

TRANSITIVE HYPERBOLIC SETS ON SURFACES

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ABSTRACT. We show that every transitive hyperbolic set on a surface is included in a locally maximal hyperbolic set.

1. HISTORY

The history of hyperbolic dynamics can be traced back to two related directions of research: First, the study of geodesic flows such as the work of Hadamard, Hedlund, and Hopf. Second, homoclinic tangles and celestial mechanics starting with the work of Poincaré and continued, for instance, by Cartwright, Littlewood, and Smale. (Additionally, Smale [10] adds a third direction of structural stability as studied by Andronov, Pontryagin, Lefschetz, Peixoto, and others.)

If M is a compact, smooth, boundaryless manifold and $f : M \rightarrow M$ is a diffeomorphism, then a compact invariant set Λ is *hyperbolic* if the tangent space of Λ splits into f -invariant subbundles $\mathbb{E}^s \oplus \mathbb{E}^u$ such that \mathbb{E}^s is uniformly exponentially contracted by the derivative of f , denoted Df , and \mathbb{E}^u is uniformly exponentially expanded by Df .

We note that hyperbolicity leads to many other interesting phenomena, such as, positive entropy and the existence of topological and measure-theoretic Markov models. In addition, for a point x if we define the *stable and unstable sets*, respectively, as follows:

$$W^s(x) = \bigcup_{n \geq 0} f^{-n}(W_\epsilon^s(f^n(x))), \text{ and} \\ W^u(x) = \bigcup_{n \geq 0} f^n(W_\epsilon^u(f^{-n}(x)))$$

where

$$W_\epsilon^s(x) = \{y \in M \mid d(f^n(x), f^n(y)) \leq \epsilon \text{ for all } n \geq 0\} \text{ and} \\ W_\epsilon^u(x) = \{y \in M \mid d(f^n(x), f^n(y)) \leq \epsilon \text{ for all } n \leq 0\},$$

then a standard result states that if $f : M \rightarrow M$ is a diffeomorphism, Λ is a hyperbolic set for f , and $x \in \Lambda$, then $W^s(x)$ and $W^u(x)$ are injectively immersed submanifolds.

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The most understood hyperbolic sets are those that are locally maximal (or isolated). A hyperbolic set Λ is *locally maximal* if there exists a neighborhood U of Λ such that

$$\Lambda = \bigcap_{i \in \mathbb{Z}} f^i(U).$$

A long standing question in dynamical systems was the following:

Question 1.1. [8, p. 272] *Let Λ be a hyperbolic set, and let V be an open neighborhood of Λ . Does there exist a locally maximal hyperbolic set $\tilde{\Lambda}$ such that $\Lambda \subset \tilde{\Lambda} \subset V$?*

This question was answered in the negative, by Crovisier [3], for a certain hyperbolic set. In [4] it was shown that every manifold of dimension greater than 1 has an open set of diffeomorphism, \mathcal{U} , such that each $f \in \mathcal{U}$ contains a hyperbolic set that is not contained in a locally maximal one.

Furthermore, Theorem 1.6 in [4] shows that there is an open set, \mathcal{U} , of diffeomorphisms of the 4-torus such that each $f \in \mathcal{U}$ contains a transitive hyperbolic set that is not contained in a locally maximal one. (A hyperbolic set Λ is transitive if there exists a point $x \in \Lambda$ whose forward orbit is dense in Λ .) The results of this theorem can naturally be extended to higher dimensions by the use of normal hyperbolicity; see [9].

2. RESULT

As surfaces often give a natural dimensional constraint for hyperbolic sets, a question is whether it can be shown that there is a transitive hyperbolic set on a surface that cannot be included in a locally maximal one. The result of the present work is that this is not possible.

Theorem 2.1. *Let $f : S \rightarrow S$ be a diffeomorphism of a compact surface and Λ be a transitive hyperbolic set for f . Then for any neighborhood V of Λ there exists a locally maximal hyperbolic set $\tilde{\Lambda}$ contained in V and containing Λ .*

Proof. Let S be a compact, boundaryless, connected, smooth surface and $\Lambda \subset S$ be a transitive hyperbolic set for a diffeomorphism $f : S \rightarrow S$. The proof proceeds by looking at the following three cases.

Case 1: Suppose that Λ has nonempty interior. Then a folklore theorem (proved, for instance, in [5]), states that $\Lambda = S$ (so f is Anosov). In this case Λ is, naturally, locally maximal.

Case 2: Suppose that Λ contains a curve γ . This proof breaks into two subcases. In the first, we assume that γ is not contained in stable

or unstable manifold for any point in Λ . In the second, we suppose that γ is contained in the stable or unstable manifold of a point in Λ .

Suppose that γ is an open curve and that for some point $x \in \Lambda$ that

- (1) $x \cap \gamma \neq \emptyset$,
- (2) there does not exist a $\delta > 0$ such that $W_\delta^s(x) \subset \gamma$, and
- (3) there does not exist a $\delta > 0$ such that $W_\delta^u(x) \subset \gamma$.

Theorem 1.5 in [4] shows that for any neighborhood V of Λ there exists a hyperbolic set $\tilde{\Lambda}$ contained in V and containing Λ that has a Markov partition (see [8, p. 591] for a definition of Markov partition). Furthermore, from [7] we see that $\tilde{\Lambda}$ can be chosen to be transitive since Λ is transitive. From the proof of Theorem 1.1 and Claim 3.2 in [6] we see that $\tilde{\Lambda}$ can be chosen to have nonempty interior.

Hence, we need only consider the case that γ is contained in the stable or unstable manifold of a point $x \in \Lambda$. We assume that $\gamma \subset W^u(x)$. Let V be a neighborhood of Λ and $\tilde{\Lambda}$ a transitive hyperbolic set with a Markov partition containing Λ and contained in V . We will show that $\tilde{\Lambda}$ is a hyperbolic attractor. (A hyperbolic set X is a hyperbolic attractor if there exists a neighborhood U of X such that $\bigcap_{n \in \mathbb{N}} f^n(U) = X$.) Let $z \in \tilde{\Lambda}$ have a dense orbit. Since a Markov partition has only a finite number of rectangles we know that one of the rectangles will contain a nonempty open segment, γ' , of γ . Choose a subsequence of the orbit of z limiting on a point of γ' . From the product structure in the rectangles of the Markov partition this implies that there exists some $\epsilon > 0$ such that for some $f^n(z)$ where $n \in \mathbb{N}$ we have $W_\epsilon^u(f^n(z)) \subset \tilde{\Lambda}$. The inclination lemma or λ -lemma (see for instance [8, p. 257]) implies that $W^u(y) \subset \tilde{\Lambda}$ for all $y \in \tilde{\Lambda}$ since z is transitive. A transitive hyperbolic set is a hyperbolic attractor if and only if the unstable manifold of each point is contained in the hyperbolic set. Therefore, we know that $\tilde{\Lambda}$ is a hyperbolic attractor.

Case 3: Suppose, that Λ does not contain a curve. Then using the arguments from Brown in [2] we know that Λ is either a finite set of points and so Λ is the orbit of a periodic point and locally maximal, or Λ is a Cantor set. If Λ is a Cantor set, then Anosov [1] has recently shown that every zero dimensional hyperbolic set, Λ , with V a neighborhood of Λ is contained in a locally maximal one that is contained in V . \square

3. QUESTIONS

We now pose a number of open problems related to Question 1.1.

Question 3.1. *Let M be a 3-manifold. Can there exist a transitive hyperbolic set that is not contained in a locally maximal hyperbolic set?*

Another question concerns locally maximal versus nonlocally maximal hyperbolic sets. In [6] it is shown that if S is a compact surface and Λ is a nontrivial topologically mixing hyperbolic attractor for a diffeomorphism f of M , and Λ is hyperbolic for a diffeomorphism g of S , then Λ is either a nontrivial topologically mixing hyperbolic attractor or repeller for g . This shows that the topology of the set ensures that the hyperbolic set has to be locally maximal. Brown [2] has extended this result to codimension one attractors and has even shown the if $f \in \text{Diff}(S)$ where S is a surface and Λ is a locally maximal transitive hyperbolic set for f , and $g \in \text{Diff}(S)$ such that Λ is hyperbolic for g , then Λ is locally maximal for g . This then raises the next question (Problem 1.4 in [6]):

Question 3.2. *Suppose that Λ is a locally maximal hyperbolic set for a diffeomorphism f and hyperbolic for a diffeomorphism g . Under what conditions does this imply that Λ is locally maximal for g ?*

We also know that some nonlocally maximal hyperbolic sets are included in locally maximal hyperbolic sets and others are not. A nonlocally maximal hyperbolic set, Λ , is *nearly locally maximal* if for any neighborhood V of Λ there exists a locally maximal hyperbolic set, $\tilde{\Lambda}$, contained in V and containing Λ . An example of a nearly locally maximal hyperbolic set is the union of a fixed hyperbolic saddle, p , and the orbit of a transverse homoclinic point for p .

Question 3.3. *Under what conditions is a hyperbolic set nearly locally maximal?*

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