\}, \dots)$ under the action of $G_{n_1,n_2,\dots,n_s}$.

Now fix $n_1, \dots, n_s \geq 1$ and let $n = \sum_{i=1}^{s} n_i$. Choose a Young diagram σ for $\{1, \dots, n\}$. Let

$$I_{n_1,\dots,n_s} = I(\{1, \dots, n_1\}, \{n_1 + 1, \dots, n_1 + n_2\}, \{n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3\}, \dots).$$

Let $\mathcal{R}_n(\sigma)$ be the $R_n^{(0)}/I_{n_1,\dots,n_s}$-module generated by all $(S|T)'$ where $|S| = |T| = \sigma$ and $S_1, T_1 \subset \{1, \dots, n_1\}, S_2, T_2 \subset \{n_1 + 1, \dots, n_1 + n_2\}, \dots$. If we set the degree of $(S|T)'$ to be 1 (and $degree(a_{ij}) = 0$), then $\mathcal{R}_n(\sigma)$ is a graded $R_n^{(0)}/I_{n_1,\dots,n_s}$-algebra, which we write as

$$\mathcal{R}_n(\sigma) = \oplus_{k=0}^{\infty} \mathcal{R}_n^k(\sigma).$$

Theorem 1.1. *Each $\mathcal{R}_n^k(\sigma)$ is a finitely-generated free $C(u)$-module which is G-invariant.*

We will study the summands of these representations. We will show that the following contribute to the existence of such summands:
(i) Multiple Laplace expansions of the determinant of (a_{ij}).
(ii) Ideals generated by the Plücker relations.
(iii) The existence of invariant involutions.
(iv) The existence of fixed forms for the action.

Examples 1.2. The bitableau $(1, 2, \dots, n|1, 2, \dots, n)$ gives the trivial representation of G. The bitableau $(1, 2, \dots, n|n + 1, n + 2, \dots, 2n)$ gives the sign representation of G.

If M is a matrix, then M^t will denote its transpose.

Theorem 1.3. *The action of $G_{n_1,\dots,n_s}$ on each $\mathcal{R}_n^k(\sigma)$ fixes a non-degenerate form which is unitary relative to the involution $*$: for fixed n_i, k there is a basis $\{b_i\}$ for $\mathcal{R}_n^k(\sigma)$ and a non-degenerate matrix E over $C(u)$ such that if M is the matrix representing $\alpha \in G_{n_1,\dots,n_s}$ relative to the basis $\{b_i\}$, then we have $ME(M^t)^* = E$. The matrix E satisfies $E^* = \pm E^t$.*

Each $\mathcal{R}_n^k(\sigma)$ splits as a sum of $G_{n_1,\dots,n_s}$-irreducible subrepresentations.

Theorem 1.4. *In the representation $\mathcal{R}_n^k(\sigma)$ the matrix representing any of the generators $\sigma_i, 1 \leq i < n$, is diagonalisable.*

Here is a complete description of one case:

Theorem 1.5. *The representation given by $(1, 2, \ldots, n|1, 2, \ldots, \hat{i}, \ldots, n, n+1)$ splits as $V_1 \oplus V_2$ where V_1 has dimension n and is irreducible and monomial, and where V_2 is irreducible and has dimension $n(n-1)$.*

We will give a branching law for the the restrictions $Res_{B_{n-1}}^{B_n} V_i, i = 1, 2$ of the above representations in Theorem 7.6.

We see how the representation theory of S_n appears in the following case where we consider the action of B_n on $R_n/I(\{1, 2, \ldots, n\})$.

Theorem 1.6. *Suppose that $\mu = \prod_k a_{i_k j_k} \in R_n^{(0)}$ is a monomial where $\{i_k\}_k \cap \{j_k\}_k = \emptyset$. Then there is a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ of n such that the $C(u)$-module generated by the orbit $B_n(\mu)$ splits into B_n-invariant summands in exactly the same way as the representation of S_n induced from the trivial representation of the subgroup*

$$S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_r}$$

does. (The summands so obtained are not necessarily B_n-irreducible.)

Before describing how these representations split we will need to describe in greater detail the action of B_n on $R_n^{(0)}$ (in §2), the action of B_n on the ideals $I(S_1, \ldots, S_r)$ (in §3) etc.

In §10 we will show that $\mathcal{R}_n(\sigma)$ is sometimes an algebra with straightening law (ASL). In [Hu5] we have shown that B_4 acts on an ASL.

Theorem 1.7. *Let $1 \leq k \leq n$. Then for $S = \{1, \ldots, k\}$ the representation $\mathcal{R}_n^1(S|\{n+1, \ldots, n+k\})$ has dimension $\binom{n}{k}^2$ and splits as $E_1 \oplus E_2$ where E_1 has dimension $\binom{n}{k}\binom{n-1}{k}$ and is irreducible and E_2 has dimension $\binom{n}{k}\binom{n-1}{k-1}$.*

The Plücker relations (defined in §11) determine certain representation spaces also:

Theorem 1.8. *Let $\mathcal{U}$ be a set of Plücker relations coming from Young diagrams with a single row of length k. Then there is an associated B_n-invariant finitely generated free $C(u)$-module $< B_n(\mathcal{U}) >$ associated to $\mathcal{U}$. If $n > 3$ is odd, then $< B_n(\mathcal{U}) >$ has an irreducible summand of dimension n. The action of B_n on this latter representation is monomial.*

A finitely generated group is *rigid* if it has only finitely many classes of irreducible complex representations in each dimension.

Theorem 1.9. *For $n \geq 3$ the braid group B_n and the braid commutator groups B_n' are not rigid.*

The braid commutator groups B_n' have been studied by Gorin and Lin [GL] and play an important role in Lin's study [L] of representations of B_n.

A result of Dyer, Formanek and Grossman [DFG] gives a connection between B_4 and $Aut(F_2)$. We use this to prove

Theorem 1.10. *The automorphism group $Aut(F_2)$ is not rigid.*

The question of whether $Aut(F_n)$ is rigid is posed in the 'Open problems in combinatorial group theory' list [P, problem F5].

§2 Action of B_n on R_n

In this section we describe in greater detail the action of B_n on $R_n^{(0)}$. A coordinate free definition [A1] of a transvection in $SL(Q^n)$ (for a commutative ring Q with identity) is as a pair $T = (\phi, d)$ where $d \in Q^n$ and ϕ is an element of the dual space of Q^n satisfying $\phi(d) = 0$. The action of T on Q^n is given by

$$T(x) = x + \phi(x)d \quad \text{for all} \quad x \in Q^n.$$

Then we have [Hu1, Lemma 2.1]

Lemma 2.1. *Let $T = (\phi, d)$ and $U = (\psi, e)$ be two transvections. Then for all $\lambda \in \mathbb{Z}$ we have*

$$U^\lambda T U^{-\lambda} = (\phi - \lambda\phi(e)\psi, U^\lambda(d)). \quad \square$$

Let $T = \{T_1 = (\phi_1, d_1), \ldots, T_n = (\phi_n, d_n)\}$ be a fixed set of transvections in $SL((R_n^{(0)})^n)$ where $\phi_i(d_j) = a_{ij}$ for all $1 \le i \ne j \le n$ as in the above. For any set of transvections

$$T' = \{T'_1 = (\phi'_1, e'_1), \ldots, T'_n = (\phi'_n, e'_n)\}$$

we let $M(T')$ denote the $n \times n$ matrix $(\phi'_i(e'_j))$ and we call $M(T')$ the *M-matrix* of the set of transvections T'.

Any monomial in $R_n^{(0)}$ that can be written in the form $a_{j_1 j_2} a_{j_2 j_3} \ldots a_{j_{r-1} j_r}$ will be called a $j_1 j_r$-word. Note that by (1.2) if $\alpha \in B_n$ and $1 \le i \ne j \le n$, then $\alpha(a_{ij})$ is a sum of rs-words, where $\alpha(T_i)$ is a conjugate of T_r and $\alpha(T_j)$ is a conjugate of T_s. Let $\alpha \in B_n$ where $\alpha(T_i) = w_i T_j w_i^{-1}$ in freely reduced form for $i = 1, ..., n$ and where $w_i = w_i(T_1, \ldots, T_n)$. Then for $i = 1, \ldots, n$ we have $w_i T_i w_i^{-1} = (\psi_i, f_i)$ for some ψ_i, f_i determined by Lemma 2.1, which result in fact shows that

$$\psi_i = q_1\phi_1 + \cdots + q_n\phi_n \quad \text{and} \quad f_i = p_1 d_1 + \cdots + p_n d_n,$$

where $p_1, \ldots, p_n, q_1, \ldots, q_n \in R_n^{(0)}$. Since the a_{ij} are algebraically independent the ϕ_i and d_j are linearly independent and so the above representation is unique. We define the action of B_n on $R_n^{(0)}$ by

$$\alpha(a_{ij}) = \psi_i(f_j).$$

One can check that this agrees with the previous definition. Thus the M-matrix is acted upon naturally by B_n:

$$\alpha(M(T)) = M(\alpha(T_1), \ldots, \alpha(T_n)).$$

From Lemma 2.3 of [Hu1] we have:

Lemma 2.2. *Let $\alpha \in B_n$ where $\alpha(T_i) = C_1 T_k C_1^{-1}, \alpha(T_j) = C_2 T_p C_2^{-1}$, with $C_1, C_2 \in < T_1, \ldots, T_n >$ and let $C = C_1^{-1} C_2 = T_{j_1}^{q_1} \ldots T_{j_r}^{q_r}$ be freely reduced with $j_r \ne p$, $j_1 \ne k, q_s \ne 0$ for $s = 1, \ldots, r$ and $j_s \ne j_{s+1}$, for $s = 1, \ldots, r-1$. Then*

$$\alpha(a_{ij}) = \sum_{h=1}^{n} A_h a_{hp}$$

where A_h is equal to the sum of all the products of the form

$$q_{r_1} q_{r_2} \dots q_{r_m} a_{kj_{r_1}} a_{j_{r_1} j_{r_2}} \dots a_{j_{r_{m-1}} j_{r_m}}$$

where $1 \le r_1 < r_2 < \cdots < r_m \le r$ and $j_{r_m} = h$. If $p \neq j_r$, then the summand of $\alpha(a_{ij})$ of highest degree is unique and is equal to

$$\pm q_1 q_2 \dots q_r a_{kj_1} a_{j_1 j_2} \dots a_{j_{r-1} j_r} a_{j_r p}. \quad \square$$

For example if $\alpha(T_1) = T_3 T_2^{-1} T_1 T_2 T_3^{-1}$ and $\alpha(T_2) = T_2^{-1} T_3 T_2$, then we would have $C = T_2 T_3^{-1} T_2^{-1}$ and

$$\alpha(a_{12}) = a_{13} + a_{13} a_{32} a_{23} + a_{12} a_{23} a_{32} a_{23}.$$

Note that (1.2) follows from Lemma 2.2 and the action of σ_r in (1.3) was already noted in [Hu4, Hu5].

§3 Lifting the determinantal representation

Here we prove:

Proposition 3.1. *There is an action of $G_{n_1,\dots,n_s}$ on the $R_n^{(0)}/I_{n_1,\dots,n_s}$-algebra generated by the $(S|T)'$ where the action of a generator σ_r is given by (1.3).*

Proof. The proof consists in showing that the braid relations (1.1) are satisfied by the rule for the σ_i given in (1.3) i.e. we need to show that $\sigma_i \sigma_{i+1} \sigma_i (S|T)' = \sigma_{i+1} \sigma_i \sigma_{i+1} (S|T)'$ for $i < n$ and that $\sigma_i \sigma_j (S|T)' = \sigma_j \sigma_i (S|T)'$ for $|i-j| > 1$. But, if $S = (S_1, \dots, S_k)$ and $T = (T_1, \dots, T_k)$, then $(S|T)' = (S_1|T_1)'(S_2|T_2)' \dots (S_k|T_k)'$ and since the σ_i act as ring homomorphisms we need only consider the case $k = 1$.

The alternative formula (1.4) can be interpreted as giving actions of B_n on the S part and on the T part (which we put into some suitable category), namely:

$$\sigma_r(S) = t_r(S - a_{rr+1} S_r^{r+1} S), \quad \sigma_r(T) = t_r(T + a_{r+1r} S_r^{r+1} T). \tag{3.1}$$

This is put in context in the following way: let V_n be a free $R_n^{(0)}/I_{n_1,\dots,n_s}$-module with basis $x_1, \dots, x_n$. We will associate to every subsequence $S = (s_1, \dots, s_k)$ of $\{1, \dots, n\}$ the element $x_{s_1} \wedge \cdots \wedge x_{s_k}$ of the exterior algebra $\bigwedge^k V_n$. Then we will check that the first equation of (3.1) gives an action on $\bigwedge^k V_n$ and the second equation gives a dual action (relative to the involution *). This will suffice to prove Proposition 3.1. The fact that these are dual actions means that we need only do the S action for example. This then amounts to showing that the action of the σ_i on the S part of (3.1) satisfies the relations in the standard presentation (1.1).

Now the relation $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| > 1$ is easy to check (since in this case t_i and t_j commute, as do t_i and S_j^{j+1} etc.). For the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ we note that this is trivially true if $S \cap \{i, i+1, i+2\} = \emptyset$. So we can assume that $S \cap \{i, i+1, i+2\} \neq \emptyset$. A further simplification is that we need only do the case $i = 1$. Also when $i = 1$ we may ignore the presence in S of any indices greater than 3 since these are unaffected by σ_1 and σ_2. There are now 7 cases to consider: (i) S contains only 1 (out of $1, 2, 3$); (ii) S contains only 2; (iii)

S contains only 3; (iv) S contains only $1, 2$; (v) S contains only $1, 3$; (vi) S contains only $2, 3$; (vii) S contains $1, 2, 3$.

For (i) by the above remarks we may (since we are ignoring all indices of S greater than 3) write $S = (1)$ and so we have

$$\begin{aligned}\sigma_1\sigma_2\sigma_1(1) &= \sigma_1\sigma_2 t_1((1) - a_{12}(2)) = \sigma_1\sigma_2((2) - a_{21}(1)) \\ &= \sigma_1((3) - a_{32}(2) - (a_{31} - a_{32}a_{21})(1)) \\ &= (3) - a_{31}(1) - (a_{32}((2) - a_{21}(1)); \\ \sigma_2\sigma_1\sigma_2(1) &= \sigma_2\sigma_1(1) = \sigma_2((2) - a_{21}(1)) \\ &= (3) - a_{32}(2) - (a_{31} - a_{32}a_{21})(1).\end{aligned}$$

Thus $\sigma_1\sigma_2\sigma_1(1) = \sigma_2\sigma_1\sigma_2(1)$. For (ii) we may similarly write $S = (2)$ and we get:

$$\begin{aligned}\sigma_1\sigma_2\sigma_1(2) &= \sigma_1\sigma_2(1) = \sigma_1(1) = (2) - a_{21}(1); \\ \sigma_2\sigma_1\sigma_2(2) &= \sigma_2\sigma_1((3) - a_{32}(2)) = \sigma_2((3) - a_{31}(1)) = (2) - a_{21}(1).\end{aligned}$$

For (iii) we write $S = (3)$ and we get:

$$\begin{aligned}\sigma_1\sigma_2\sigma_1(3) &= \sigma_1\sigma_2(3) = \sigma_1(2) = (1); \\ \sigma_2\sigma_1\sigma_2(3) &= \sigma_2\sigma_1(2) = \sigma_2(1) = (1).\end{aligned}$$

For (iv) we write $S = (12)$ and we get:

$$\begin{aligned}\sigma_1\sigma_2\sigma_1(12) &= \sigma_1\sigma_2(-(12)) = -\sigma_1((13) - a_{32}(12)) = -((23) - a_{21}(13) + a_{31}(12)); \\ \sigma_2\sigma_1\sigma_2(12) &= \sigma_2\sigma_1((13) - a_{32}(12))) = \sigma_2((23) - a_{21}(13) + a_{31}(12)) \\ &= (32) - (a_{31} - a_{32}a_{21})(12) + a_{21}((13) - a_{32}(12)) \\ &= -((23) - a_{21}(13) + a_{31}(12)).\end{aligned}$$

For (v) we write $S = (13)$ and we get:

$$\begin{aligned}\sigma_1\sigma_2\sigma_1(13) &= \sigma_1\sigma_2((23) - a_{21}(13)) = \sigma_1((32) - (a_{31} - a_{32}a_{21})(12)) \\ &= (31) + a_{32}(12); \\ \sigma_2\sigma_1\sigma_2(13) &= \sigma_2\sigma_1(12) = -\sigma_2(12) = -(13) + a_{32}(12).\end{aligned}$$

For (vi) we write $S = (23)$ and we get:

$$\begin{aligned}\sigma_1\sigma_2\sigma_1(23) &= \sigma_1\sigma_2(13) = \sigma_1(12) = (21); \\ \sigma_2\sigma_1\sigma_2(23) &= \sigma_2\sigma_1(32) = \sigma_2(31) = (21).\end{aligned}$$

For (vii) we write $S = (123)$ and we get:

$$\begin{aligned}\sigma_1\sigma_2\sigma_1(123) &= \sigma_1\sigma_2(213) = \sigma_1(312) = (321); \\ \sigma_2\sigma_1\sigma_2(123) &= \sigma_2\sigma_1(132) = \sigma_2(231) = (321).\end{aligned}$$

This concludes the proof of all cases $\square$

The proof of the above result immediately gives:

Corollary 3.2. *For all $1 \leq k \leq n$ we have an action of B_n on $\bigwedge^k V_n$.* $\square$

§4 Invariant ideals

Lemma 4.1. *Let S be the sequence $(1, 2, \dots, n)$. Then*
(i) The ideal $I(S)$ is B_n-invariant.
(ii) If $c_I = a_{ij}a_{jk}\dots a_{rs}$ is an is-word where $i, j, k, \dots, r, s \in S$, then there are two cases:
a) if $i = s$, then $c_I + I(S) = r + I(S)$, where $r \in C(u)$;
b) if $i \neq s$, then $c_I + I(S) = ra_{is} + I(S)$ for $r \in C(u)$.
(iii) If $i \neq j$, then $a_{ij} + I(S) \neq r + I(S)$ for $r \in C(u)$.

Proof. (i) The proof is to check that if $e = a_{ij}a_{ji} - \frac{u}{u+1}$, $e = a_{ij}a_{jk} - \frac{1}{u+1}a_{ik}$ or $e = a_{ij}a_{jk} - \frac{1}{u}a_{ik}$ is one of the ideal generators (as in (1.5)), then $\sigma_r(e) \in I(S)$. Note that if $\{i, j, k\} \cap \{r, r+1\} = \emptyset$, then $\sigma_r(e) = e$. Thus one may assume that $\{i, j, k\} \cap \{r, r+1\} \neq \emptyset$ so that $card(\{i, j, k\} \cup \{r, r+1\}) \leq 4$. One easily sees that in fact one may renumber so that $i, j, k \in \{1, 2, 3, 4\}$ and $r = 1, 2, 3$, thus reducing the checking to a finite number of cases, as indicated below. (One may also use the invariance of the B_n-action under the involution * to further reduce the number of cases to be checked.) For example,

$$\sigma_1(a_{12}a_{23} - \frac{1}{u}a_{13}) = (1 + \frac{1}{u})(a_{21}a_{13} - \frac{1}{u+1}a_{23});$$
$$\sigma_2(a_{12}a_{23} - \frac{1}{u}a_{13}) = (a_{13}a_{32} - \frac{1}{1+u}a_{12}) + a_{12}(a_{23}a_{32} - \frac{1}{u(1+u)});$$
$$\sigma_3(a_{12}a_{23} - \frac{1}{u}a_{13}) = (a_{12}a_{24} - \frac{1}{u}a_{14}) + a_{34}(a_{12}a_{23} - \frac{1}{u}a_{13});$$
$$\sigma_1(a_{13}a_{32} - \frac{1}{u+1}a_{12}) = (a_{23}a_{31} - \frac{1}{u}a_{21}) - a_{21}(a_{13}a_{31} - \frac{1}{u(u+1)});$$
$$\sigma_2(a_{13}a_{32} - \frac{1}{u+1}a_{12}) = \frac{u}{u+1}(a_{12}a_{23} - \frac{1}{u}a_{13});$$
$$\begin{aligned}\sigma_3(a_{13}a_{32} - \frac{1}{u+1}a_{12}) = {} & (a_{14}a_{42} - \frac{1}{u+1}a_{12}) + a_{13}(a_{34}a_{42} - \frac{1}{u}a_{32}) \\ & - a_{32}(a_{14}a_{43} - \frac{1}{u+1}a_{13}) - a_{13}a_{32}(a_{34}a_{43} - \frac{1}{u(u+1)});\end{aligned}$$
$$\sigma_1(a_{12}a_{21} - \frac{1}{u(u+1)}) = a_{12}a_{21} - \frac{1}{u(u+1)};$$
$$\begin{aligned}\sigma_2(a_{12}a_{21} - \frac{1}{u(u+1)}) = {} & (a_{13}a_{31} - \frac{1}{u(u+1)}) - a_{13}(a_{32}a_{21} - \frac{1}{u+1}a_{31}) \\ & + a_{31}(a_{12}a_{23} - \frac{1}{u}a_{13}) - (a_{12}a_{21} - \frac{1}{u(u+1)})a_{23}a_{32} + \frac{1}{u(u+1)}(a_{13}a_{31} - a_{23}a_{32}); \quad \text{etc.}\end{aligned}$$

Alternatively, for a fixed ring $C = \mathbb{Q}, \mathbb{F}_q$, one can do these calculations (faster) using a Gröbner basis algorithm, as implemented in, for example, Magma [MA], since, as we have already noticed, one only has to deal with the case $n = 4$.

(ii) Given a cycle $c_I = a_{ij}a_{jk}\dots a_{rs}$ of degree d one can use the relations in $I(S)$ to replace, for example, $a_{ij}a_{jk}$ by a non-zero $C(u)$-multiple of a_{ik}, thus reducing the degree, while the resulting monomial of degree $d-1$ is still an *is*-word. (ii) follows.

(iii) Define a ring homomorphism $\eta = \eta_S : R_n^{(0)} \to R_n^{(0)}$ by its action on generators:

$$\eta(a_{ij}) = \frac{1}{u} \text{ if } \ i, j \in S \ \text{ and } i < j; \quad \eta(a_{ij}) = \frac{1}{u+1} \text{ if } \ i, j \in S \ \text{ and } i > j. \tag{4.1}$$

Then one checks that $\eta(I(S))=0$. But clearly $\eta(a_{ij})\neq 0$ and this gives (iii) $\square$

Lemma 4.2. *Suppose that S,T have a single row. Let $(S|T)^{(0)}$ be the minor of $M_n^{(0)}$ with row indices from S and column indices from T. Then, when expanded out, each monomial of $(S|T)^{(0)}$ is a product of rs-words and ii words for distinct choices of $r\in S\setminus T$ and $s\in T\setminus S$.*

If $\alpha\in B_n$ and $T\cap\{1,2,\dots,n\}=\emptyset$, then each monomial of $\alpha(S|T)'$ has the form $w(S'|T)'$ where $(S'|T)'$ has the same shape as $(S|T)'$ and where either $w\in C$ or $w\in R_n^{(0)}$ is a product of r_is_i-words, $i=1,\dots,k$, where the r_i and the s_i are all distinct and $s_i\in S'$ and $r_i\notin S'$.

If $\alpha\in B_n$ and $S\cap\{1,2,\dots,n\}=\emptyset$, then each monomial of $\alpha(S|T)'$ has the form $(S|T')'w'$ where $(S|T')'$ has the same shape as $(S|T)'$ and where either $w'\in C$ or $w'\in R_n^{(0)}$ is a product of r_is_i-words, $i=1,\dots,k$, where the r_i and the s_i are all distinct and $r_i\in T'$ and $s_i\notin T'$.

If $\alpha\in B_n$, then each monomial of $\alpha(S|T)'$ has the form $w(S'|T')'w'$ where $(S'|T')'$ has the same shape as $(S|T)'$ and where w and w' are as in the last two paragraphs.

Proof. The first statement follows from elementary properties of determinants. The rest follows from using (1.3) by induction on the length of α as a product of the standard generators. $\square$.

Remark 4.3. Now most of the time we will reduce the monomials w,w' referred to in Lemma 4.2 mod $I_{n_1,\dots,n_s}$ and only deal with representatives which are products of r_is_i-words of smallest degree (see Lemma 4.1). We note that for $w(S'|T')'w'$ as in the last paragraph of Lemma 4.2, Lemma 4.2 then places a bound on the degree of such monomials w, w' (when so reduced). However one should note that there may well be further reductions for ww' e.g. $\dots a_{ij}(\dots,j,\dots|\dots,j,\dots)a_{jk}\dots$ could be reduced to $\dots a_{ik}(\dots,j,\dots|\dots,j,\dots)\dots.$ Note that this latter form may not look like it has the form indicated in Lemma 4.2.

We note that if σ is a Young diagram with a single row of length k, then there are only a finite number of the $(S|T)'$ with $|S|=|T|=k$ and by Lemma 4.2 there are only a finite number of monomials $w(U|V)'$ in the B_n-orbit of such $(S|T)'$. Thus $\mathcal{R}_n^1(\sigma)$ is a finite-dimensional free $C(u)$-module and so $\mathcal{R}_n^m(\sigma)$ is a finite-dimensional free $C(u)$-module since it is a quotient of the mth symmetric power of $\mathcal{R}_n^1(\sigma)$ where a basis consists of all $w(S_1|T_1)'\dots(S_m|T_m)'$ with w satisfying conditions similar to those of Lemma 4.2. The case where σ has more than one row is similar. This proves Theorem 1.1. $\square$

From (1.5) we see that if $(j-i)(k-i)(k-j)>0$ then mod $I_{n_1,\dots,n_s}$ we have $a_{ij}a_{jk}=\frac{1}{u}a_{ik}$. Acting on this latter equation by $*$ we get $a_{ji}a_{kj}=-\frac{1}{u^*}a_{ki}$ and comparing this with (1.5) again we see that it is natural to define

$$u^*=-(u+1).$$

One then checks:

Lemma 4.4. *For all $w\in I_{n_1,\dots,n_s}$ we have $w^*\in I_{n_1,\dots,n_s}$.*

Proof. One need only consider the case where w is one of the generators of $I_{n_1,\dots,n_s}$ as in (1.5) and we have already done one case above. The rest are also easily checked. $\square$

We now define the action of the involution * on the generators $(S|T)'$, where S,T have a single row, by

$$((S|T)')^*=(T|S)'.$$

We extend this action naturally: $(w_1S|w_2T)^*=w_1^*w_2^*(T|S)$, and then $C(u)$-linearly over monomials. This now gives:

Lemma 4.5. *For $\alpha \in B_n$ we have*

$$\alpha((S|T)')^* = \alpha(((S|T)')^*).$$

In particular, for all $x \in \mathcal{R}_n$ and all $\alpha \in B_n$ we have $\alpha(x^*) = \alpha(x)^*$.

Proof. We need only prove the first statement, and this only in the case $\alpha = \sigma_r$, $1 \leq r < n$. Using (1.4) we have:

$$\begin{aligned}
(\sigma_r(S|T)')^* &= (t_r(S - a_{rr+1}S_r^{r+1}S|T + S_r^{r+1}Ta_{r+1r}))^* \\
&= (t_rS - a_{r+1r}t_rS_r^{r+1}S|t_rT + t_rS_r^{r+1}Ta_{rr+1})^* \\
&= (t_rT - t_rS_r^{r+1}Ta_{r+1r}|t_rS + a_{rr+1}t_rS_r^{r+1}S) \\
&= t_r(T - S_r^{r+1}Ta_{r+1r}|S + a_{rr+1}S_r^{r+1}S) \\
&= \sigma_r(T|S)' = \sigma_r(((S|T)')^*),
\end{aligned}$$

as required □

Lemma 4.6. *Let $S = (1, \ldots, n)$ and choose distinct $u_1, \ldots, u_r, v_1, \ldots, v_r \in \{1, \ldots, n\}, r \geq 2$. Then for any $\zeta \in S_n$ there is $c \neq 0 \in C$ such that*

$$a_{u_1v_1}a_{u_2v_2}\ldots a_{u_rv_r} = c \times a_{u_1v_{\zeta 1}}a_{u_2v_{\zeta 2}}\ldots a_{u_rv_{\zeta r}} \ \mathit{mod}\ I(S).$$

Proof. It will suffice to do the case $r = 2$, since transpositions generate S_n. Now u_1, u_2, v_1, v_2 are distinct and so there are $c_1, c_2 \in C(u)$ such that:

$$a_{u_1v_1}a_{u_2v_2} = c_1a_{u_1u_2}a_{u_2v_1}a_{u_2v_2} = c_1a_{u_1u_2}a_{u_2v_2}a_{u_2v_1} = c_1c_2a_{u_1v_2}a_{u_2v_1},$$

as required. □

Remark 4.7. For those who like their ring involutions to look like complex conjugation we can (in the situation where $C = \mathbb{C}$) put $u = -\frac{1}{2} + iy$, where $i^2 = -1$.

Proposition 4.8. *Suppose 2 is invertible in C and that V is a B_n-invariant subrepresentation of $\mathcal{R}_n^k(\sigma)$ with $V^* = V$. Then V splits as $V^+ \oplus V^-$ where*

$$V^\pm = \{b \in V | b^* = \pm b\}.$$

Here $V^\pm$ are both B_n-invariant.

Proof. Lemma 4.5 shows that each of $V^\pm$ are invariant under the action of B_n. The rest follows since for $b \in V$ we can write $b = (b + b^*)/2 + (b - b^*)/2 = b^+ + b^-$, where $b^\pm \in V^\pm$. □

Let $w(S|T)$ be a monomial where S, T have a single row. We assume that w is in normal form (see Lemma 4.2) so that $w = w_1w_2$ with $w_1 = a_{r_1s_1}\ldots a_{r_zs_z}, w_2 = a_{p_1q_1}\ldots a_{p_yq_y}$ with $s_i \in S, p_i \in T, s_i, q_i \notin S \cup T$. Then we let

$$E^- = (S \cup \{r_1, \ldots, r_z\}) \setminus \{s_1, \ldots, s_z\}, \quad E^+ = (T \cup \{q_1, \ldots, q_y\}) \setminus \{p_1, \ldots, p_z\}.$$

Lemma 4.9. *Suppose that S, T have a single row. Let $\alpha \in B_n$ and let μ be a monomial in $\alpha(w(S|T)')$. Then*

$$|E^+(w(S|T)') \cap E^-(w(S|T)')| = |E^+(\mu) \cap E^-(\mu)|.$$

If σ is a Young diagram, then we have the following B_n-invariant splitting

$$\mathcal{R}^1_n(\sigma) = \oplus_{i=0}^{|S|} W_i$$

where W_i is spanned by all monomials $\mu \in \mathcal{R}^1_n(\sigma)$ with $|E^+(\mu) \cap E^-(\mu)| = i$. Further we have $W_i = W_i^+ \oplus W_i^-$.

Proof. The last sentence follows from Proposition 4.8. The first statement follows from (1.3) and (1.4): induct on the length of α as a word in the standard braid generators. The rest follows from the first sentence. ☐

Construction 4.10. We now indicate another way to get B_n-invariant summands. Fix $k < n$. Suppose that V_k is the B_n-space generated by all $\alpha(S|T)'$ where $\alpha \in B_n$ and S, T are subsequences of $\{1, \ldots, n\}$, thought of as tableau with a single row. As in Example 1.2 we note that the element $([1, \ldots, n]|[1, \ldots, n])$ is fixed by the B_n action. Now for S as above we let e_S be the element of V_k which is obtained by expanding $([1, \ldots, n]|[1, \ldots, n])$ along all rows labeled i where $i \notin S$. Then each monomial in e_S has the form $w(S|T)'$ for some w, T. For example, if $n = 4$ and $S = [2, 3, 4]$, then

$$e_S = -a_{12}([2,3,4]|[1,3,4])' + a_{13}([2,3,4]|[1,2,4])' - a_{14}([2,3,4]|[1,2,3])'.$$

Then the B_n-orbit of all such $e_S, |S| = k$ generates a B_n-invariant $C(u)$-submodule E_k of V_k. It is clear that E_k is invariant under the involution * and so Proposition 4.8 shows that we have the B_n-invariant splitting: $E_k = E_K^- \oplus E_k^+$ (if 2 is invertible in C).

§5 Invariant forms

We will first consider the case where the Young diagram has a single row.

By the above we have an action of B_n on the ring $R_n^{(0)}$. This can be extended to an action of B_n on a ring

$$R_n^\infty = C[a_{ij}|i, j \geq 1, a_{ii} = 0 \text{ for } i \leq n].$$

The action is still given by (1.2) so that $\alpha(a_{ij}) = a_{ij}$ for $\alpha \in B_n$ and $i, j > n$. As usual, we will think of the a_{ij} as entries in a matrix of sufficiently large degree.

Given finite subsequences S, T, U, V of $\mathbb{N}$ of the same size and $w_1, w_2 \in R_n$ we define

$$< w_1(S|T)', w_2(U|V)' >' = w_1 w_2^*(S|U)^{(0)}(V|T)^{(0)} \bmod I_n = I(\{1, \ldots, n\}).$$

We will say that S, T, U, V and w_1, w_2 are *compatible* if $< w_1(S|T)', w_2(U|V)' >'$ is in the subring $C(u)[a_{ij}|i, j > n]$ of R_n^∞ (so that it is fixed by the action of B_n). We define

$$< w_1(S|T)', w_2(U|V)' >= \begin{cases} < w_1(S|T)', w_2(U|V)' >' & \text{if } S, T, U, V, w_1, w_2 \text{ are compatible;} \\ 0 & \text{otherwise} \end{cases}$$

We extend $<,>$ $C(u)$-linearly to act on $\mathcal{R}^1_n$. For notational convenience we will sometimes use $det(S|T)$ for $(S|T)^{(0)}$.

Example 5.1. If $n=4, S=[1], T=[5]$, then the only $w(U|V)'$ which are compatible with $(S|T)'$ are $b_1=(S|T)', b_2=a_{12}([2],[5])', b_3=a_{13}([3],[5])', b_4=a_{14}([4],[5])'$ and the values of $< b_i, b_j >$ are given in the following matrix (where we suppress the $det([5],[5])=a_{55}$ factor) :

$$\begin{pmatrix} 0 & -\frac{1}{u(u+1)} & -\frac{1}{u(u+1)} & -\frac{1}{u(u+1)} \\ \frac{1}{u(u+1)} & 0 & -\frac{1}{u(u+1)^2} & -\frac{1}{u(u+1)^2} \\ \frac{1}{u(u+1)} & -\frac{1}{u^2(u+1)} & 0 & -\frac{1}{u(u+1)^2} \\ \frac{1}{u(u+1)} & -\frac{1}{u^2(u+1)} & -\frac{1}{u^2(u+1)} & 0 \end{pmatrix}.$$

Proposition 5.2. *The form $<,>$ is $C(u)$-linear in both entries and for $w \in R_n^\infty, x, y \in \mathcal{R}_n^1$ satisfies*

$$< wx, y >= w < x, y >, \quad < x, wy >= w^* < x, y >, \quad < x, y >^* =< y, x > .$$

Further, $<,>$ is B_n-invariant: for all $\alpha \in B_n, x, y \in \mathcal{R}_n^1$ we have $< \alpha(x), \alpha(y) >=< x, y >$.

Proof. The linearity and the first two properties are clear. To show that $< x, y >^* =< y, x >$ we need only do the case where $x=(S|T), y=(U|V)$. We need to note that $det(S|T)^* = (-1)^{|S|} det(T|S)$ and then we have:

$$\begin{aligned} < (S|T)', (U|V)' >^* &= (det(S|U)det(V|T))^* \mod I_n \\ &= det(S|U)^* det(V|T)^* \mod I_n \\ &= det(U|S)det(T|V) \mod I_n \\ &=< (U|V)', (S|T)' > . \end{aligned}$$

We now prove the invariance under the B_n-action, again noting that it suffices to check this for $\alpha=\sigma_r, 1 \le r < n$, and $x=(S|T)', y=(U|V)'$. First note that by (1.4) we have

$$\sigma_r(S) = t_r S - a_{r+1r} t_r S_r^{r+1} S, \quad \sigma_r(U) = t_r U - a_{r+1r} t_r S_r^{r+1} U$$

and so

$$\begin{aligned} < \sigma_r(S|T), \sigma_r(U|V) > &=< (\sigma_r S|\sigma_r T), (\sigma_r U|\sigma_r V) > \\ &= det(\sigma_r S, \sigma_r U)det(\sigma_r V, \sigma_r T) \mod I_n. \end{aligned}$$

We will now prove that $det(\sigma_r S, \sigma_r U) = \sigma_r det(S, U)$:

$$\begin{aligned} det(\sigma_r S, \sigma_r U) &= det(t_r S - a_{r+1r} t_r S_r^{r+1} S, t_r U - a_{r+1r} t_r S_r^{r+1} U) \\ &= det(t_r S, t_r U) - a_{r+1r} det(t_r S_r^{r+1} S, t_r U) - det(t_r S, a_{r+1r} t_r S_r^{r+1} U) \\ &\qquad + a_{r+1r} det(t_r S_r^{r+1} S, a_{r+1r} t_r S_r^{r+1} U) \\ &= det(t_r S, t_r U) - a_{r+1r} det(t_r S_r^{r+1} S, t_r U) + a_{rr+1} det(t_r S, t_r S_r^{r+1} U) \\ &\qquad - a_{r+1r} a_{rr+1} det(t_r S_r^{r+1} S, t_r S_r^{r+1} U) \\ &= \sigma_r det(S|U). \end{aligned}$$

We similarly have $det(\sigma_r V, \sigma_r T)' = \sigma_r det(V, T)'$. Now combining these results we get

$$\begin{aligned} < \sigma_r(S|T), \sigma_r(U|V) > &= det(\sigma_r S, \sigma_r U)det(\sigma_r V, \sigma_r T) \mod I_n \\ &= \sigma_r det(S, U) \sigma_r det(V, T) \mod I_n \\ &= \sigma_r (det(S, U)det(V, T)) \mod I_n \\ &= det(S, U)det(V, T) \mod I_n, \end{aligned}$$

the last equality coming from the fact that S, T, U, V are compatible. $\square$

Proposition 5.3. *The form* $<,>$ *is non-degenerate.*

Proof. We will need:

Lemma 5.4. *Let* $s_0(x,y) = 1$ *and for* $n > 0$ *let* $s_n(x,y) = x^n + x^{n-1}y + x^{n-2}y^2 + \cdots + y^n$ *and*

$$X_n = \begin{pmatrix} 0 & \frac{1}{u} & \frac{1}{u} & \dots & \frac{1}{u} \\ \frac{1}{u+1} & 0 & \frac{1}{u} & \dots & \frac{1}{u} \\ \frac{1}{u+1} & \frac{1}{u+1} & 0 & \dots & \frac{1}{u} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{u+1} & \frac{1}{u+1} & \frac{1}{u+1} & \dots & 0 \end{pmatrix}.$$

Then $det(X_n) = (-1)^{n-1}\frac{1}{u(u+1)}s_{n-2}(\frac{1}{u}, \frac{1}{u+1})$.

Proof. This follows directly from the last exercise in [Mu, §828, p. 764]. □

Now we have noted above that we may find a basis of $\mathcal{R}_n^1$ of the form $\{b_i = w_i(S_i|T_i)'\}_i$. We may order this basis so that $b_1, \dots, b_{N_1}$ are all compatible, $b_{N_1+1}, \dots, b_{N_1+N_2}$ are all compatible (but not compatible with b_1), etc. In fact the number of b_i compatible with a given b_j is the same, so we have $N_i = N_j$. Relative to this basis the matrix representing the form $<,>$ has block form

$$\begin{pmatrix} C_1 & 0 & 0 & \dots \\ 0 & C_2 & 0 & \dots \\ 0 & 0 & C_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each C_i is an $N_1 \times N_1$ matrix. Thus to show that the form $<,>$ is non-degenerate it suffices to show that each of the matrices $C_k = (< b_i, b_j > |(k-1)N_1 \leq i,j \leq kN_1)$ is non-degenerate.

Fix $(S|T)'$ with $|S| = |T| = k$ and consider all b_i which are compatible with $(S|T)'$. We will first consider the case of arbitrary S and $T = (n+1, n+2, \dots, 2n)$. In fact there is little loss in this case in assuming that $S = (1, 2, \dots, k)$. Now note that given the compatibility of each such $b_i = w(S'|T)'$ with $(S|T)'$ we see that S' completely determines w (and vice-versa). Further, for each $S' \subset \{1, 2, \dots, n\}$ with $|S'| = k$ there is w' such that $w'(S'|T)'$ is a basis element. It follows that there are exactly $\binom{n}{k}$ of the b_i which are compatible with $(S|T)'$, one for each subset of $\{1, 2, \dots, n\}$ of cardinality k. Thus $N_1 = \binom{n}{k}$.

Recall the ring homomorphism $\eta = \eta_{\{1,\dots,n\}} : R_n^{(0)} \to C(u)$ defined in the proof of Lemma 4.1 (iii). We there showed that it satisfies: $\eta(I_n) = 0$. We can extend η as follows:

$$\eta(a_{ij}) = \frac{1}{u} \text{ if } \{i,j\} \cap \{1,\dots,n\} \neq \emptyset \text{ and } i < j;$$
$$\eta(a_{ij}) = \frac{1}{u+1} \text{ if } \{i,j\} \cap \{1,\dots,n\} \neq \emptyset \text{ and } i > j;$$
$$\eta(a_{ij}) = a_{ij} \text{ if } \{i,j\} \cap \{1,\dots,n\} = \emptyset.$$

Lemma 5.5. *For compatible* $b_i = w_i(S_i|T)$ *and* $b_j = w_j(S_j|T)$ *we have*

$$< w_i(S_i|T), w_j(S_j|T) >= \eta(w_i w_j^* det(S_i|S_j)det(T|T)).$$

Proof. Since $\eta(I(S)) = 0$, and b_i and b_j are compatible we have

$$\eta(w_1 w_2 det(S_1, S_2) det(T|T)) = \eta(< b_i, b_j >) =< b_i, b_j >,$$

as required. $\square$

Now we wish to show that $det(C_k) \neq 0$ where $(C_k)_{ij}$ is $< w_i(S_i|T), w_j(S_j|T) >$; but by Lemma 5.5 and the fact that η is a ring homomorphism, it suffices to show that the matrix E with i, j entry equal to $\eta(< (S_i|T), (S_j|T) >)$ is non-degenerate. But since $det(T|T)$ is a constant and non-zero factor this latter fact will follow if we can show that the matrix D with i, j entry equal to $\eta(det(S_i|S_j))$ is non-degenerate.

Lemma 5.6. *Fix $1 \leq k \leq n$ and let X_n be as in Lemma 5.4. Let $S_1, \dots, S_{\binom{n}{k}}$ be the subsets of $\{1, \dots, n\}$ of cardinality k and let D be the $\binom{n}{k} \times \binom{n}{k}$ matrix $(\eta(det(S_i, S_j)))$. Then D is invertible.*

Proof. Lemma 5.4 shows that X_n is invertible. We can think of X_n as acting on a $C(u)$-vector space V_n with basis $x_1, \dots, x_n$. Then by [Bo, Prop. 10 p. 529; see also Ex. 11 p. 640 (watch for the misprint!)] we see that the matrix D represents the action of X_n on the exterior algebra $\bigwedge^k V_n$. Since X_n is invertible we see that D is also. $\square$

Conjecture 5.7. We conjecture that the determinants $det(C_i)$ have the form $\frac{(u+1)^m - u^m}{(u(u+1))^p}$ for some m, p. If this were the case and one solves $det(C_i) = 0$, then one obtains $(u+1)^m - u^m = 0$ and finds (over $\mathbb{C}$) that the solutions are:

$$u = -\frac{1}{2} - i\frac{sin(2k\pi/m)}{1 - cos(2k\pi/m)},$$

for $1 \leq k < m$. We compare these solutions with Remark 4.7.

Lemma 5.8. *Assume that $|S| = |T| = k$. Then the action of B_n on $(S|T)'$ is the same as the action of B_n on the elements $(S|[n+1, \dots, n+k])'([n+1, \dots, n+k]|T)'$.*

Proof. We need only check that for $\alpha \in B_n$ we have

$$\alpha(S|T)' = \alpha(S|[n+1, \dots, n+k])'\alpha([n+1, \dots, n+k]|T)',$$

and in fact we need only check this for $\alpha = \sigma_i, i < n$. However this latter fact in this case follows from (1.3). $\square$

We now show how the above implies the non-degeneracy for general S, T.

Now the action of B_n on the $w([n+1, \dots, n+k]|T)$ is dual to the action on the $w^*(S|[n+1, \dots, n+k])$. Thus by the above the action of B_n on the tensor product (over $C(u)$) generated by all $w(S|[n+1, \dots, n+k]) \otimes w'([n+1, \dots, n+k]|T)$ is also a B_n-representation space with the B_n action fixing a non-degenerate B_n-invariant form; denote this space by $U_n \otimes U_n^*$. Then $U_n \otimes U_n^*$ splits as a sum of B_n-irreducibles.

Now by Lemma 5.8 we see that the B_n-representation space that we are interested in is a quotient of this tensor product; denote it by Q. Thus, due to the above splitting property, this quotient can be identified with a summand of $U_n \otimes U_n^*$ i.e. $U_n \otimes U_n^* \cong Q \oplus Y$. Then the form on $U_n \otimes U_n^*$ restricts to a form on Q, which, since Q and Y are orthogonal relative to the form on $U_n \otimes U_n^*$, is also non-degenerate. This does the case where S, T have a single row. The general case follows by a similar argument since $\mathcal{R}_n^1(\sigma\sigma')$ is a quotient of $\mathcal{R}_n^1(\sigma) \otimes \mathcal{R}_n^1(\sigma')$. $\square$

§6 Diagonalisability

In this section we prove

Theorem 6.1. *For all Young diagrams σ of $n \geq 2$ and all $1 \leq i < n$ the matrix representing the action of σ_i on $\mathcal{R}_n^1(\sigma)$ is diagonalisable over a finite extension of $C(u)$.*

Proof. It clearly suffices to prove the result in the case where σ has a single row. Now fix a monomial μ. We will show that the orbit $O_i(\mu) = \{\sigma_i^k(\mu)\}_{k\in\mathbb{Z}}$ is spanned by a certain finite set $M_i(\mu)$ of monomials and that the action on the subspace $V_i(\mu)$ spanned by these monomials is diagonalisable. Now by Lemma 4.2 we may assume that $\mu = w_1(S|T)'w_2$ with w_1, w_2 as in Lemma 4.2. Since $\sigma_i(a_{rs}) = a_{rs}$ for all $r, s \neq i, i+1$ we see that there are a finite number of cases to be checked, depending upon whether $i, i+1$ occur in S or T or as subscripts of factors of w_1, w_2. Here, for example, $a_{ji}(i|-)$ will indicate a monomial where a_{ji} is a divisor of w_1, $j \neq i, i+1$, (but none of $a_{ij}, a_{i+1j}, a_{ji+1}, a_{ii+1}, a_{i+1i}$ are) and $i \in S$ (but $i+1 \notin S$), and $i, i+1 \notin T$. The cases are: $\mu =$

(i) $(-|-)$, $(ii+1|-)$, $(-|ii+1)$; *(ii)* $(i|-)$; *(iii)*$(i+1|-)$; *(iv)* $(i|i)$; *(v)* $(i|i+1)$; *(vi)* $(ii+1|-)$; *(vii)* $(ii+1|i)$; *(viii)* $(ii+1|i+1)$; *(ix)* $a_{ji}(i|-)$; *(x)* $a_{ji}(i|i+1)$; *(xi)* $a_{ji}(ii+1|-)$; *(xii)* $a_{ji}(ii+1|i)$; *(xiii)*$a_{ji}(ii+1|i+1)$; *(xv)* $a_{ij}(|-)$; *(xvi)*$a_{ij}(|i)$; *(xvii)* $a_{ij}(-|i)$; *(xviii)* $a_{i+1i}(i|-)$; *(ixx)* $a_{ii+1}(i+1|-)$; *(xx)* $a_{i+1i}(i|i+1)$; *(xxi)* $a_{ii+1}(i+1|i)$; *(xxii)* $a_{i+1i}(i+1|i+1)$; *(xxiii)* $a_{i+1i}(i|i)$; *(xxiv)* $a_{i+1i}^2(i|i+1)$; *(xxv)*$a_{ii+1}^2(i+1|i)$; *(xxvi)* $a_{ji}(i|i+1)a_{i+1i}$; *(xxvii)* $a_{ji+1}(i+1|i+1)$; *(xxviii)* $a_{ji+1}(i+1|i)$; *(xxix)* $a_{ji+1}(i|i)$; *(xxx)* $a_{ji}(i+1|i+1)$; $a_{ji}(i|i)$; *(xxxi)* $a_{ji+1}a_{ii+1}(i+1|i)$; *(xxxii)* $a_{ji}a_{i+1i}(i|i+1)$; *(xxxiii)* $a_{ji}a_{i+1k}(-|-)$; *etc.*

Here we have only indicated some of the cases, other cases will follow by duality.

We now indicate how each case can be checked. Of course $\sigma_i(-|-) = (-|-)$ and similarly $\sigma_i(ii+1|-) = -(ii+1|-)$ and so these cases are easy. For (ii) $\mu = (i|-)$, we have

$$M(\mu) = \{(i|-), (i+1|-), a_{i+1i}(i|-), a_{ii+1}(i+1|-)$$

and relative to this basis the matrix representing the action of σ_i is

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & \frac{-1}{u(u+1)} & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which has characteristic polynomial $(z^2 - \frac{u}{u+1})(z^2 - \frac{u+1}{u})$ and so M is diagonalisable over a finite extension of $C(u)$. This is case (ii), but we note that this also takes care of cases (iii), (xviii) and (ixx).

For (iv) we have

$$M(\mu) = \{(i+1|i+1), (i|i), a_{i+1i}(i|i+1); a_{ii+1}(i+1|i)\}$$

and relative to this basis the matrix representing the action of σ_i is

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \frac{-1}{u(u+1)} & -1 & 1 \\ 0 & \frac{-1}{u(u+1)} & 0 & 1 \\ 0 & \frac{1}{u(u+1)} & 1 & 0 \end{pmatrix}$$

and this has characteristic polynomial $(z-1)^2(z+\frac{u}{u+1})(z+\frac{u+1}{u})$. To obtain the result in this case we just need to note that $(1,1,0,1),(0,0,1,1)$ span the 1-eigenspace. This does (iv) and also (xx) and (xxi).

For (v) we have

$$M(\mu) = \{(i+1|i),(i|i+1),a_{i+1i}(i+1|i+1),a_{i+1i}(i|i),a_{ii+1}(i+1|i+1),$$
$$a_{ii+1}(i|i); a_{i+1i}^2(i|i+1),a_{ii+1}^2(i+1|i)\},$$

and relative to this basis the matrix representing the action of σ_i is

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{-1}{u(u+1)} & 0 & 0 & 1 & \frac{-1}{u(u+1)} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{u(u+1)} & 0 & 1 & \frac{-1}{u(u+1)} & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{u(u+1)} & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{u(u+1)} & 0 & 0 & 1 & 0 \end{pmatrix}.$$

This has characteristic polynomial

$$(z-1)^2(z+1)^2(z-\frac{u}{u+1})(z+\frac{u}{u+1})(z-\frac{u+1}{u})(z+\frac{u+1}{u}).$$

Here we note that the eigenspaces for the ± 1 eigenvectors are generated (respectively) by

$$(1,0,u(u+1),u(u+1),u(u+1),u(u+1),0,u(u+1)),$$
$$(0,1,-u(u+1),-u(u+1),-u(u+1),-u(u+1),u(u+1),0),$$
$$(1,0,u(u+1),u(u+1),-u(u+1),-u(u+1),0,-u(u+1)),$$
$$(0,1,u(u+1),u(u+1),-u(u+1),-u(u+1),-u(u+1),0).$$

This does this case and (xxii), (xxiii), (xxiv), (xxv).

For (ix) we have $M(\mu)=\{a_{ji}(i|-),a_{ji+1}(i+1|-)\}$ and the matrix is $\begin{pmatrix} 0 & 1 \\ \frac{u+1}{u} & \frac{-1}{u} \end{pmatrix}$ which has distinct eigenvalues.

For (x) we have

$$M(\mu) = \{a_{ji+1}(i+1|i+1),a_{ji+1}(i+1|i),a_{ji+1}(i|i),a_{ji}(i+1|i+1),a_{ji}(i|i+1),a_{ji}(i|i),$$
$$a_{ji}a_{i+1i}(i|i+1),a_{ji+1}a_{ii+1}(i+1|i)\}.$$

Here the σ_1 matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{u} & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{u} & 0 & 1 & 0 & \frac{-1}{u(u+1)} & -1 & 0 \\ 0 & 0 & \frac{u+1}{u} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{u+1}{u} & 0 & 0 & 0 & \frac{-1}{u} & 0 & 0 \\ \frac{u+1}{u} & 0 & -\frac{1}{u^2} & 0 & 0 & 0 & 0 & \frac{u+1}{u} \\ 0 & 0 & -\frac{1}{u^2} & 0 & 0 & 0 & 0 & \frac{u+1}{u} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{u(u+1)} & 1 & 0 \end{pmatrix}.$$

This has characteristic polynomial

$$(z^2 - \frac{u}{u+1})(z^2 - \frac{u+1}{u})^2(z^2 - \frac{(u+1)^3}{u^3})$$

and the eigenvectors for the squared factor are:

$$\begin{aligned} &(-u\sqrt{(u+1)u}, 0, -u\sqrt{(u+1)u}, -(u+1)\,u, \sqrt{(u+1)\,u}, -(u+1)\,u, 1, 0), \\ &((u+1)\,u, \sqrt{(u+1)\,u}, (u+1)\,u, \sqrt{(u+1)\,u}\,(u+1)\,, 0, \sqrt{(u+1)\,u}\,(u+1)\,, 0, 1), \\ &((u+1)\,u, -\sqrt{(u+1)\,u}, (u+1)\,u, -\sqrt{(u+1)\,u}\,(u+1)\,, 0, -\sqrt{(u+1)\,u}\,(u+1)\,, 0, 1), \\ &(u\sqrt{(u+1)\,u}, 0, u\sqrt{(u+1)\,u}, -(u+1)\,u, -\sqrt{(u+1)\,u}, -(u+1)\,u, 1, 0). \end{aligned}$$

This again shows diagonalisability for (x) and (xxvi)-(xxxii).

The rest of the cases are similarly checked, giving Theorems 6.1 and 1.4. $\square$

§7 The $(1, 2, \ldots, n|1, 2, \ldots, \hat{i}, \ldots, n, n+1)$ representation.

In this section we prove Theorem 1.5. Let $\mu_i = \mu_i^{(n)} = (1, 2, \ldots, n|1, 2, \ldots, \hat{i}, \ldots, n, n+1)$.

Lemma 7.1. *For any $1 \leq i \leq n$ the $C(u)$-module V_n generated by the B_n-orbit of μ_i is freely generated by $\mu_k, a_{ij}\mu_j$ for $i, j, k = 1, \ldots, n$ with $i \neq j$. It has dimension n^2.*

Proof. We note the following:

$$\begin{aligned} &\sigma_i(\mu_i) = -\mu_{i+1}; \quad \sigma_i(a_{ii+1}\mu_{i+1}) = -a_{i+1i}\mu_i - \frac{u}{u+1}\mu_{i+1}; \\ &\sigma_i(a_{ij}\mu_j) = \frac{u}{u+1}a_{i+1j}\mu_j; \quad \sigma_i(a_{i+1j}\mu_j) = a_{ij}\mu_j; \quad \sigma_i(\mu_{i+1}) = -\mu_i - a_{ii+1}\mu_{i+1}; \\ &\sigma_i(a_{ji}\mu_i) = -\frac{(u+1)}{u}a_{ji+1}\mu_{i+1}; \quad \sigma_i(a_{ji+1}\mu_{i+1}) = -a_{ji}\mu_i - \frac{1}{u}a_{ji+1}\mu_{i+1}. \end{aligned} \tag{7.1}$$

Here $j \neq i, i+1$. The first of these equations shows that all the $\pm\mu_j$ are in the orbit of μ_1, for example. For $i < j < n$ the 5th equation shows that we can get $a_{jj+1}\mu_{j+1}$ and then

$$\sigma_i\sigma_{i+1}\ldots\sigma_{j-1}(a_{jj+1}\mu_{j+1}) = a_{ij+1}\mu_{j+1}.$$

We can similarly get all $a_{ij}\mu_j$ for $i > j$. $\square$

Define the following vectors:

$$\begin{aligned}
v_1 &= \mu_1 - ua_{12}\mu_2 + ua_{13}\mu_3 - ua_{14}\mu_4 + ua_{15}\mu_5 - \dots + (-1)^{n+1}ua_{1n}\mu_n \\
v_2 &= -(u+1)a_{21}\mu_1 + \mu_2 - ua_{23}\mu_3 + ua_{24}\mu_4 - ua_{25}\mu_5 + \dots + (-1)^{n}ua_{2n}\mu_n \\
v_3 &= (u+1)a_{31}\mu_1 - (u+1)a_{32}\mu_2 + \mu_3 - ua_{34}\mu_4 + ua_{35}\mu_5 - \dots + (-1)^{n+1}ua_{3n}\mu_n; \\
v_4 &= -(u+1)a_{41}\mu_1 + (u+1)a_{42}\mu_2 - (u+1)a_{43}\mu_3 + \mu_4 - ua_{45}\mu_5 + \dots + (-1)^{n}ua_{4n}\mu_n; \\
&\vdots \\
v_n &= \pm(u+1)a_{n1}\mu_1 \mp (u+1)a_{n2}\mu_2 \pm (u+1)a_{n3}\mu_3 \mp (u+1)a_{n4}\mu_4 \\
&\qquad \pm (u+1)a_{n5}\mu_5 \mp \dots + \mu_n. \qquad (7.2)
\end{aligned}$$

We now note that $\sigma_i(a_{jk}) = a_{jk}$ and $\sigma_i(\mu_j) = \mu_j$ for all $j, k \neq i, i+1$. From (7.1) we see that for $j \neq i, i+1$ the only monomials in v_j which are not fixed are $a_{ji}\mu_i$ and $a_{ji+1}\mu_{i+1}$, both having the same coefficients only differing in sign; so we have:

$$\sigma_i(a_{ji}\mu_i - a_{ji+1}\mu_{i+1}) = -\frac{(u+1)}{u}a_{ji+1}\mu_{i+1} + a_{ji}\mu_i + \frac{1}{u}a_{ji+1}\mu_{i+1} = a_{ji}\mu_i - a_{ji+1}\mu_{i+1},$$

showing that $\sigma_i(v_j) = v_j$ for all $j \neq i, i+1$. We also have (for $j \neq i, i+1$):

$$\begin{aligned}
&\sigma_i(a_{ij}\mu_j) = \frac{u}{u+1}a_{i+1j}\mu_j, \text{ and} \\
&\sigma_i(\mu_i - ua_{ii+1}\mu_{i+1}) = -\mu_{i+1} + u(a_{i+1i}\mu_i + \frac{1}{u(u+1)}\mu_{i+1}) = ua_{i+1i}\mu_i - \frac{u}{u+1}\mu_{i+1}.
\end{aligned}$$

This shows that $\sigma_i(v_i) = -\frac{u}{u+1}v_{i+1}$. Similarly we have $\sigma_i(a_{i+1j}\mu_j) = a_{ij}\mu_j$ for $j \neq i, i+1$ and

$$\sigma_i((u+1)a_{i+1i}\mu_i - \mu_{i+1}) = -(u+1)a_{ii+1}\mu_{i+1} + \mu_i + a_{ii+1}\mu_{i+1} = \mu_i - ua_{ii+1}\mu_{i+1}.$$

Which shows that $\sigma_i(v_{i+1}) = -v_i$. Thus we get a monomial representation ρ of degree n where

$$\rho(\sigma_1) = \begin{pmatrix} 0 & -\frac{u}{u+1} & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \rho(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -\frac{u}{u+1} & 0 & \dots \\ 0 & -1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{etc.} \qquad (7.3)$$

We recall that a *monomial representation* of a group G is a representation $\rho : G \to GL(V)$, where for each $g \in G$ the matrix $\rho(g)$ has only one entry in each row and each column; such a matrix is called a monomial matrix. General results about monomial groups and representations can be found in [O, Sc].

We now show that this representation $V =< v_1, \dots, v_n >$ is irreducible. For suppose that W is an invariant subspace and let $v \in W, 0 \neq v = \sum_{i=1}^{n} \lambda_i v_i$. Let $r = r(v) = min\{i|\lambda_i \neq 0\}$. From the above we see that the action of σ_i^2 is represented relative to the basis $v_1, \dots, v_n$ by the diagonal matrix $diag(1,\ldots,1,\frac{u}{u+1},\frac{u}{u+1},1,\ldots,1)$, where the $\frac{u}{u+1}$ entries are in the i and $i+1$ positions. Thus if $r=r(v)>1$, then the span of v and $\sigma_{r-1}^2(v)$ contains v_r. Since V is a monomial representation whose corresponding permutation representation is transitive we see that $v_i\in W$ for all $i\le n$ and so $W=V$.

Similarly, if W is not 1-dimensional, then there is $0\neq v\in W$ with $r(v)>1$ and so as in the above we are done. Thus we may assume that $dim(W)=1, W=<v>$ and $r(v)=1$ and in this situation the span $<v,\sigma_1^2(v)>$ contains an element of the form $w=\lambda_1v_1+\lambda_2v_2$. If $\lambda_1\lambda_2=0$, then $v_i\in W$ for some $i=1,2$; whereas if $\lambda_2\neq 0$, then $v_2\in<w,\sigma_2^2(w)>$. In either case we again see that $W=V$ and we are done.

Remark 7.2. The action of B_n on $<v_1,\ldots,v_n>$ is not faithful since, for example, one can show that the images of σ_1^2 and of σ_2^2 are both diagonal and so commute. However it is well-known [Bi] that the subgroup $<\sigma_1^2,\sigma_2^2>$ of B_3 is free on the two given generators.

Now any μ_i can be evaluated as a minor of the $(n+1)\times(n+1)$ matrix (a_{rs}) and then we can look at this element mod $I_{n+1})$. This map we denote by $\mathcal{I}_{n+1}$.

Lemma 7.3. *(i) Let $n\ge 2$ and $1\le j\le n$. Then*

$$\mathcal{I}_{n+1}(\mu_j)=(-1)^{j+1}\frac{1}{(u+1)^{j-1}u^{n-j}}a_{jn+1}.$$

(ii) For $n\ge 2$ we have

$$\mathcal{I}_{n+1}(det(a_{ij})_{n\times n})=(-1)^{n+1}\left(\frac{1}{u^{n-1}}-\frac{1}{(u+1)^{n-1}}\right).$$

(iii) For $n\ge 2$ and $1\le i<j\le n$ we have

$$\mathcal{I}_n((1,\ldots\hat{i},\ldots,n|1,\ldots\hat{j},\ldots,n))=(-1)^{n+i+j+1}\frac{1}{u^{n+i-j-1}(u+1)^{j-i-1}}a_{ji}$$

Proof. We first show that (ii) for n follows from (i) for $n-1$. Expanding $det(a_{ij})$ along the last row we get (remembering that $a_{ii}=0$):

$$\begin{aligned}
det(a_{ij}) &= \sum_{i=1}^{n-1}(-1)^{n+i}a_{ni}(1,2,\ldots,n-1|1,2,\ldots,\hat{i},\ldots,n)\\
&= \sum_{i=1}^{n-1}(-1)^{n+i}\mu_i^{(n-1)} = \sum_{i=1}^{n-1}(-1)^{n+i}a_{ni}(-1)^{i+1}\frac{a_{in}}{(u+1)^{i-1}u^{n-1-i}}\\
&= (-1)^{n+1}\sum_{i=1}^{n-1}\frac{1}{(u+1)^iu^{n-i}} = (-1)^{n+1}\frac{1}{u^n}\frac{u}{u+1}\sum_{i=0}^{n-2}\left(\frac{u}{u+1}\right)^i\\
&= (-1)^{n+1}\frac{1}{u^n}\frac{u}{u+1}\frac{\left(1-\left(\frac{u}{u+1}\right)^{n-1}\right)}{1-\frac{u}{u+1}}\\
&= (-1)^{n+1}\left(\frac{1}{u^{n-1}}-\frac{1}{(u+1)^{n-1}}\right).
\end{aligned}$$

Proof of (i). This is by induction on $n \geq 2$, the case $n = 2$ being easy to check. So assume that the lemma is true for $n-1 \geq 2$ and for all $j \leq n-1$. Then expanding along the jth row we have:

$$
\begin{aligned}
\mu_j^{(n)} &= \sum_{i=1}^{j-1}(-1)^{j+i}a_{ji}(1,2,\ldots,\hat{j},\ldots,n|1,2,\ldots,\hat{i},\ldots,\hat{j},\ldots,n,n+1) \\
&\qquad + \sum_{i=j+1}^{n}(-1)^{j+i-1}a_{ji}(1,2,\ldots,\hat{j},\ldots,n|1,2,\ldots,\hat{j},\ldots,\hat{i},\ldots,n,n+1) \\
&\qquad + (-1)^{j+n+1}a_{jn+1}(1,2,\ldots,\hat{j},\ldots,n|1,2,\ldots,\hat{j},\ldots,n) \\
&= \sum_{i=1}^{j-1}(-1)^{j+i}a_{ji}\mu_i^{(n-1)} + \sum_{i=j+1}^{n}(-1)^{j+i-1}a_{ji}\mu_{i-1}^{(n-1)} + (-1)^{j+n+1}a_{jn+1}det((a_{ij})_{(n-1)\times(n-1)}) \\
&= \sum_{i=1}^{j-1}(-1)^{j+i}a_{ji}\frac{(-1)^{i+1}a_{in+1}}{(u+1)^{i-1}u^{n-1-i}} + \sum_{i=j+1}^{n}(-1)^{j+i-1}a_{ji}\frac{(-1)^{i}a_{in+1}}{(u+1)^{i-2}u^{n-i}} \\
&\qquad + (-1)^{j+n}a_{jn+1}(-1)^n\left(\frac{1}{u^{n-2}} - \frac{1}{(u+1)^{n-2}}\right) \\
&= (-1)^{j+1}\sum_{i=1}^{j-1}\frac{a_{jn+1}}{(u+1)^{i}u^{n-i-1}} + (-1)^{j+1}\sum_{i=j+1}^{n}\frac{a_{jn+1}}{(u+1)^{i-2}u^{n-i+1}} \\
&\qquad + (-1)^{j}a_{jn+1}\left(\frac{1}{u^{n-2}} - \frac{1}{(u+1)^{n-2}}\right) \\
&= (-1)^{j+1}a_{jn+1}\left(\sum_{i=1}^{j-1}\frac{1}{(u+1)^{i}u^{n-i-1}} + \sum_{i=j+1}^{n}\frac{1}{(u+1)^{i-2}u^{n-i+1}} - \frac{1}{u^{n-2}} + \frac{1}{(u+1)^{n-2}}\right) \\
&= (-1)^{j+1}a_{jn+1}\left(\frac{1}{u^{n-1}}\sum_{i=1}^{j-1}\left(\frac{u}{u+1}\right)^{i} + \frac{(u+1)^2}{u^{n+1}}\sum_{i=j+1}^{n}\left(\frac{u}{u+1}\right)^{i} - \frac{1}{u^{n-2}} + \frac{1}{(u+1)^{n-2}}\right) \\
&= (-1)^{j+1}a_{jn+1}\left(\frac{1}{u^{n-1}}\frac{\left(\frac{u}{u+1} - \frac{u^j}{(u+1)^j}\right)}{(1-\frac{u}{u+1})} + \frac{(u+1)^2}{u^{n+1}}\frac{\left(\frac{u^{j+1}}{(u+1)^{j+1}} - \frac{u^n}{(u+1)^n}\right)}{(1-\frac{u}{u+1})} - \frac{1}{u^{n-2}} + \frac{1}{(u+1)^{n-2}}\right) \\
&= (-1)^{j+1}a_{jn+1}(\frac{u}{u^{n-1}}\left(1-\left(\frac{u}{u+1}\right)^{j-1}\right) + \frac{(u+1)^3}{u^{n+1}}\frac{u^{j+1}}{(u+1)^{j+1}}\left(1-\left(\frac{u}{u+1}\right)^{n-j}\right) \\
&\qquad - \frac{1}{u^{n-2}} + \frac{1}{(u+1)^{n-2}}) \\
&= (-1)^{j+1}a_{jn+1}(\frac{1}{u^{n-2}} - \frac{1}{(u+1)^{j-1}u^{n-1-j}} + \frac{1}{u^{n-j}(u+1)^{j-2}} - \frac{1}{(u+1)^{n-2}} \\
&\qquad - \frac{1}{u^{n-2}} + \frac{1}{(u+1)^{n-2}}) \\
&= (-1)^{j+1}a_{jn+1}\frac{1}{(u+1)^{j-1}u^{n-j}},
\end{aligned}
$$

as required for (i). (iii) is just a variation of (i). □

Note that as a $C(u)$-module the image of $\mathcal{I}_{n+1}$ has dimension n with generators a_{1n+1}, $a_{2n+1}, \ldots, a_{nn+1}$. Now let $\mathcal{K}_n = \{x \in V_n | \mathcal{I}_{n+1}(x) = 0\}$. Then by Lemma 4.1 $\mathcal{K}_n$ is B_n-invariant and by Lemma 7.1 and the above remark it has dimension $n(n-1)$. Now we have

$$\begin{aligned}
\mathcal{I}_{n+1}(v_1) &= \mathcal{I}_{n+1}(\mu_1 - ua_{12}\mu_2 + ua_{13}\mu_3 - ua_{14}\mu_4 + ua_{15}\mu_5 - \cdots + (-1)^{n+1}ua_{1n}\mu_n) \\
&= \frac{1}{u^{n-1}} + \frac{u}{(u+1)u^{n-2}}a_{12}a_{2n+1} + \frac{u}{(u+1)^2u^{n-3}}a_{13}a_{3n+1} + \ldots \\
&\qquad\qquad + \frac{u}{(u+1)^{n-1}}a_{1n}a_{nn+1} \\
&= \left(\frac{1}{u^{n-1}} + \frac{1}{(u+1)u^{n-2}} + \frac{1}{(u+1)^2u^{n-3}} + \cdots + \frac{1}{(u+1)^{n-1}}\right)a_{1n+1},
\end{aligned}$$

which is clearly non-zero. One similarly (or even directly from this) sees that $\mathcal{I}_{n+1}(v_j)$ is a non-zero multiple of a_{jn+1} for all $j \leq n$. Thus $< v_1, \ldots, v_n > \cap \mathcal{K}_n = \{0\}$. It follows that $< v_1, \ldots, v_n >$ is a complement to $\mathcal{K}_n$, both being B_n-invariant.

We now show that $\mathcal{K}_n$ is an irreducible representation of B_n. For this we define the following basis: for $1 \leq i \neq j \leq n$ let

$$\begin{aligned}
\gamma_{ij} &= a_{ij}\mu_j - \frac{(-1)^{i+j}}{u}\left(\frac{u}{u+1}\right)^{j-i}\mu_i \text{ if } i < j; \\
\gamma_{ij} &= a_{ij}\mu_j - \frac{(-1)^{i+j}}{u+1}\left(\frac{u+1}{u}\right)^{i-j}\mu_i \text{ if } i > j.
\end{aligned}$$

It will be convenient to put $\gamma_{ii} = 0$ for all i. Then using Lemma 7.3 we see that $\mathcal{I}_{n+1}(\gamma_{ij}) = 0$ for all $i \neq j$. Since the γ_{ij} are clearly independent, they form a basis for $\mathcal{K}_n$. We will find it convenient to write $\gamma_{ij} = a_{ij}\mu_j - \lambda_{ij}\mu_i$, which thus defines the $\lambda_{ij} \in C(u)$.

Lemma 7.4. *(i) For $1 \leq i < n$ and $j \neq i, i+1$ we have*

$$\begin{aligned}
&\sigma_i(\gamma_{ii+1}) = -\gamma_{i+1i}; \quad \sigma_i(\gamma_{i+1i}) = (\lambda_{i+1i} - 1)\gamma_{ii+1}; \quad \sigma_i(\gamma_{ij}) = \frac{u}{u+1}\gamma_{i+1j}; \\
&\sigma_i(\gamma_{i+1j}) = \gamma_{ij} + \lambda_{i+1j}\gamma_{ii+1}; \quad \sigma_i(\gamma_{ji}) = -\frac{u+1}{u}\gamma_{ji+1}; \\
&\sigma_i(\gamma_{ji+1}) = -\gamma_{ji} - \frac{1}{u}\gamma_{ji+1}.
\end{aligned}$$

(ii) For the action of σ_i^2 we have:

$$\begin{aligned}
&\sigma_i^2(\gamma_{ii+1}) = \frac{u+1}{u}\gamma_{ii+1}; \quad \sigma_i^2(\gamma_j) = \frac{u}{u+1}(\gamma_{ij} + \lambda_{i+1j}\gamma_{ii+1}); \\
&\sigma_i^2(\gamma_{ji}) = \frac{u+1}{u}(\gamma_{ji} + \frac{1}{u}\gamma_{ji+1}); \quad \sigma_i^2(\gamma_{i+1j}) = \frac{u}{u+1}\gamma_{i+1j} - \lambda_{i+1j}\gamma_{i+1i}; \\
&\sigma_i^2(\gamma_{ji+1}) = \frac{1}{u} + \frac{u^2+u+1}{u^2}\gamma_{ji+1}; \quad \sigma_i^2(\gamma_{i+1i}) = \frac{1+u}{u}\gamma_{i+1i}.
\end{aligned}$$

Proof. (i) follows from (1.2) and (7.1), and (ii) follows from (i). □

Lemma 7.5. *Let $1 \leq i < n$. The matrix m_i^2 representing the action of σ_i^2 on $\mathcal{K}_n$ is diagonalisable and has eigenvalues $1, \frac{u}{u+1}, \frac{u+1}{u}, \frac{(u+1)^2}{u^2}$.*

Proof. We need only consider $i = 1$ and so we will give a basis for the eigenspaces of m_1^2 corresponding to these eigenvalues. The dimensions will be seen to sum to $n(n-1)$ and so the result will follow.

The elements γ_{rs} for $r, s \neq i, i+1$ are 1 eigenvectors; there are $(n-2)(n-3)$ of these. For $i > 2$ the elements $\gamma_{i1} - \gamma_{i2}$ are also fixed; there are $n-2$ of these.

The elements $\gamma_{ii+1}, \gamma_{i+1i}$ are eigenvectors for the eigenvalue $\frac{u+1}{u}$.

For $i > 2$ the elements $\gamma_{i1} + \frac{u+1}{u}\gamma_{i2}$ are eigenvectors for the eigenvalue $\frac{(u+1)^2}{u^2}$; there are $n-2$ of these.

For $1 < i < n$ the elements $\gamma_{1i} - \frac{\lambda_{2j}}{\lambda_{2n}}\gamma_{1n}$ are eigenvectors for $\frac{u}{u+1}$; there are $n-2$ of these. For $i \neq 2, n$ the elements $\gamma_{2i} - \frac{\lambda_{2j}}{\lambda_{2n}}\gamma_{2n}$ are eigenvectors for $\frac{u}{u+1}$; there are $n-2$ of these. $\square$

This last result shows that this representation (over $\mathbb{C}$ say) has at least 4 eigenvalues for each σ_i. Since we will show that it is irreducible, it follows that it is not a summand of the Jones representation [J].

Let $0 \neq b \in \mathcal{K}_n$ and write $b = \sum c_{ij}\gamma_{ij}$ with $c_{ij} \in C(u)$. Let $i' = min\{i | c_{ij} \neq 0 \text{ for some } j\}, j' = min\{j | c_{ij} \neq 0 \text{ for some } i\}$. Let $r = min\{i', j'\}$. Assume $r = i'$ (the other case is similar). Let $s = min\{j | c_{rj} \neq 0\}$. Note that if $r > 1$, then $\sigma_{r-1}(\gamma_{rs}) = \gamma_{r-1s} + \lambda_{is}\gamma_{r-1r}$ and so we have a smaller r in $\sigma_{r-1}(b)$. Thus we may assume that $r = 1$. Similarly, if $s \neq 2$, then we can lower s by acting on b by σ_{s-1}. It follows that we may assume that $c_{12} \neq 0$.

For this b we can now let $b = \sum_{j=1}^4 b_j$ where each b_j is an eigenvector for σ_i^2; namely $\sigma_1^2(b_1) = b_1, \sigma_1^2(b_2) = \frac{u+1}{u}b_2$ etc. But $c_{12} \neq 0$ shows that $b_2 \neq 0$ and so we see that some $C(u)$-combination of $\sigma_1^{2k}(b)$ contains a non-zero element in the $\frac{u+1}{u}$-eigenspace of σ_1^2. This eigenspace is spanned by γ_{12} and γ_{21} and so we may assume that $b = c_1\gamma_{12} + c_2\gamma_{21}$. But one now checks that either γ_{12} or γ_{21} is a linear combination of $b, \sigma_2^2(b), \sigma_2^4(b), \sigma_2^6(b)$ (use Lemma 7.4). Thus Lemma 7.4 shows that $C(u)(B_n(b))$ contains γ_{12}. But $C(u)(B_n(\gamma_{12}))$ contains all the elements γ_{ij} and so $\mathcal{K}_n$ is irreducible. This proves Theorem 1.5. We will denote the $\mathcal{K}_n$ representation of B_n by V_{n,n^2-n}.

We now consider how these two irreducible B_n-representations split when considered as B_{n-1}-modules. First, the representation $< v_1, \dots, v_n >$ clearly splits as $< v_1, \dots, v_{n-1} > \oplus < v_n >$, both of which are irreducible B_{n-1} representations. We will denote the trivial representation of B_{n-1} by V_{n-11} and the $< v_1, \dots, v_{n-1} >$ representation of B_{n-1} by V_{n-1n-1}.

For the $\mathcal{K}_n$ representation we note that the element $\sum_{i=1}^{n-1}(-1)^i\gamma_{ni}$ is fixed by B_{n-1}. This gives a 1-dimensional summand. From the above we clearly see that the span of $\{\gamma_{ij} | 1 \leq i, j \leq n-1\}$ is an irreducible B_{n-1}-module; it has dimension $(n-1)(n-2)$.

Now let $w_i = \gamma_{ni} + \frac{u+1}{u}\gamma_{ni+1}$ for $1 < i < n$ and $W =< w_2, \dots, w_{n-1} >$. We will show that

W is an irreducible B_{n-1}-module. We first note:

$$\sigma_i(w_{i-1}) = \sigma_i(\gamma_{ni-1} + \frac{u+1}{u}\gamma_{ni}) = \gamma_{ni-1} + \frac{u+1}{u}\left(-\frac{u+1}{u}\gamma_{ni+1}\right)$$
$$= \gamma_{ni-1} + \frac{u+1}{u}\gamma_{ni} - \frac{u+1}{u}\left(\gamma_{ni} + \frac{u+1}{u}\gamma_{ni+1}\right) = w_{i-1} - \frac{u+1}{u}w_i;$$
$$\sigma_i(w_i) = \sigma_i(\gamma_{ni} + \frac{u+1}{u}\gamma_{ni+1}) = -\frac{u+1}{u}\gamma_{ni+1} + \frac{u+1}{u}(-\gamma_{ni} - \frac{1}{u}\gamma_{ni+1}) = -\frac{u+1}{u}w_i;$$
$$\sigma_i(w_{i+1}) = \sigma_i(\gamma_{ni+1} + \frac{u+1}{u}\gamma_{ni+2}) = -\gamma_{ni} - \frac{1}{u}\gamma_{ni+1} + \frac{u+1}{u}\gamma_{ni+1} = -w_i + w_{i+1}.$$

Thus the $(n-2)\times(n-2)$ matrices $m_i, i < n-1$ representing σ_i relative to the basis w_i are (where $t = \frac{u+1}{u}$):

$$m_1 = \begin{pmatrix} -t & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad m_2 = \begin{pmatrix} 1 & -t & 0 & 0 & \dots \\ 0 & -t & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$m_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & -t & 0 & \dots \\ 0 & 0 & -t & 0 & \dots \\ 0 & 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad etc.$$

We now show that this gives an irreducible representation of B_{n-1}. It will suffice to show that the action of these matrices on the row space is irreducible. Let U be a non-trivial subrepresentation and let $w \in U$. We first show that U contains some $e_i = (0,\dots,0,1,0,\dots,0)$. This is certainly the case if $wm_i \neq w$ for some i. Now if $wm_i = w$ for all i, then $wM = 0$, where

$$M = m_1 + \dots + m_{n-2} - (n-2)I_{n-2} = \begin{pmatrix} -t & -t & 0 & 0 & 0 & \dots \\ -1 & -t & -t & 0 & 0 & \dots \\ 0 & -1 & -t & -t & 0 & \dots \\ 0 & 0 & -1 & -t & -t & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

But this matrix has non-zero determinant and so $w = 0$; thus $e_i \in W$ for some i.

Now suppose that $e_i \in W$. Then $e_{i+1} \in < e_i, e_i m_{i+1} >$ and one easily sees that $W =< e_1,\dots,e_{n-2} >$. This gives the irreducibility. We will denote this representation of B_{n-1} by V_{n-1n-2}.

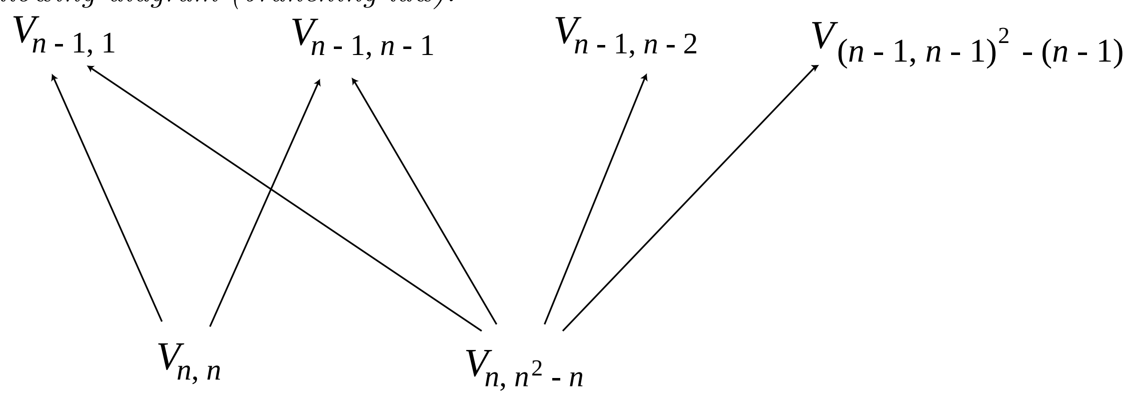
For the last representation we let

$$z_1 = -u\gamma_{12} + u\gamma_{13} - u\gamma_{14} + u\gamma_{15} - \dots$$
$$z_2 = -(u+1)\gamma_{21} - u\gamma_{23} + u\gamma_{24} - u\gamma_{25} + \dots$$
$$z_3 = (u+1)\gamma_{31} - (u+1)\gamma_{32} - u\gamma_{34} + u\gamma_{35-\dots}$$
$$\dots$$
$$z_{n-1} = \pm(u+1)\gamma_{n-11} \mp (u+1)\gamma_{n-12} \pm (u+1)\gamma_{n-13} \mp (u+1)\gamma_{n-14}\dots$$

exactly in analogy to how we defined the v_i in (7.2). Then the same argument used there shows that $Z =< z_1, \dots, z_{n-1} >$ is an irreducible B_{n-1}-module with the $(n-1) \times (n-1)$ matrices given by (7.3). We denote this representation by $V_{n-1,n-1}$.

We have now proved that the restrictions of $V_{n,n}$ and of V_{n,n^2-n} to B_{n-1} are multiplicity free:

Theorem 7.6. *The restrictions $Res^{B_n}_{B_{n-1}} V_{nn}$ and $Res^{B_n}_{B_{n-1}} V_{nn^2-n}$ decompose according to the following diagram (branching law):*



The restriction $Res^{B_n}_{B_{n-1}} V_{n,n-1}$ decomposes as $V_{n-1,1} \oplus V_{n-1,n-2}$. $\square$

For results concerning the existence of branching laws for the classical groups see [GW, Ch. 8].

§8 The action of B_n on $(S|\{n+1, \dots, n+|S|\})$

By Corollary 3.2 and Lemma 5.8 the action on the elements in the B_n-orbit of $(S|\{n+1, \dots, n+|S|\})$ is the same as the action on a submodule of $R_n^{(0)}$-module $\bigwedge^{|S|} V_n$. By Lemma 4.2 we see that every monomial in the B_n-orbit of $(S|\{n+1, \dots, n+|S|\})$ has the form $a_{r_1 s_1} \dots a_{r_k s_k}(S'|\{n+1, \dots, n+|S|\})$ where $s_i \in S'$ and $r_i \notin S'$. Let $V(S)$ denote the $C(u)$-module generated by all such elements.

Lemma 8.1. *The dimension of $V(S)$ is $\binom{n}{|S|}^2$.*

Proof. We will need the following:

Lemma 8.2. *Let $a_{r_1 s_1} \dots a_{r_k s_k}$ be given as above. Let π be any permutation of the set $\{r_1, \dots, r_k\}$. Then there is $c \in C(u)$ such that*

$$a_{\pi(r_1)s_1} \dots a_{\pi(r_k)s_k} = c a_{r_1 s_1} \dots a_{r_k s_k} mod I(\{1, 2, \dots, n\}).$$

Proof. From the defining relations for $I(\{1, \dots, n\})$ we see that for any distinct i, j, k, m there are non-zero $c, c' \in C(u)$ such that $c a_{im} a_{mj} = a_{ij}$ and $a_{km} a_{mj} = c' a_{kj}$. Thus in $R_n / I(\{1, \dots, n\})$ we have

$$a_{ij} a_{km} = c a_{im} a_{mj} a_{km} = c a_{im} a_{km} a_{mj} = c c' a_{im} a_{kj} = c c' a_{kj} a_{im}.$$

Thus in $R_n / I(\{1, \dots, n\})$ we can interchange i and k in any product of the form $a_{ij} a_{km}$. The result easily follows. $\square$

We count the number of elements of the form $a_{r_1 s_1} \dots a_{r_k s_k}(S'|\{n+1,\dots,n+|S|\})$ with the r_i, s_i as described above and with S' fixed. Note that there exactly $\binom{n}{|S|}$ of the $(S'|\{n+1,\dots,n+|S|\})$s.

Now for fixed $|S| \geq k \geq 0$ there are $\binom{n-s}{k}$ choices of the r_i and $\binom{s}{k}$ choices of the s_i and so there are $\binom{n-s}{k}\binom{s}{k}$ total such choices. Summing over the various k gives

$$dimV(S) = \sum_{k=0}^{|S|} \binom{n-s}{k}\binom{s}{k} = \binom{n}{|S|},$$

the last equality being a well-known binomial identity [R]. This proves Lemma 8.1. □

We will need the following construction. Let S be a subsequence of $\{1,2,\dots,n\}$ with $|S| = k$. For $m > 0$ we let $N_m = [n+1, n+2, \dots, n+m]$. Then to the element $(S|N_k)'$ we associate

$$\omega(S|N_k)' = (S, N_{n-k}|1,2,\dots,n)'.$$

We extend the action of ω so as to obtain an R_n^∞-module map, also denoted by ω.

The action of B_n on $(N_n|1,2,\dots n)'$ gives the sign permutation $\epsilon : B_n \to S_n \to \{\pm 1\}$. Thus the action of $\alpha \in B_n$ on the $\omega(S|N_k)'$ is given by

$$\alpha(\omega(S|N_k)') = \epsilon(\alpha)\omega(\alpha(S|N_k)').$$

Thus the representation theory for the $(S|N_k)'$ is the same as for the $\omega(S|N_k)'$.

Now we define a map $\mathcal{J} = \mathcal{J}_{n+1}$ by

$$\mathcal{J}_{n+1}(w(S|N_k)') = w \times det(\omega(S|N_k)') \bmod I_n.$$

Since the ideal I_n is B_n-invariant we see that the B_n-action commutes with $\mathcal{J}$: for all $\alpha \in B_n, b \in \mathcal{R}_n^k$ we have $\mathcal{J}\alpha(b) = \alpha\mathcal{J}(b)$. Thus the image of $\mathcal{J}$ is a B_n-representation space which is isomorphic to a direct sum of the B_n-irreducible summands of $\mathcal{R}_n^1$. We will next show that $\mathcal{J}$ is not the zero homomorphism:

Lemma 8.3. *For $1 \leq k < n$ we have*

$$\mathcal{J}_{n+1}(([1,2,\dots,k]|N_k)') = \frac{(-1)^k}{u^k} det(n+1, n+2, \dots, n|k+1, k+2, \dots, n).$$

Proof. Consider the matrix $M_k = (1,2,\dots,k,n+1,n+2,\dots n+(n-k)|1,2,\dots,n)$. Then

$$M_k = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1k} & a_{1k+1} & \dots & a_{1n} \\ a_{21} & 0 & a_{23} & \dots & a_{2k} & a_{2k+1} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \dots & 0 & a_{k,k+1} & \dots & a_{k,n} \\ a_{n+1,1} & a_{n+12} & a_{n+13} & \dots & a_{n+1k} & a_{n+1,k+1} & \dots & a_{n+1,n} \\ a_{n+21} & a_{n+22} & a_{n+23} & \dots & a_{n+2,k} & a_{n+2,k+1} & \dots & a_{n+2,n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ a_{n+n-k,1} & a_{n+n-k,2} & a_{n+n-k,3} & \dots & a_{n+n-k,k} & a_{n+n-k,k+1} & \dots & a_{n+n-k,n} \end{pmatrix}$$

Now $a_{12}a_{2p} = \frac{1}{u}a_{1p}$ for $p > 2$ and so adding $-ua_{12}$ times the second row to the first row produces the matrix whose first row is $(\frac{-1}{u+1}, a_{12}, 0, 0, \dots, 0)$. Similarly, adding $-ua_{23}$ times the third row to the second row produces the matrix whose second row is $(0, \frac{-1}{u+1}, a_{23}, 0, \dots, 0)$. Repeating this process $k-1$ times and then adding $-(u+1)a_{kn+1}$ times the $k+1$th row to the kth row produces the matrix

$$\begin{pmatrix}
\frac{-1}{u+1} & a_{12} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & \frac{-1}{u+1} & a_{23} & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & 0 & \frac{-1}{u+1} & a_{34} & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\
0 & 0 & 0 & 0 & \dots & \frac{-1}{u+1} & a_{k-1,k} & \dots & 0 \\
\frac{-1}{u}a_{k1} & \frac{-1}{u}a_{k2} & \frac{-1}{u}a_{k3} & \frac{-1}{u}a_{k4} & \dots & \frac{-1}{u}a_{k,k-1} & \frac{-1}{u} & \dots & 0 \\
a_{n+11} & a_{n+12} & a_{n+13} & a_{n+14} & \dots & a_{n+1,k-1} & a_{n+1k} & \dots & a_{n+1n} \\
a_{n+21} & a_{n+22} & a_{n+23} & a_{n+24} & \dots & a_{n+2,k-1} & a_{n+2k} & \dots & a_{n+2n} \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\
a_{2n-k,1} & a_{2n-k,2} & a_{2n-k,3} & a_{2n-k,4} & \dots & a_{2n-k,k-1} & a_{2n-k,k} & \dots & a_{2n-k,n}
\end{pmatrix}$$

One can now see that the determinant of this matrix is $det(n+1, n+2, \dots, n+n-k|k+1, k+2, \dots, n)$ multiplied by the determinant of the principal $k \times k$ matrix. To find this latter determinant, which we prove is $(-1/u)^k$, we induct on $k \geq 1$ the case $k = 1$ being clear (look at the (k,k) entry, not the (1,1) entry). For $k > 1$ we note that by the $k-1$ case the $(1,1)$ entry of the adjoint matrix is $(-1/u)^{k-1}$; thus expanding along the first row we get:

$$\begin{aligned}
&\frac{-1}{u+1}\left(\frac{-1}{u}\right)^{k-1} - a_{12}a_{23}a_{34}\dots a_{k-1k}\left(\frac{-1}{u}a_{k1}\right)(-1)^{k-1} \\
&= \frac{-1}{u+1}\left(\frac{-1}{u}\right)^{k-1} - \frac{1}{u^{k-2}}\left(\frac{-1}{u}\right)\frac{1}{u(u+1)}(-1)^k \\
&= \frac{(-1)^k}{u^{k-1}(u+1)}\left(1+\frac{1}{u}\right) = \frac{(-1)^k}{u^k},
\end{aligned}$$

as required. □

Proposition 8.4. *The image and the kernel of $\mathcal{J}$ are non-trivial.*

Proof. That the image is non-trivial follows from Lemma 8.3. Let $\eta = \eta_{\{1,\dots,n,n+1\}}$ as in (4.1).

Lemma 8.5. *A Gröbner basis for the ideal I_n relative to the 'degree lexicographical' order [AL, p. 19] consists of all elements of the following forms (where $1 \leq i, j, k \leq n$):*

$$\begin{aligned}
&a_{ij}a_{ji} - \frac{1}{u(u+1)}, \quad &&\textit{for} \quad i \neq j; \\
&a_{ij}a_{jk} - \frac{1}{u}a_{ik}, &&\textit{if } (j-i)(k-i)(k-j) > 0; \\
&a_{ij}a_{jk} - \frac{1}{u+1}a_{ik} &&\textit{if } (j-i)(k-i)(k-j) < 0 \\
&a_{ij}a_{rs} - \frac{\eta(a_{ij}a_{rs})}{\eta(a_{is}a_{rj})}a_{is}a_{rj} &&\textit{if} \quad i,j,r,s \textit{ are distinct and } i < r, j < s.
\end{aligned}$$

Proof. We should here also note that we are ordering the polynomial ring generators a_{ij} in decreasing order with $a_{12} > a_{13} > \cdots > a_{1n} > a_{21} > a_{23} > \cdots > a_{n,n-1}$. One can now check the the elements given satisfy the requirements for a Gröbner basis relative to the degree lexicographical order [AL, §1.6]. (Note that this order is called 'glex' in [MA]). $\square$

Now note that if S is a subsequence of $\{1,\ldots,n\}$ with $|S| = k$, then $\mathcal{J}(S|N_k)$ is the determinant of a certain matrix, which determinant can be expanded along the row labeled $n+1$:

$$\mathcal{J}(S|N_k)' = \sum_{i=1}^{n}(-1)^{n+1+i}a_{n+1i}det(S, n+2, \ldots n+n-1|1, \ldots, \hat{i}, \ldots, n)$$

and then the monomials can be reduced mod I_{n+1}, so that each monomial looks like $\mu = a_{n+1i}b_{n+2c_2}b_{n+3c_3}\ldots b_{n+n-kc_{n-k}}$ where b_{n+ic_i} either has the form $a_{n+vj(v)}a_{j(v)e(v)}$ or the form $a_{n+vuk(v)}$. In the first case we will call the $j(v)$ the *middle indices* of the monomial. We note that no two b_{n+ic_i} have the same middle indices.

We will say that such a monomial μ has *end set* $\{i, e(2), e(3), \ldots, e(n-k)\}$. We note that for a monomial of $\mathcal{J}(S|N_k)$ as in the above we must have $\{i, e(2), e(3), \ldots, e(n-k)\} = \{1,\ldots,n\} \setminus S$. We can collect together all such terms and so are able to write

$$\mathcal{J}_{n+1}((S|N_k)') = \sum_{i \notin S} a_{n+1i}\mu_i \tag{8.1}$$

where each μ_i is a sum of monomials all having the same end set, and each monomial of μ_i is in normal form $a_{n+1i}b_{n+2,c_2}b_{n+3,c_3}\ldots b_{n+n-k,c_{n-k}}$ (as in the above) relative to the Gröbner basis of Lemma 8.5.

It is now easy to see from Lemma 8.3 that $\mathcal{J}_{n+1}(([1,2,\ldots,k]|N_k)' \neq 0$. We will next show that the kernel of $\mathcal{J}$ is non-trivial.

Note that given any subsequence $S \subset \{1,\ldots,n\}, |S| = k$ and any end set $E, |E| = n-k$, there is $w \in R_n^{(0)}$ such that all monomials in $w \times det(S, n+1, \ldots n+n-k|1,\ldots,n)$ have the end set E.

Now note that if S, S' are subsequences of $\{1,\ldots,n\}$ with $|S| = |S'| = k$, then $\mathcal{J}(S|N_k)' \neq \mathcal{J}(S'|N_k)'$ whenever $S \neq S'$, as they have different end sets. Since there are $\binom{n}{k}$ of the Ss we see that we must check for relations among the $w\mathcal{J}(S|N_k)'$ only in the set of such which have the same end sets.

For S a subsequence of $\{1,\ldots,n\}$ and $\{1,\ldots,n\} \setminus S = \{s_1, \ldots s_{n-k}\}$ let

$$\delta_S = \delta_{s_1,s_2,\ldots,s_{n-k}} = det(n+1, n+2, \ldots, n+n-k|s_1, s_2, \ldots, s_{n-k}) \bmod I_{n+1}.$$

Then Lemma 8.3 shows that we have $\delta_{k+1,k+2,\ldots,n}$ in the image of $\mathcal{J}$. From the above we see that for any end set E and any sequence $S \subset \{1,\ldots,n+1\}$ with $|S| = k$ there is $w \in R_n^{(0)}$ such that each monomial of the expanded form of $w\delta_S$ has the end set E.

We also see from (1.3) that

$$\begin{aligned}\sigma_k^2\delta_{k+1,k+2,\ldots,n} &= \sigma_k\delta_{k,k+2,\ldots,n}\\ &= \delta_{k+1,k+2,\ldots,n} + a_{k,k+1}\delta_{k,k+2,\ldots,n}.\end{aligned}$$

Similarly we have

$$\sigma_{k-1}\sigma_k^2\delta_{k+1,k+2,\dots,n} = \delta_{k+1,k+2,\dots,n} + a_{k-1,k+1}\delta_{k-1,k+2,\dots,n},$$
$$\sigma_{k-2}\sigma_{k-1}\sigma_k^2\delta_{k+1,k+2,\dots,n} = \delta_{k+1,k+2,\dots,n} + a_{k-2,k+1}\delta_{k-2,k+2,\dots,n},$$

etc. Thus we can get all $a_{hk+1}\delta_{h,k+2,\dots,n}$ for $h < k+1$. We also have

$$\begin{aligned}
\sigma_k^2\sigma_{k+1}^2(a_{hk+1}\delta_{(h,k+2,\dots,n)}) &= \frac{u+1}{u^2}a_{h,k+2}a_{k,k+1}\delta_{h,k,k+3,\dots,n} \\
&\quad + \frac{(u+1)^2}{u^2}a_{h,k+1}\delta_{h,k+2,\dots,n} + \frac{u+1}{u^2}a_{h,k+2}\delta_{h,k+1,k+3,\dots,n}; \\
\sigma_k^4\sigma_{k+1}^2(a_{hk+1}\delta_{(h,k+2,\dots,n)}) &= \frac{2u^2+2u+1}{u^3}a_{h,k+2}a_{k,k+1}\delta_{h,k,k+3,\dots,n} \\
&\quad + \frac{(u+1)^3}{u^3}a_{h,k+1}\delta_{h,k+2,\dots,n} + \frac{u^2+u+1}{u^3}a_{h,k+2}\delta_{h,k+1,k+3,\dots,n},
\end{aligned}$$

from which we see that we can get $a_{h,k+2}a_{k,k+1}\delta_{h,k,k+3,\dots,n}$ and $a_{h,k+2}\delta_{h,k+1,k+3,\dots,n}$. Continuing in this way we get the first sentence of:

Lemma 8.6. *The $C(u)$-span of the B_n-orbit of $\delta_{k+1,\dots,n}$ contains all $w\delta_S$ with the end set $\{k+1,\dots,n\}$. The $C(u)$-span of the B_n-orbit of $\delta_{k+1,\dots,n}$ contains all $w\delta_S$ with any end set.*

Proof. For the second sentence we let V_E be the $C(u)$-span of all $w\delta_S$ with end set E. Then this follows from the fact that for $\alpha \in B_n$ we have $\alpha(V_E) = V_{\Pi_n(\alpha)(E)}$. $\square$

Next we note that if $E = \{s_1,\dots,s_{n-k+1}\}$ with $s_1 < s_2 < \dots < s_{n-k+1}$, then we can evaluate $det(s_1, n+1, n+2, \dots, n+n-k|s_1,\dots,s_{n-k+1})$ in two ways: (i) by Lemma 8.3 we see that it is equal to $-\frac{1}{u}\delta_{s_2,\dots,s_{n-k+1}}$; (ii) expanding along the row labeled s_1 we have

$$\begin{aligned}
det(s_1, n+1, n+2, \dots, n+n-k|s_1,\dots,s_{n-k+1}) &= \sum_{i=1}^{n-k+1}(-1)^i a_{s_1s_i}\delta_{s_1,\dots,\hat{s}_i,\dots s_{n-k+1}} \\
&= \sum_{i=2}^{n-k+1}(-1)^i a_{s_1s_i}\delta_{s_1,\dots,\hat{s}_i,\dots s_{n-k+1}}.
\end{aligned}$$

Thus we have the relation

$$\sum_{i=2}^{n-k+1}(-1)^i a_{s_1s_i}\delta_{s_1,\dots,\hat{s}_i,\dots s_{n-k+1}} + \frac{1}{u}\delta_{s_2,\dots,s_{n-k+1}} = 0, \tag{8.2}$$

among the $w\delta_S$. This is a relation involving terms all having the same end set, namely $\{s_2,\dots,s_{n-k+1}\}$. However, given any end set E' we can multiply such an expression (8.2) by some $w \in R_n^\infty$ so that the resulting product has end set E'. There are $\binom{n}{k}$ end sets. This proves Proposition 8.4. $\square$

We now show that there are exactly $\binom{n-1}{k}$ of the $w\delta_S$ with $|S| = k$ and all having the same end set E. We may clearly take $E = \{k+1,\dots,n\}$ and we count them according to the degree of w (where we always take w reduced as in the above). For $deg(w) = 0$, there is just one possibility. If $deg(w) = 1$, then $w = a_{n+z,i}$, where $z = 2, \ldots n-k, 1 \leq i \leq k$ and there are $\binom{n-k-1}{1} \times \binom{k}{1}$ possibilities. If $deg(w) = 2$, then $w = a_{n+z_1,i}a_{n+z_2,j}$, where $z_1, z_2 = 2, \ldots n-k, z_1 \neq z_2, 1 \leq i \neq j \leq k$ and there are $\binom{n-k-1}{2} \times \binom{k}{2}$ possibilities (by Lemma 8.5 or Lemma 4.6). Continuing in this way we see that the total number is

$$\sum_{i=0}^{k} \binom{n-k-1}{i} \times \binom{k}{i} = \binom{n-1}{k},$$

as required. It follows from Lemma 8.5 that these are all linearly independent over $C(u)$. Thus the image of $\mathcal{J}_{n+1}$ has dimension $\binom{n}{k}\binom{n-1}{k}$, proving the first part of Theorem 1.7.

We now show that this image is an irreducible representation. We have the following actions:

$$\begin{aligned}
&\sigma_i(a_{n+z,i}) = \frac{u+1}{u} a_{n+z,i+1}, \quad \sigma_i(a_{n+z,i+1}) = a_{n+z,i},\\
&\sigma_i(a_{n+z,j}) = a_{n+z,j} \text{ for } j \neq i, i+1,\\
&\sigma_i(\delta_{s_1,\ldots,s_m}) = \delta_{s_1,\ldots,s_m} \text{ if } \{s_1, \ldots, s_m\} \cap \{i, i+1\} = \emptyset,\\
&\sigma_i(\delta_{s_1,\ldots,s_m}) = -\delta_{s_1,\ldots,s_m} \text{ if } \{s_1, \ldots, s_m\} \cap \{i, i+1\} = \{i, i+1\},\\
&\sigma_i(\delta_{s_1,\ldots,s_j=i,\ldots,s_m}) = \delta_{s_1,\ldots,i+1,\ldots,s_m} + a_{i+1,i}\delta_{s_1,\ldots,i,\ldots,s_m},\\
&\sigma_i(\delta_{s_1,\ldots,s_j=i+1,\ldots,s_m}) = \delta_{s_1,\ldots,i,\ldots,s_m}.
\end{aligned} \tag{8.3}$$

Recall that a generator $w\delta_S$ is completely determined by its end set together with w (always chosen in normal form). Choose $b \neq 0$, an element in an irreducible subrepresentation of $C(u) < w\delta_S >$. We will show that the $C(u)$-span of the B_n-orbit of b contains some $w\delta_S$; this will show irreducibility of $C(u) < w\delta_S >$. Then by (8.3) we can assume that (some image of) b has a non-trivial monomial of the form $w\delta_{1,2,\ldots,m}$; and then that the span of $b, \sigma_m^2(b), \sigma_m^4(b), \ldots$ contains a single monomial. This proves the irreducibility. □

Conjecture 8.7. We conjecture that the kernel of $\mathcal{J}$ (having dimension $\binom{n}{k}\binom{n-1}{k-1}$) is also irreducible.

§9 Plücker relations

Here we prove Theorem 1.8. We first describe the Plücker relations. These give the relations between the minors of generic matrices:

Lemma 9.1. *[BH Lemma 7.2.3] For every $m \times n$ matrix X, $m \leq n$, with entries in a commutative ring A and all indices*

$$a_1, \ldots, a_p, b_q, \ldots, b_m, c_1, \ldots, c_s \in \{1, 2, \ldots, n\}$$

such that $s = m - p + q - 1 > m$ and $t = m - p > 0$, we have

$$\sum_{\substack{i_1 < \cdots < i_t \\ i_{t+1} < \cdots < i_s \\ \{i_1, \ldots, i_s\} = \{1, \ldots, s\}}} \sigma(i_1, \ldots, i_s)[a_1, \ldots, a_p, c_{i_1}, \ldots, c_{i_t}][c_{i_{t+1}}, \ldots, c_{i_s}, b_q, \ldots, b_m] = 0.$$

Here if $\{1, \ldots, s\} = \{i_1, \ldots, i_s\}$, then $\sigma(i_1, \ldots, i_s)$ is the sign of the permutation determined by $(i_1, \ldots, i_s)$ and $[x_1, \ldots, x_m]$ is the minor of X using columns indexed by $x_1, \ldots, x_m$.

Example 9.2. The simplest non-trivial Plücker relation is the following 'Pfaffian':

$$[1,2][3,4]-[1,3][2,4]+[1,4][2,3]=0.$$

Now given any set $\mathcal{U}$ of Plücker relations we can look at the corresponding elements of $(R_n/I_n)[(S|N)'|S\subset\{1,\dots,n\},|S|=k]$. These are obtained by replacing each $[S]$ in the Plücker relation by $(S|N)'$. For example for $n=4, S=[1,2], N=[5,6]$ and $[1,2][3,4]-[1,3][2,4]+[1,4][2,3]\in\mathcal{U}$ (as in 9.2) then the element of $\mathcal{R}_4^2$ would be

$$(1,2|5,6)'(3,4|5,6)'-(1,3|5,6)'(2,4|5,6)'+(1,4|5,6)'(2,3|5,6)'.$$

The orbit of such elements under the action of B_n would then generate a B_n-invariant $C(u)$-submodule $< B_n(\mathcal{U}) >\subset\mathcal{R}_n^2$.

Lemma 9.3. *If $\mathcal{U}$ is finite, then $< B_n(\mathcal{U}) >$ is a finitely-generated free $C(u)$-module.*

Proof. We need only consider the case where $\mathcal{U}=\{a\}$ has a single element. But now each monomial in a has the same form and there are only a finite number of monomials of the same form as the monomials in a. The result follows. □

If π is a Plücker relation, then the corresponding ring element will be denotes by $(\pi|N)'$. In the next result we will give the action of B_n on these Plücker relations.

Lemma 9.4. *Let $S,T\subset\{1,2,\dots,n\}$ with $|S|=|T|=k$ and let $N=N_k$. Let π be the Plücker relation determined by $(S|N)'(T|N)'$ and let $1\le r<n$. Then*

$$\begin{aligned}
&\sigma_r(\pi|N)'=-(\pi|N)' \quad \textit{if} \quad r,r+1\in S\cup T;\\
&\sigma_r(\pi|N)'=t_r(\pi|N)'-a_{r+1r}(\pi|N)' \quad \textit{if} \quad r\in S\cup T, r+1\notin S\cup T;\\
&\sigma_r(\pi|N)'=t_r(\pi|N)' \quad \textit{if} \quad r\notin S\cup T, r+1\in S\cup T;\\
&\sigma_r(\pi|N)'=(\pi|N)' \quad \textit{if} \quad r,r+1\notin S\cup T.
\end{aligned}$$

Here $t_r\in S_n$ is the transposition.

Proof. We consider the action of the generators σ_r on the monomial summands of $(\pi|N)$. Let $(S|N)(T|N)$ represent one of these monomial summands. First from (1.3) we note that

$$\begin{aligned}
&\sigma_r(S|N)'=(S|N)' \quad \text{if} \quad r,r+1\notin S;\\
&\sigma_r(S|N)'=-(S|N)' \quad \text{if} \quad r,r+1\in S;\\
&\sigma_r(S|N)'=(t_rS|N)'-a_{r+1r}(S|N)' \quad \text{if} \quad r\in S, r+1\notin S;\\
&\sigma_r(S|N)'=(t_rS|N)' \quad \text{if} \quad r+1\in S, r\notin S.
\end{aligned}\tag{9.1}$$

Now consider the situation where $r,r+1\in S\cup T$. Then there are four cases: (i) $r,r+1\in S$; (ii) $r,r+1\in T$; (iii) $r\in S,r+1\in T$; (iv) $r\in T,r+1\in S$.

For (i) (9.1) shows that $\sigma(S|N)'=-(S|N)',\sigma(T|N)'=-(T|N)'$ and so the result follows. (ii) is similar.

For (iii) (9.1) gives

$$\sigma_r(S|N)'=(t_rS|N);\quad \sigma_r(T|N)'=(t_rT|N)'-a_{r+1r}(T|N)'.$$

Now let $U = t_r S, V = t_r T$. Then the monomial summand $(U|N)'(V|N)'$ also occurs in $(\pi|N)'$, only with sign opposite to the sign of $(S|N)'(T|N)'$. Now we note that

$$\sigma_r(U|N)' = (t_r U|N)' \quad \text{and} \quad \sigma_r(V|N)' = (t_r V|N)' - a_{r+1r}(V|N)'$$

form which it follows that

$$\sigma_r((S|N)'(T|N)' - (U|N)'(V|N)') = -((S|N)'(T|N)' - (U|N)'(V|N)').$$

Thus $\sigma_r(\pi|N)' = -(\pi|N)'$ as required. Case (iv) is similar (interchange S, T with U, V) and so this proves the first statement in Lemma 9.4. The rest is similar. $\square$

Proposition 9.5. *Let $n = 2m+1 > 3$ be odd. Then there is an n-dimensional proper subrepresentation of the the Plücker representation which is B_n-invariant. This representation is irreducible and monomial.*

Proof. For $i \leq m$ let π_i be the Plücker generator corresponding to $(S_i|N)(T_i|N)$ where $S_i \cap T_i = \emptyset, S_i \cup T_i = \{1, 2, \ldots, n\} \setminus \{i\}, |S_i| = |T_i| = m$. Thus, for example, when $n = 5$, we would have $\pi_5 = (1,2|5,6)'(3,4|5,6)' - (1,3|5,6)'(2,4|5,6)' + (1,4|5,6)'(2,3|5,6)'$. Now for $i \leq n$ we let

$$\begin{aligned}
v_1 &= \frac{1}{u}\pi_1 - \frac{1+u}{u}a_{21}\pi_2 + \frac{1+u}{u}a_{31}\pi_3 - \ldots \\
&\vdots \\
v_i &= a_{1i}\pi_1 - a_{2i}\pi_2 + \cdots + (-1)^{i+1}\frac{1}{u}\pi_i - (-1)^{i+1}\frac{1+u}{u}a_{i+1i}\pi_{i+1} + \ldots \\
&\vdots \\
v_n &= a_{1n}\pi_1 - a_{2n}\pi_2 + \cdots + (-1)^n a_{n-1n}\pi_{n-1} - (-1)^n\frac{1}{u}\pi_n.
\end{aligned}$$

Now using Lemma 9.4 one can now check the following actions:

$$\begin{aligned}
&\sigma_i(v_i) = -\frac{1+u}{u}v_{i+1}; \quad \sigma_i(v_{i+1}) = -v_i; \quad \text{and} \\
&\sigma_i(v_j) = -v_j \quad \text{for all } j \neq i, i+1.
\end{aligned}$$

It is now clear that we have a representation and that the representation is monomial relative to the basis $v_1, v_2, \ldots, v_n$. The irreducibility is proved in the same way that the matrices in (7.3) were proved to generate an irreducible representation. $\square$

§10. Algebras with straightening law

Let R be a commutative ring, let A be an R-algebra and $\Pi \subset A$ a finite subset with a partial order $\leq$. Then A is a *graded algebra with straightening law (on* Π, *over* R*)* (shortened to ASL most of the time) if we have:

(1) $A = \bigoplus_{i \geq 0} A_i$ is a graded R-algebra such that $A_0 = R$, the poset Π consists of homogeneous elements of positive degree which generate A as an R-algebra.

(2) The products $\psi_1 \dots \psi_m, m \in \mathbb{N}, \psi_i \in \Pi$, such that $\psi_1 \leq \psi_2 \leq \cdots \leq \psi_m$ are linearly independent. They are called the *standard monomials*.
(3) For all incomparable $\psi, \nu \in \Pi$ the product $\psi\nu$ has a representation

$$\psi\nu = \sum a_\mu \mu, \quad a_\mu \in R, \quad \mu \text{ a standard monomial}$$

where the μ's occuring in the above sum with non-zero coefficients each contain a factor $\zeta \in \Pi$ such that $\zeta \leq \psi, \zeta \leq \nu$.

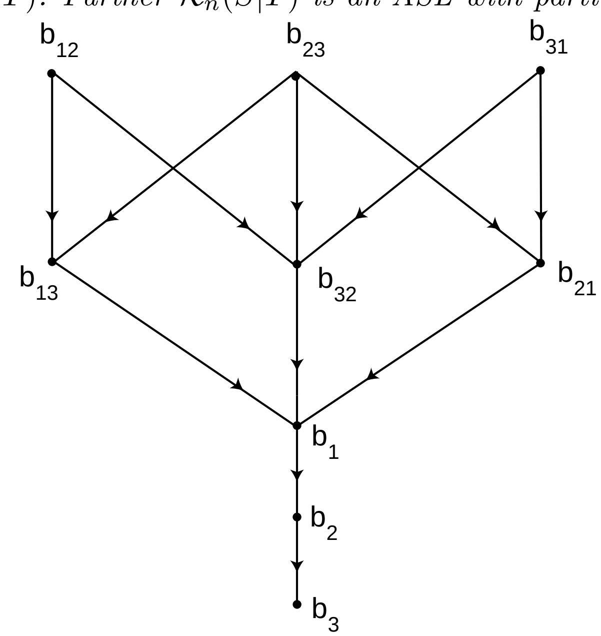
The standard monomials form a basis for A as an R-module. For a proof of this fact and general information about ASL's see [BV]. ASL's are called *ordinal Hodge algebras* in [DEP2].

In this section we will point out various situations where $\mathcal{R}_n$ is an ASL.

Lemma 10.1. *Let $n = 3$ and $(S|T) = (1,2|4,5)$. Putting $(i,j) = (i,j|4,5)$ then*

$$b_1 = (2,3), b_2 = (1,3), b_3 = (1,2), b_{12} = a_{12}(2,3), b_{13} = a_{13}(2,3),$$
$$b_{21} = a_{21}(1,3), b_{23} = a_{23}(1,3), b_{31} = a_{31}(1,2), b_{32} = a_{32}(1,2)$$

are a basis for $\mathcal{R}_3^1(S|T)$. Further $\mathcal{R}_n(S|T)$ is an ASL with partial order as follows:



Proof. Standard monomials will be products of the b_i, b_{ij} where we do not have b_{ij} and b_{jk} in such a product. Note that for distinct $1 \leq i,j,k \leq 3$ we see that $b_{ij}b_{jk} \in b_{ik}b_jC(u)$ and that $b_{ij}b_{ji} \in b_ib_jC(u)$. This shows that every monomial in the b_i, b_{jk} is a $C(u)$-multiple of a standard monomial. We have also thus shown that incompatible products satisfy (3) of the above definition. The rest is easy. $\square$

§11 The $R_n^{(0)}/I_n$ Representations

In this section we look at the situation where B_n acts on $R_n^{(0)}/I_n$.

We need to recall the standard generators for P_n [Bi p. 20]: For $1 \leq i < j \leq n$ we let

$$A_{ij} = \sigma_{j-1}\sigma_{j-2}\dots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\dots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}.$$

The action of the generators of B_n and P_n on $R_n^{(0)}/I_n$ is given in

Lemma 11.1. *(i) For all $1 \le i < n$ we have*

$$\begin{aligned}
&\sigma_i(a_{i\,i+1}) = a_{i+1\,i}, \quad \sigma_i(a_{i+1\,i}) = a_{i\,i+1}, \quad \sigma_i(a_{h\,i+1}) = a_{h\,i},\\
&\sigma_i(a_{h\,i}) = \frac{u+1}{u} a_{h\,i+1}, \quad \sigma_i(a_{i+1\,j}) = a_{i\,j},\\
&\sigma_i(a_{i\,j}) = \frac{u}{u+1} a_{i+1\,j}, \quad \sigma_i(a_{h\,j}) = a_{h\,j},
\end{aligned}$$

where $h, j \neq i,\, i+1$.

(ii) For all $1 \le i < j \le n$ we have

$$\begin{aligned}
&A_{ij}(a_{ij}) = a_{ij},\ \ A_{ij}(a_{ji}) = a_{ji},\ \ A_{ij}(a_{ih}) = \frac{u}{u+1} a_{ih},\ \ A_{ij}(a_{jh}) = \frac{u}{u+1} a_{jh},\\
&A_{ij}(a_{hi}) = \frac{u+1}{u} a_{hi},\ \ A_{ij}(a_{hj}) = \frac{u+1}{u} a_{hj},\ \ A_{ij}(a_{rs}) = a_{rs},
\end{aligned}$$

for all $r, s, h \neq i, j$.

(iii) Let $1 \le k < n$. Then

$$(\sigma_1\sigma_2 \dots \sigma_k)^{k+1}(a_{ij}) = \frac{u^k}{(u+1)^k} a_{ij},\ \ (\sigma_1\sigma_2 \dots \sigma_k)^{k+1}(a_{ji}) = \frac{(u+1)^k}{u^k} a_{ji}$$

for all $1 \le i \le k, j > k$.

Proof. (i) follows immediately from (1.2) and the definition of I_n. (ii) follows from (i) by induction on $j - i \ge 1$. For (iii) we use (ii) and the formula

$$(\sigma_1\sigma_2 \dots \sigma_k)^{k+1} = A_{12} \times A_{13}A_{23} \times \dots \times A_{1\,k+1}A_{2\,k+1} \dots A_{k\,k+1}$$

found in [Bi, p.28] as follows. We note that for fixed $1 \le i < j \le n$ there are exactly $k+1$ occurrences of the generators A_{rs} in this product where $\{r, s\} \cap \{i, j\} \neq \emptyset$, one of these being A_{ij}. The action of each such A_{rs} on a_{ij} is given in (ii), with the action of A_{ij} being trivial, and so we get the factor of $\frac{u^k}{(u+1)^k}$ or its reciprocal. □

Now given any monomial $\mu \in R_n^{(0)}$ we let

$$I(\mu) = \{i | a_{ij} \text{ divides } \mu \text{ for some } j \le n\}, \quad J(\mu) = \{i | a_{ji} \text{ divides } \mu \text{ for some } j \le n\}.$$

For $I, J \subseteq \{1, 2, \dots, n\}$ we will say that μ has *type* IJ if $I(\mu) = I$ and $J(\mu) = J$. For any $b \in R_n^{(0)}$ we may write

$$b = \sum_{I,J \subseteq \{1,2,\dots,n\}} \mu_{IJ}$$

where μ_{IJ} is a sum of $C(u)$-multiples of monomials of type IJ. Further, given such a monomial $\mu \in R_n$ there are $c \in C(u)$ and $\mu' \in R_n$ such that $\mu \equiv c\mu' \bmod I_n$ and where μ' has type IJ with $I \cap J = \emptyset$; for if $j \in I \cap J$, then there are a_{ij} and a_{jk} both dividing μ and we can replace the product $a_{ij}a_{jk}$ in μ by a $C(u)$-multiple of a_{ik}, thus reducing the degree of the monomial. In this section we will always assume that all monomials in $R_n^{(0)}/I(\{1, 2, \dots, n\})$ are so represented.

Further, to any such monomial $\mu \in R_n^{(0)}/I(\{1,2,\dots,n\})$ (or any non-trivial $C(u)$-multiple of μ) we may associate the directed, weighted graph $\Gamma(\mu)$ whose vertices are the numbers $1,2,\dots,n$ and where we have an edge from i to j of weight k if a_{ij}^k divides μ (but a_{ij}^{k+1} doesn't). We note that by Lemma 11.1 for $\alpha \in B_n$ the graphs $\Gamma(\mu)$ and $\Gamma(\alpha(\mu))$ are isomorphic as directed, weighted graphs; in fact we have $\Gamma(\alpha(\mu)) = \Pi_n(\alpha)\Gamma(\mu)$ where S_n acts on the graphs $\Gamma(\mu)$ in the obvious way.

It is easily seen from Lemma 11.1 that the B_n-orbit of a monomial μ consists of $C(u)$-multiples of monomials μ' such that $\Gamma(\mu) \cong \Gamma(\mu')$ and that, conversely, if μ, μ' are monomials with $\Gamma(\mu) \cong \Gamma(\mu')$, then there is $\alpha \in B_n$ and $c \in C(u)$ with $\alpha(\mu) = c\mu'$. Thus the $C(u)$-module generated by all μ' with $\Gamma(\mu) \cong \Gamma(\mu')$ is B_n-invariant. We denote it by $C(u)(B_n(\mu))$. Elementary group theory shows that

$$dim_{C(u)}(C(u)(B_n(\mu))) = \frac{n!}{|Sym(\Gamma(\mu))|}$$

where $Sym(\Gamma(\mu)) \subseteq S_n$ is the group of all symmetries of the directed, labeled graph $\Gamma(\mu)$.

Now the action of B_n on $R_n^{(0)}/I_n$ thus splits as a sum of irreducible summands of these $C(u)(B_n(\mu))$, which we now investigate.

Note that $\Gamma(\mu)$ is a bipartite graph with vertices being either *sources* or *sinks* (if a_{ij} divides μ, then the vertex i is a source and j is a sink). Now to each vertex i of $\Gamma(\mu)$ we associate its signed degree (the sum of the weights of the adjacent edges) which we denote by $d_i(\mu) = d_i(\Gamma(\mu))$. Here we let $d_i(\mu)$ be positive if the vertex i is a source, and negative otherwise. Note then that by Lemma 11.1 (ii) we have

$$A_{ij}^w(\mu) = \left(\frac{u}{u+1}\right)^{w(d_i(\mu)+d_j(\mu))} \mu \tag{11.1}$$

for all $1 \le i < j \le n$ and $w \in \mathbb{Z}$.

For a monomial $\mu \in R_n/I(\{1,2,\dots,n\})$ we let

$$Win(\mu) = \sum_{i \in J(\mu)} d_i(\mu), \quad Wout(\mu) = \sum_{i \in I(\mu)} d_i(\mu),$$

so that $Win(\mu) \le 0$ and $Wout(\mu) \ge 0$.

Lemma 11.2. *Let $0 \ne b \in R_n/I(\{1,2,\dots,n\})$. Then there is $k \ge 1$ such that in the $C(u)$-span of the B_n-orbit of b there is some b' which is a sum of monomials all of type $\{1,2,\dots,k\},\{k+1,\dots,n\}$ and all having the same Win and $Wout$ values.*

Proof. Choose a monomial μ in b such that (i) $Wout(\mu)$ is maximal among all monomials of b; and (ii) among all monomials μ satisfying (i) we choose a μ with $|J(\mu)|$ smallest. For such a μ let $k = |J(\mu)|$. Let $I = I(\mu), J = J(\mu)$. Now the fact that B_n surjects onto the symmetric group S_n, together with Lemma 11.1 shows that there is $\alpha \in B_n$ such that $\alpha(\mu)$ is a $C(u)$-multiple of a monomial μ' of type $\{1,2,\dots,k\},\{k+1,\dots,t\}$ for some $t \le n$.

Now write $b' = \alpha(b) = \sum b_{I,J,r,s}$, where $b_{I,J,r,s}$ is the sum of all $c\mu'$, $c \in C(u)$, such that $I(\mu') = I, J(\mu') = J, Wout(\mu') = r, Win(\mu') = s$. Let $\beta = (\sigma_1\sigma_2\dots\sigma_k)^{(k+1)w}$ and consider the action of β. Lemma 11.1 (iii) shows that $b_{I,J,r,s}$ gets multiplied by $\frac{u^{kw}}{(u+1)^{kw}}$ while every other $\mu_{I',J',r',s'}$ is multiplied by a strictly smaller power of $\frac{u^k}{(u+1)^k}$. Thus $b_{I,J,r,s}$ is in the $C(u)$-span of the B_n-orbit of b. $\square$

Now suppose that b is a $C(u)$-sum of monomials all of type $\{1,2,\ldots,k\},\{k+1,\ldots,n\}$: $b=\sum_{m=1}^{r} c_m\mu_m$ where $c_m\neq 0, m=1,\ldots,r$. We will call $r=r(b)$ the *length* of b. Suppose that $1\leq i\neq j\leq n$ and that $d_i(\mu_s)+d_j(\mu_s)\neq d_i(\mu_t)+d_j(\mu_t)$ for some $s\neq t\leq r$. Then (11.1) above shows that the $C(u)$-linear span of the elements $A^w_{ij}(b)$ contains a non-zero element b' with $r(b')<r(b)$. We thus have the first part of

Lemma 11.3. *For $0\neq b\in R_n^{(0)}/I_n$ and k as in Lemma 11.2 we see that there is $b'\in C(u)(B_n(b))$ such that if $b'=\sum_{m=1}^{r} c_m\mu_m$, where $0\neq c_m\in C(u)$, then*

$$d_i(\mu_s)+d_j(\mu_s)=d_i(\mu_t)+d_j(\mu_t),$$

for all $s\neq t\leq r$ and for all $1\leq i\neq j\leq n$. In particular, the μ_m satisfy $I(\mu_s)=I(\mu_t)$ and $J(\mu_s)=J(\mu_t)$ for all $1\leq s,t\leq r$.

Further, we may suppose that for all $\alpha\in P_n$ we have $\alpha(b')\in C(u)b'$.

Lastly, if $d_i(\mu_s)=d_j(\mu_s)$, then $d_i(\mu_t)=d_j(\mu_t)$ for all t.

Proof. We only need to prove the statement in the last paragraph. Since $n\geq 3$ we can choose $k\neq i,j$ and from the above we obtain

$$\begin{aligned} d_i(\mu_s)+d_k(\mu_s)&=d_i(\mu_t)+d_k(\mu_t) \text{ and}\\ d_j(\mu_s)+d_k(\mu_s)&=d_j(\mu_t)+d_k(\mu_t). \end{aligned}$$

Thus if $d_i(\mu_s)=d_j(\mu_s)$, then $d_i(\mu_t)=d_j(\mu_t)$ as required. $\square$

Now S_n clearly acts on the algebra $R_n^{(0)}$ and so on monomials $\mu\in R_n^{(0)}$ and on the graphs $\Gamma(\mu)$. Given any monomial μ we let

$$Aut(\mu)=\{\alpha\in S_n|\alpha(\mu)=\mu\},\quad DAut(\mu)=\{\alpha\in S_n|d_{\alpha(i)}(\mu)=d_i(\mu),\text{ for all } i=1,\ldots,n\}.$$

Note that if $0\neq b\in R_n^{(0)}/I_n$, then Lemma 11.3 shows that there is $0\neq b'\in C(u)(B_n(b))$ with $b'=\sum_{m=1}^{r} c_m\mu_m$ where $I(\mu_m)=\{1,2,\ldots,k\}$ for every m and $J(\mu_s)=J(\mu_t)$ for all s,t. We also have: if $d_i(\mu_s)=d_j(\mu_s)$, then $d_i(\mu_t)=d_j(\mu_t)$ for all t. It follows that for $1\leq i\leq r$ there is $\alpha\in DAut(\mu_1)$ with $\alpha(\mu_1)=\mu_i$. In particular we see that if $r>1$, then $DAut(\mu_1)\neq Aut(\mu_1)$.

We now notice that if $C(u)(B_n(b)), b=\sum_{m=1}^{r} c_m\mu_m$, is to give an irreducible representation of B_n, then all the monomials μ_m with $c_m\neq 0$ are isomorphic under the action of S_n in the sense that there is some $\alpha\in S_n$ with $\alpha(\mu)=\mu'$. For otherwise we could write $b=b_1+b_2+\cdots+b_y, y>1$, where each of the b_i is a $C(u)$-sum of isomorphic monomials, and this would give a splitting of $C(u)(B_n(b))$. We have:

Lemma 11.4. *Let $b=\sum_{m=1}^{r} c_m\mu_m, c_m\neq 0$. If $C(u)(B_n(b))$ is an irreducible representation of B_n, then all the μ_m are isomophic monomials.* $\square$

Proposition 11.5. *Let W be an irreducible subrepresentation of $R_n^{(0)}/I_n$. Then W is contained in $C(u)(B_n(\mu))$ for some monomial $\mu \in R_n$.*

Let $\mu \in R_n^{(0)}/I_n$ be a monomial. Suppose that $DAut(\mu) = Aut(\mu)$. Then $C(u)(B_n(\mu))$ is an irreducible, monomial representation of B_n of degree $n!/|Aut(\mu)|$.

Proof. The first statement follows directly from Lemma 11.4.

We note that for any monomial μ the representation $C(u)(B_n(\mu))$ is a monomial representation (see (11.1)). Now suppose that there is a subrepresentation W of $C(u)(B_n(\mu))$ and that $0 \neq b \in W, b = \sum_{m=1}^{r} c_m \mu_m$, where $c_m \neq 0$. Then by Lemma 11.3 and Lemma 11.4 we may assume that $I(\mu_i) = I(\mu_j)$ and $J(\mu_i) = J(\mu_j)$ for all i, j, and that if for some i, j, s we have $d_i(\mu_s) = d_j(\mu_s)$, then $d_i(\mu_t) = d_j(\mu_t)$ for all t. Now if $r > 1$, then there would be an element of $DAut(\mu) \setminus Aut(\mu)$, a contradiction. The rest is standard. $\square$

Now for a monomial μ and $d \geq 1$ we let $I_d(\mu)$ be the set of vertices of $\Gamma(\mu)$ having degree d; similarly for $J_d(\mu)$. Now note that

$$DAut(\mu) = \sum_d Sym(I_d(\mu)) \times \sum_d Sym(J_d(\mu)).$$

Let $N \subset \mathbb{N}$ be finite. We recall the basis facts about representations of symmetric groups $Sym(N)$ [FH]. Given any tableau T with entries from N we let

$$\begin{aligned} P = P_T &= \{g \in S_r | g \text{ preserves each row of } T\}, \\ Q = Q_T &= \{g \in S_r | g \text{ preserves each column of } T\}; \\ a_T &= \sum_{g \in P} g \quad \text{and} \quad b_T = \sum_{g \in Q} sgn(g) g. \end{aligned}$$

The *Young symmetriser* is $c_T = a_T b_T \in \mathbb{C}Sym(N)$, the fundamental fact being that a complex multiple of c_T is an idempotent. Further $c_T \mathbb{C}Sym(N)$ is an irreducible representation of $Sym(N)$, where different tableau corresponding to the same Young diagram give isomorphic representations.

Let μ be a monomial such that each of $I_d(\mu), J_d(\mu)$ is an interval of distinct positive integers. Let $X(\mu)$ be the space generated by all μ' such that $I_d(\mu') = I_d(\mu), J_d(\mu') = J_d(\mu), d = 1, 2, \ldots$. Then $X(\mu)$ is an S_n-module. Further, if $\alpha \in P_n$ and $x \in X(\mu)$, then (11.1) shows that $\alpha(x) = c(\alpha)x$ for some $c(\alpha) \in C(u)$. Thus the representation of P_n on $X(\mu)$ is 1-dimensional, and so any representation induced from it is monomial [S]. Define the following subgroup of B_n:

$$B_n(\mu) =< \sigma_i | \text{there is } d \geq 1 \text{ with } i, i+1 \in I_d(\mu) \text{ or } i, i+1 \in J_d(\mu) > .$$

By (11.1) we see that $X(\mu)$ is invariant under the action of P_n. Further, by Lemma 11.1 we see that $X(\mu)$ is invariant under the action of $B_n(\mu)$. Thus $X(\mu)$ is invariant under the action of $\bar{B}_n(\mu) =< B_n(\mu), P_n >$. But the index of $\bar{B}_n(\mu)$ in B_n is clearly finite and so we get an induced action $Ind_{\bar{B}_n(\mu)}^{B_n}$ [S]. Now note that if Y is an S_n-invariant subspace of $X(\mu)$ with matrices $M(\alpha), \alpha \in S_n$, then Y is also a $\bar{B}_n(\mu)$-invariant subspace of W (see Lemma 11.1) and the matrices for the elements $\alpha' \in \bar{B}_n(\mu)$ have the form $M(\pi_n(\alpha'))D(\alpha')$ where $D(\alpha')$ is a diagonal matrix. Thus if Y is irreducible as an S_n-module, then Y is irreducible as a $\bar{B}_n(\mu)$-module.

Now given any irreducible B_n-invariant subspace W of $V = C(u)(B_n(\mu))$ Lemmas 11.3 and 11.4 show that there is $b' = \sum_m c_m\mu_m \in W$ where all the μ_m are isomorphic and where $I(\mu_m) = I(\mu_{m'}), J(\mu_m) = J(\mu_{m'})$ for all m, m'. We may also assume that each of $I_d(\mu), J_d(\mu)$ is an interval of positive integers. Note that the $I_d(\mu), J_d(\mu)$ determine a partition of n. Thus $X(\mu_1) \cap W \neq \{0\}$ and so the action of B_n on W is induced by a non-trivial action of $\bar{B}_n(\mu_1)$. Putting this all together gives:

Proposition 11.6. *Let $\mu \in R_n^{(0)}/I_n$ be a monomial. Then $X(\mu)$ is a $\Pi_n(\bar{B}_n(\mu))$-module and let $X(\mu) = \oplus_i X_i$ be a decomposition into irreducibles as a $\Pi_n(\bar{B}_n(\mu))$-module. Then, there is a corresponding decomposition of the action of B_n on $V(\mu) = C(u)(B_n(\mu))$, as $V = \oplus_i V_i$ where $V_i = Ind_{\bar{B}_n(\mu)}^{B_n}$.* $\square$

Example 11.7. One situation that has received a lot of attention is that, in our context, where $Aut(\mu) = \{id\}$, so that $S_\lambda = \Pi_n(\bar{B}_n(\mu)) = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r}$. Recalling that the irreducible representations of S_n correspond to Young diagrams, we see that the multiplicities of the irreducible components (corresponding to a Young diagram κ) of the induced representations $Ind_{S_\lambda}^{S_n}$ are given by the Kostka numbers $K_{\lambda\kappa}$. Here $K_{\lambda\kappa}$ is defined to be the coefficient of the monomial $X^\lambda = x_1^{\lambda_1}x_2^{\lambda_2}\dots x_r^{\lambda_r}$ in the Schur polynomial S_κ [FH, p. 56]. Explicit bases for these irreducible subspaces are given in [E].

The situation where $Aut(\mu) \neq \{id\}$ is more complicated.

§12 Rigidity

It is well-known that there is an epimorphism $\psi_n : P_n \to P_{n-1}$ obtained by "pulling out the nth string" [Bi, p. 23]. Composing $\psi_n, \psi_{n-1}, \dots, \psi_4$ we see that there is an epimorphism $\phi_n : P_n \to P_3$. Now using the presentation for P_3 given in [Bi, p. 20] or [Ha] with generators

$$A_{ij} = \sigma_i\sigma_{i+1}\dots\sigma_{j-1}\sigma_j^2\sigma_{j-1}^{-1}\dots\sigma_{i+1}^{-1}\sigma_i^{-1}$$

for $1 \leq i < j \leq n$ we see that

$$P_3 =< A_{12}, A_{13}, A_{23} >=< A_{12}, A_{23} > \times < A_{23}A_{13}A_{12} >\cong F_2 \times \mathbb{Z}.$$

Here $z_3 = A_{23}A_{13}A_{12}$ generates the centre of B_3 [Bi, p.28]. This allows the construction of an epimorphism $\tau_n : P_n \to F_2$ for each $n \geq 3$. Now F_2 has infinitely many irreducible representations in dimension 2; for example we could just choose the degree 2 irreducible representations of the dihedral groups $D_m =< r, s|r^m, s^2, (rs)^2 >$ of order $2m$. These are described in [S]. For simplicity let us only consider the case where $m = 2k$ is even. Let $w = e^{2\pi i/m}$ and $h \in \mathbb{Z}$. Then a representation ρ^h of D_m is defined by

$$\rho^h(r^k) = \begin{pmatrix} w^{hk} & 0 \\ 0 & w^{-hk} \end{pmatrix}, \quad \rho^h(sr^k) = \begin{pmatrix} 0 & w^{-hk} \\ w^{hk} & 0 \end{pmatrix}.$$

Then for $0 < h < m/2$ the representation ρ^h is irreducible and these account for all such degree 2 irreducible representations of D_m [S]. Let W_{mh} be the corresponding representation space.

Now the induced representation $Ind_{P_n}^{B_n} W_{mh}$ has degree $2 \times n!$ and if the irreducibles so obtained (for varying m and h) were finite in number, then only a finite number of primes would show up in the orders of the matrix groups $Ind_{P_n}^{B_n} W_{mh}$ so induced. But m is arbitrary and the order of $Ind_{P_n}^{B_n} W_{mh}$ is clearly divisible by m. Thus we must have infinitely many distinct representations of B_n, all of degree at most $2 \times n!$. This proves the B_n case of Theorem 1.9.

Now for the B_n' case we note that Gorin and Lin [GL] show that B_n' is finitely generated and is perfect for $n > 4$. We have $B_n/B_n' \cong \mathbb{Z}$ and so B_n' consists of those braids having zero exponent sum in the standard generators σ_i. Also $[B_n' : B_n' \cap P_n] = n!/2$. Using the above we have maps

$$B_n' \cap P_n \hookrightarrow P_n \to F_2 \times \mathbb{Z} \to F_2.$$

Call this composite ζ_n. The generator of this central $\mathbb{Z}$ in $F_2 \times \mathbb{Z}$ is $z_3 = A_{23}A_{13}A_{12}$ which has exponent 6 in the standard generators. Thus any word $w \in F_2 =< A_{12}, A_{23} >$ of exponent a multiple of 6 is in the image of ζ_n. The set of such words is a subgroup of F_2 of finite index and so the image of ζ_n is a finitely generated free group [MKS]. This now allows the construction of an epimorphism $B_n' \cap P_n \to F_2$ and an argument similar to that used to prove that B_n is not rigid now proves the B_n' case of Theorem 1.9. $\square$

We now explain the relationship between B_4 and $Aut(F_2)$ described in [DFG] that will allow us to prove Theorem 1.10. Recall Artin's embedding of B_n in $Aut(F_n)$. The fact that each $\alpha \in B_n \subset Aut(F_n)$ fixes $T_1 \dots T_n$ implies that there is a representation $B_n \to B_n^* < Aut(F_{n-1})$. The kernel of this representation is the centre $Z(B_n) \cong \mathbb{Z}$ [DFG]. The connection between B_4 and $Aut(F_2)$ is now expressed as: $B_4^* (\cong B_4/Z(B_4))$ is isomorphic to a subgroup $Aut^+(F_2)$ of $Aut(F_2)$ of index 2. In [DFG] they use this result to show that B_4 has a faithful representation over $\mathbb{C}$ if and only if $Aut(F_2)$ does. To prove Theorem 1.10 it will thus suffice to show that $B_4/Z(B_4)$ is not rigid.

Now there is an epimorphism $\beta : B_4 \to B_3$ given by its action on the generators σ_i:

$$\beta(\sigma_1) = \sigma_1, \beta(\sigma_2) = \sigma_2, \beta(\sigma_3) = \sigma_1.$$

We obtain by composition an epimorphism

$$P_4 \to P_3 =< A_{12}, A_{13} > \times < A_{23}A_{13}A_{12} > \to < A_{12}, A_{13} > \cong F_2.$$

Call this $\alpha : P_4 \to F_2$ and note that $\alpha(A_{23}A_{13}A_{12}) = id$. Now the cyclic generator of $Z(B_4) = Z(P_4)$ is $z_4 = (\sigma_1\sigma_2\sigma_3)^4 = A_{34}A_{24}A_{23}A_{14}A_{13}A_{12}$ [Bi, p. 28] and so the above gives

$$\beta(z_4) = \beta((\sigma_1\sigma_2\sigma_3)^4) = (\sigma_1\sigma_2\sigma_1)^4 = (z_3)^2.$$

Thus $\alpha(z_4) = id$, showing that α induces an epimorphism $\alpha : P_4/Z(P_4) \to F_2$. Since $[B_4^* : P_4/Z(P_4)] = 24$ we can now construct infinitely many distinct irreducible degree 2 representations of $B_4^* = B_4/ < z_4 >$ as in the B_n case. This proves Theorem 1.10. $\square$

Remarks 12.1. 1. The group $H(n)$ of *symmetric automorphisms* of the free group F_n is the subgroup of automorphisms α such that $\alpha(x_i)$ is a conjugate of x_j, where $i \mapsto j$ is a permutation of $\{1, \dots, n\}$. A set of relations for $H(n)$ is given by McCool [Mc]. Let $PH(n)$ be those corresponding to the identity permutation. Then $PH(n)$ is a subgroup of index $n!$ in $H(n)$ and there are epimorphisms $PH(n) \to PH(n-1)$. From the presentation given by McCool one can see that $H(3)$ is an extension of F_3 by F_3 and one easily uses ideas similiar to those used in the above to show that $PH(n)$ and $H(n)$ are not rigid for $n \geq 3$.

2. In each of Theorem 1.9, 1.10 we obtained our infinite set of irreducible representations in some fixed degree by using the epimorphisms $F_2 \to D_m$. This meant that the degrees were necessarily bounded by a function of n. However we could also use epimorphisms $F_2 \to SL_r(F_p)$ for any $r \geq 3$ and prime p, since $SL_r(F_p)$ is a 2-generator group. Using these representations the above methods show that there are an infinite number of degrees d such that B_n, B_n' and $Aut(F_2)$ have an infinite number of irreducible representations of degree d.

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