Math 547

MIDTERM

Name: ______________________________

March 3, 2025

Answer all questions and show all your work carefully. **Please write clearly, clean, use large fonts.** Do not answer two different questions in the same page. It means start a new question in a new page. There is a LIMIT OF FOUR HOURS for this test. This is an OPEN BOOK Exam. When you are done upload your work into Gradescope as a regular homework. make sure you reserve at least 10 mins to upload your work. Submit good quality scan of your work.

"The more I consider what learning is, the more I see it as a sacred privilege, an act of wonder. I believe that when we are learning we feel most alive. When we are learning we feel closest to God"
Russ Osguthorpe, 1996

Prof. Vianey Villamizar

Problema No.	**Puntos**
1.	
2.	
3.	
4.	
5.	
Total	

1. (20 points)]

(a) Use the divergence theorem to determine an alternative expression for

$$\int\int\int_V u\nabla^2 u\, dv$$

(b) Consider the boundary value problem (BVP):
Find $u \in C^2(V) \cap C^1(\overline{\partial V})$ satisfying

$$\nabla^2 u(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in V, \tag{1}$$

$$\frac{\partial u}{\partial n}(\mathbf{x}) + hu(\mathbf{x}) = g(\mathbf{x}) \qquad \mathbf{x} \in \partial V, \tag{2}$$

where $g(\mathbf{x})$ is continuous in ∂V, $f(\mathbf{x})$ is continuous in V, and $h > 0$ (constant). If this BVP (1)-(2) has a solution u, then this solution is unique. Prove it.
Hint: *Use part (a)*

(c) Can you also conclude that this BVP HAS a SOLUTION? Explain.

a) $\nabla\cdot(u\nabla u) = u\,\nabla\cdot(\nabla u) + \nabla u\cdot\nabla u$

$= u\nabla^2 u + \|\nabla u\|^2$

$\int_\Omega$:

$$\int_V u\nabla^2 u\, dv = \int_V \nabla\cdot(u\nabla u)\, dv - \int_V \|\nabla u\|^2 dv$$

$$\overset{\text{div Thm}}{=} \int_{\partial V} u(\nabla u\cdot\hat{n})\, ds - \int_V \|\nabla u\|^2 dv$$

or

$$\boxed{\int_\Omega u\nabla^2 u\, dv = \int_{\partial V} u\frac{\partial u}{\partial n}\, ds - \int_V \|\nabla u\|^2 dv} \qquad (1.1)$$

b) Consider the Corresponding homogeneous problem.

$$\begin{cases} \nabla^2 w = 0, & \text{in } V. \quad (2.1) \\ \dfrac{\partial w}{\partial n} + hw = 0, & \text{on } \partial V \quad (2.2) \end{cases}$$

Using (1.1)

$$0 = \int_V w \overbrace{\nabla^2 w}^{=0} \, dv = \int_{\partial V} w \frac{\partial w}{\partial n} ds - \int_V \|\nabla w\|^2 dv \quad (2.3)$$

From (2.2)

$$\boxed{\frac{\partial w}{\partial n} = -hw}$$

Substituling it into (2.3)

$$0 = \int_{\partial V} w \frac{\partial w}{\partial n} ds - \int_V \|\nabla w\|^2 dv = -\int_{\partial V} h w^2 ds - \int \|\nabla w\|^2 dv$$

This implies 1) $\boxed{w(\bar{x}) = 0, \text{ on } \partial V}$

2) $\boxed{\nabla w = 0, \text{ on } V} \Rightarrow W(\bar{x}) = K$ (Const) in V.

But, $W(\bar{x}) = 0$ on ∂V and $W(x)$ is continuous in $V \cup \partial V \Rightarrow \boxed{W(\bar{x}) = 0 \text{ in } V.}$

c) Because the related eigenvalue problem is a regular Sturm-Liouville eigenvalue problem, there are a complete set of orthogonal eigenfunctions and as a consequence

$$\boxed{\hat{u}(\vec{x}) = \sum_{\lambda} a_\lambda \phi_\lambda} \qquad (3.0)$$

for $\hat{u}$ satisfying

$$\begin{cases} \nabla^2 \hat{u} = f(\vec{x}), & \vec{x} \in V \qquad (3.1) \\ \dfrac{\partial \hat{u}}{\partial n} + h\hat{u} = 0, & \vec{x} \in \partial V. \qquad (3.2) \end{cases}$$

Upon substitution of (3.0) into (3.1)

$$\nabla^2 u \overset{\substack{\text{same homog BCs.} \\ \downarrow}}{=} \sum_{\lambda} a_\lambda \nabla^2 \phi_\lambda = f(\vec{x})$$

or

$$-\sum_{\lambda} a_\lambda \lambda \phi_\lambda(\vec{x}) = f(\vec{x})$$

Using orthog.

$$-a_\lambda \lambda \int_V \phi_\lambda^2 \, dx = \int_V f(\vec{x}_0)\, \phi_\lambda(\vec{x}_0)\, dv$$

or

$$\boxed{-\lambda a_\lambda = \frac{\int_V f(\vec{x}_0)\, \phi_\lambda(\vec{x}_0)\, dv}{\int_V \phi_\lambda^2(x_0)\, dv}}$$

a_λ is uniquely determined if $\lambda \neq 0$. for all λ

c) (Cont.) If $\lambda \neq 0$, as in our case (3.0) represents the Soln. of (3.1)-(3.2)

Now, to construct the soln. of the original nonhomogeneous problem (1)-(2), we use Green's formula:

$$\int_V u \underbrace{\nabla^2 G}_{=\delta(\vec{x}-\vec{x}_0)} dv - \int_V G \nabla^2 u \, dv$$

$$= \int_{\partial V} \left[u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right] dv$$

$$= \cancel{\int_{\partial V} -h u \, G \, dv} + \int_{\partial V} (\cancel{h u} - g(\vec{x})) G \, dv$$

$$= -\int_{\partial V} G(\vec{x}, \vec{x}_0) g(\vec{x}) dv$$

Using Symm. of G and reversing roles

$$\boxed{u(\vec{x}) = \int_V f(\vec{x}_0) G(\vec{x}, \vec{x}_0) dv_{x_0} - \int_{\partial V} G(\vec{x}, \vec{x}_0) g(\vec{x}_0) ds_{x_0}} \qquad (4.1)$$

2. (a) Consider the BVP:

$$u'' + 9\,u = 1 + \alpha\,x \tag{3}$$
$$u'(0) = 0, \qquad u'(\pi) = 0 \tag{4}$$

i. (8 points)
Solve the following BVP, using elementary techniques for linear second order ODEs. Then answer the following questions: what value should α have for a solution to exist? In case a solution exists, is that unique?

ii. (8 points) Apply the Fredholm alternative. Is it consistent with the results found in part (i)?

(b) (6 points) Consider the BVP:

$$u'' + 9\,u = \beta + \cos x \tag{5}$$
$$u'(0) = 0, \qquad u'(\pi) = 0 \tag{6}$$

Do not solve this BVP, instead apply the Fredholm alternative to determine the value(s) of β for a solution to exist. Is there any value of β that gives a unique solution? Explain.

(c) (6 points) Consider the BVP:

$$u'' + 8\,u = 1 + \alpha x \tag{7}$$
$$u'(0) = 0, \qquad u'(\pi) = 0 \tag{8}$$

Do not solve this BVP, instead apply the Fredholm alternative to determine the values of α for a solution to exist. Is there any value of α that gives a unique solution? Explain.

2

a

$$\begin{cases} u'' + 9u = 1 + \alpha x & (1) \\ u'(0) = 0, \quad u'(\pi) = 0 & (2) \end{cases}$$

Gen. Soln. $u(x) = A\cos 3x + B\sin 3x + u_p(x)$

$$u_p(x) = a_0 + a_1 x$$

$$u_p'' + 9u_p = 9a_0 + 9a_1 x = 1 + \alpha x$$

$$\Rightarrow \quad a_0 = 1/9, \quad a_1 = \alpha/9$$

i

$$\Rightarrow \boxed{u(x) = A\cos 3x + B\sin 3x + \frac{1}{9} + \frac{\alpha}{9}x}$$

4

$$0 = u'(0) = -A3\,\cancel{\sin}^{0}(0) + 3B\cos(0) + \frac{\alpha}{9}$$

$$\Rightarrow \quad 3B + \frac{\alpha}{9} = 0 \quad \Rightarrow \quad \boxed{B = -\alpha/27}$$

$$0 = u'(\pi) = -3A\,\cancel{\sin(3\pi)}^{0} + 3B\underbrace{\cos(3\pi)}_{=-1} + \frac{\alpha}{9}$$

$$-3B + \frac{\alpha}{9} = 0 \quad \Rightarrow \quad \boxed{B = \alpha/27}$$

$$B = 0 \Leftrightarrow \alpha = 0$$

2

If $\alpha \neq 0$ there is no soln.

$$\Rightarrow \boxed{u(x) = A\cos 3x + \frac{1}{9} + \cancel{\frac{\alpha}{9}x}}$$

2

infinitely many solns. for

$$u'' + 9u = 1, \quad u'(0) = 0, \quad u'(\pi) = 0$$

Using Fredholm Alternative

(ii) EVP:

$$\begin{cases} \phi'' + 9\phi = -\lambda\phi \\ \phi'(0) = 0, \quad \phi'(\pi) = 0 \end{cases}$$

$$\phi'' + (\lambda + 9)\phi = 0$$

(I) If $\lambda + 9 > 0$

$$\phi(x) = A\cos\sqrt{\lambda+9}\,x + B\sin(\sqrt{\lambda+9}\,x)$$

$$\phi'(x) = -\sqrt{\lambda+9}\,A\sin(\sqrt{\lambda+9}\,x) + B\sqrt{\lambda+9}\cos(\sqrt{\lambda+9}\,x)$$

$$0 = \phi'(0) = B\sqrt{\lambda+9} \Rightarrow \underline{\underline{B=0}}$$

$$0 = \phi'(\pi) = \sqrt{\lambda+9}\,A\sin(\sqrt{\lambda+9}\,\pi) \Rightarrow \sqrt{\lambda+9}\,\pi = n\pi, \quad n = 0, 1, 2, \ldots$$

$$\lambda + 9 = n^2$$

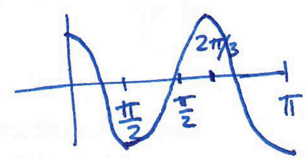
$$\Rightarrow \boxed{\lambda_n = n^2 - 9}$$

$$\boxed{\phi_n(x) \doteq \cos(nx)}$$

In particular $\Rightarrow n = 3, \ \lambda_3 = 0$

Therefore, $\phi_3(x) \equiv \cos(3x)$

nontrivial soln. for associated homog probl.



(II) If $\lambda + 9 = 0 \Rightarrow \underline{\lambda = -9}$

$\Rightarrow \phi(x) = A + Bx$

and $\phi(x) \equiv 1$ eigenfn.

For (1) – (2) to have a soln.

$$\phi_3(x) \perp (1 + \alpha x)$$

$$0 = \int_0^\pi \phi_3(x)(1 + \alpha x)\,dx = \int_0^\pi \underbrace{\cos 3x\,dx}_{=0} + \alpha \int_0^\pi \underbrace{x\cos 3x\,dx}_{\neq 0}$$

$$\Rightarrow \underline{\underline{\alpha = 0}} \quad \text{for inf. many solns.}$$

b) If

$$\begin{cases} u'' + 9u = \beta + \cos x \\ u'(0) = 0, \quad u'(\pi) = 0 \end{cases}$$

$$0 = \int_0^\pi \cos 3x\,(\beta + \cos x)\,dx =$$

$$= \beta \underbrace{\int_0^\pi \cos 3x\,dx}_{=0} + \underbrace{\int_0^\pi \cos 3x \cos x\,dx}_{=0}$$

Any β would work.
But not unique
Fredholm Alternative
~~$u(x) = A\cos 2\sqrt{2}x + B\sin 2\sqrt{2}x$~~

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c) Ask also for

$$\begin{cases} u'' + 8u = 1 + \alpha x \\ u'(0) = 0, \quad u'(\pi) = 0 \end{cases}$$

$\phi(x) = A\cos(\sqrt{\lambda+8}\,x) + B\sin(\sqrt{\lambda+8}\,x)$

$\phi'(0) = B\underbrace{\sqrt{\lambda+8}}_{\neq 0} = 0 \Rightarrow B = 0$

$\phi'(\pi) = A\underbrace{\sqrt{\lambda+8}}_{\neq 0}\sin(\sqrt{\lambda+8}\,\pi) = 0$

$\Rightarrow \sqrt{\lambda+8}\,\cancel{\pi} = n\cancel{\pi}$

$\lambda + 8 = n^2$

$n = 1, 2, \ldots$ never.

Homog. probl. only trivial soln.

$\lambda = 0$ not eigenvalue

~~$u_p(x) = a_0 + a_1 x$~~

$u_p(x) = 1/8 + \alpha/8\, x$

$$u(x) = A\cos 2\sqrt{2}x + B\sin 2\sqrt{2}x + \frac{1}{8} + \frac{\alpha}{8}x$$

$$u'(x) = -2\sqrt{2}A\sin 2\sqrt{2}x + 2\sqrt{2}B\cos(2\sqrt{2}x) + \frac{\alpha}{8}$$

$$0 = u'(0) = 2\sqrt{2}B + \frac{\alpha}{8} \Rightarrow B = -\frac{\alpha}{16\sqrt{2}} \longrightarrow = -\frac{\alpha}{8}$$

$$0 = u'(\pi) = -2\sqrt{2}A\sin 2\sqrt{2}\pi + 2\sqrt{2}B\cos(2\sqrt{2}\pi) + \frac{\alpha}{8}$$

$$\Rightarrow A = \frac{\frac{\alpha}{8}\left(\cos(2\sqrt{2}\pi - 1)\right)}{-2\sqrt{2}\sin 2\sqrt{2}\pi} = -\frac{\alpha}{16\sqrt{2}}\frac{[\cos(2\sqrt{2}\pi) - 1]}{\sin(2\sqrt{2}\pi)}$$

Unique Soln.

(2)

c) For Fredholm alternative look for solns for homogeneous problem.

$$\begin{cases} u'' + 8u = 0 \\ u'(0) = 0, \quad u'(\pi) = 0 \end{cases}$$

$\therefore \quad r^2 + 8 = 0 \quad \Rightarrow \quad r = \pm 2\sqrt{2}$

$\Rightarrow \quad u(x) = A \cos 2\sqrt{2}x + B \sin 2\sqrt{2}x$

$u'(x) = -2\sqrt{2} A \sin(2\sqrt{2}x) + 2\sqrt{2} B \cos(2\sqrt{2}x)$

$0 = u'(0) = 2\sqrt{2} B \quad \Rightarrow \quad \underline{\underline{B = 0}}$

$0 = u'(\pi) = -2\sqrt{2} A \sin(2\sqrt{2}\pi) \Rightarrow \underline{\underline{A = 0}}$

therefore, only soln.

$\underline{\underline{u(x) \equiv 0}}$ $\quad$ <u>Trivial soln.</u>

3. Consider a one-dimensional acoustic scattering problem of an incident plane wave $p_{inc}(x) = e^{-ikx}e^{-i\omega t}$ (propagating to the left) from a rigid wall located at $x = 0$. There is an external continuous force $f(x)$ ($f \in C[0,1]$).

$$u''(x) + k^2 u(x) = f(x), \qquad 0 < x < 1, \quad k \neq 0 \text{ in } \mathcal{R} \tag{9}$$
$$u'(0) = ik, \qquad u'(1) - iku(1) = 0, \qquad i = \sqrt{-1} \tag{10}$$

(a) (5 points)
Show directly that the homogeneous BVP problem corresponding to BVP (9)-(10) has only the trivial solution. Recall that the corresponding homogeneous BVP is defined by the same equation with $f(x) = 0$ and has the same type of BC's, but they are homogeneous.
Hint: *Use linearly independent complex-valued functions to express the general solution of the homogeneous equation associated to (9).*

(b) (4 points) Show directly (do not apply previous theorems) that the above part (a) implies that if BVP (9)-(10) has a solution then this solution is unique.

(c) (10 points) The result in part (a) also implies the existence of a Green's function for BVP (9)-(10). It means a function $G(x, x_0)$ satisfying:

$$\frac{\partial^2 G}{\partial x^2}(x, x_0) + k^2 G(x) = \delta(x, x_0), \qquad 0 < x < 1, \quad k \in R \tag{11}$$
$$\frac{\partial G}{\partial x}(0, x_0) = 0, \qquad \frac{\partial G}{\partial x}(1, x_0) - ikG(1, x_0) = 0. \tag{12}$$

Solve for this Green's function directly from (11)-(12).

(d) (6 points)] Obtain a representation of the solution for BVP (9)-(10) in terms of the Green's function derived in part (c). As a validation of your result, if $f(x) = 0$ your solution should reduce to $u(x) = e^{ikx}$
Hint: *Use Green's formula for a Sturm-Liouville operator with $v = G(x, x_0)$.*

For the following BVP:

$$\begin{cases} u''(x) + k^2 u(x) = f(x), & 0<x<1, \quad k \in \mathbb{R} \qquad (1) \\ u'(0) = ik, & u'(1) - iku(1) = 0 \qquad (2) \end{cases}$$

(Scattering problem from a rigid wall with a source function $f(x)$)

a) Show that $\lambda = 0$ is not an eigenvalue of the corresponding eigenvalue problem. What conclusion can you draw by applying Fredholm alternative?

b) Find the Green's function of BVP (1) and (2).

c) Determine a representation of the solution of BVP (1) and (2) in terms of the Green's function found in part (b).

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a) It's equivalent to prove that the homogeneous problem has only the trivial solution.

$$\phi_h'' + k^2 \phi_h = 0, \quad \phi_h'(0) = 0, \quad \phi_h'(1) - ik\phi_h(1) = 0 \qquad (1)$$

In fact, $\boxed{\phi_h(x) = C_1 e^{ikx} + C_2 e^{-ikx}}$

$$\phi_h'(x) = ikC_1 e^{ikx} - ikC_2 e^{-ikx}$$

BC's imply

$$0 = \phi_h'(0) = ikc_1 - ikc_2 = ik(c_1 - c_2) \quad (2)$$

and

$$0 = \phi_h'(1) - ik\phi_h(1) = ik\cancel{c_1}e^{ik} - ikc_2e^{-ik} - ikc_1\cancel{e^{ik}} - ikc_2e^{-ik} = -2ikc_2e^{-ik} \Rightarrow \boxed{c_2 = 0}$$

Since $e^{-ik} \neq 0$ for all k

From (2) $c_1 = c_2 \Rightarrow \boxed{c_1 = 0}$

$\Rightarrow \boxed{\phi_h(x) \equiv 0}$ only trivial solution.

b) The Green's function of BVP (1) and (2) satisfies

$$\begin{cases} G''(x,x_0) + k^2G(x,x_0) = \delta(x-x_0), & 0<x<1 \\ G'(0,x_0) = 0, & G'(1,x_0) - ikG(1,x_0) = 0 \end{cases}$$

Therefore,

$$G(x,x_0) = \begin{cases} c_1(x_0)e^{ikx} + c_2(x_0)e^{-ikx}, & x>x_0 \\ d_1(x_0)e^{ikx} + d_2(x_0)e^{-ikx}, & x<x_0 \end{cases}$$

Using B.C. at $x=1$

$$0 = G'(1,x_0) - ikG(1,x_0) = ik\cancel{c_1}e^{ik} - ikc_2e^{-ik} - ik\cancel{c_1}e^{ik} - ikc_2e^{-ik}$$

$$= -2ikc_2e^{-ik}$$

$\Rightarrow \boxed{c_2(x_0) = 0}$, since $e^{-ik} \neq 0$, for all k.

Using B.C. at $x=0$: $0 = G'(0,x_0) = \left(ik\, d_1(x_0)\, e^{ikx} - ik\, d_2(x_0)\, e^{-ikx} \right)\Big|_{x=0}$

$$\Rightarrow \quad 0 = ik\, d_1(x_0) - ik\, d_2(x_0) \Rightarrow \boxed{d_1^{(x_0)} = d_2^{(x_0)}}$$

Thus,

$$G(x,x_0) = \begin{cases} c_1(x_0)\, e^{ikx}, & x > x_0 \\ d_1(x_0)\left(e^{ikx} + e^{-ikx} \right), & x < x_0 \end{cases}$$

Constants $c_1(x_0)$ and $d_1(x_0)$ can be determined from Cont. of Green's fn. at $x = x_0$ and the jump cond. in the derivative as follows:

Cont: $G(x_0^+, x_0) = G(x_0^-, x_0)$

$$\boxed{c_1(x_0)\, e^{ikx_0} - d_1(x_0)\left[e^{ikx_0} + e^{-ikx_0} \right] = 0} \qquad (3)$$

Jump Cond: By $\int_{x_0-\varepsilon}^{x_0+\varepsilon}$ and taking $\lim_{\varepsilon \to 0}$

$$G'(x_0^+, x_0) - G'(x_0^-, x_0) = 1$$

$$\boxed{\cancel{ik}\, c_1(x_0)\, e^{ikx_0} - d_1^{(x_0)}\left(\cancel{ik}\, e^{ikx_0} - \cancel{ik}\, e^{-ikx_0} \right) = \frac{1}{ik}} \qquad (4)$$

$$\lim_{\varepsilon \to 0} \int_{x_0-\varepsilon}^{x_0+\varepsilon} G''(x,x_0)\, dx + \lim_{\varepsilon \to 0} \underbrace{\int_{x_0-\varepsilon}^{x_0+\varepsilon} k^2 G(x,x_0)\, dx}_{\xrightarrow{\varepsilon \to 0}\ 0} = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x-x_0)\, dx = 1$$

because G is conts at x_0 $\Rightarrow \int G$ is also cont at x_0.

$$G'(x_0^+, x_0) - G'(x_0^-, x_0) = 1$$

Therefore, to determine $C_1(x_0)$ and $d_1(x_0)$ we need to solve the Linear system:

$$C_1(x_0)\, e^{ikx_0} - d_1(x_0)\left[e^{ikx_0} + e^{-ikx_0}\right] = 0 \quad (3)$$

$$(-1) * \cancel{C_1(x_0)\, e^{ikx_0}} - d_1(x_0)\left[\cancel{e^{ikx_0}} - e^{-ikx_0}\right] = 1/ik \quad (4)$$

$$-2d_1(x_0)\, e^{-ikx_0} = -1/ik$$

$$\Rightarrow \boxed{d_1(x_0) = \frac{1}{2ik} e^{ikx_0}} \Rightarrow \boxed{d_2(x_0) = \frac{1}{2ik} e^{ikx_0}}$$

from (3)

$$\Rightarrow C_1(x_0) = \cancel{e^{-ikx_0}}\, \frac{1}{2ik} \cancel{e^{ikx_0}} \left[e^{ikx_0} + e^{-ikx_0}\right]$$

$$\therefore \boxed{C_1(x_0) = \frac{1}{2ik}\left[e^{ikx_0} + e^{-ikx_0}\right]}$$

Thus,

$$G(x,x_0) = \begin{cases} \frac{1}{2ik}\left[e^{ik(x+x_0)} + e^{+ik(x-x_0)}\right], & x > x_0 \\ \frac{1}{2ik}\left[e^{ik(x+x_0)} + \underbrace{e^{-ik(x-x_0)}}_{= e^{ik(x_0-x)}}\right], & x < x_0 \end{cases} \quad (5)$$

Checking symmetry.:

$x<x_0$ $$G(x,x_0) = \frac{1}{2ik}\left[e^{ik(x+x_0)} + e^{-ik(x-x_0)}\right] = \frac{1}{2ik}\left[e^{ik(x_0+x)} + e^{ik(x_0-x)}\right]$$

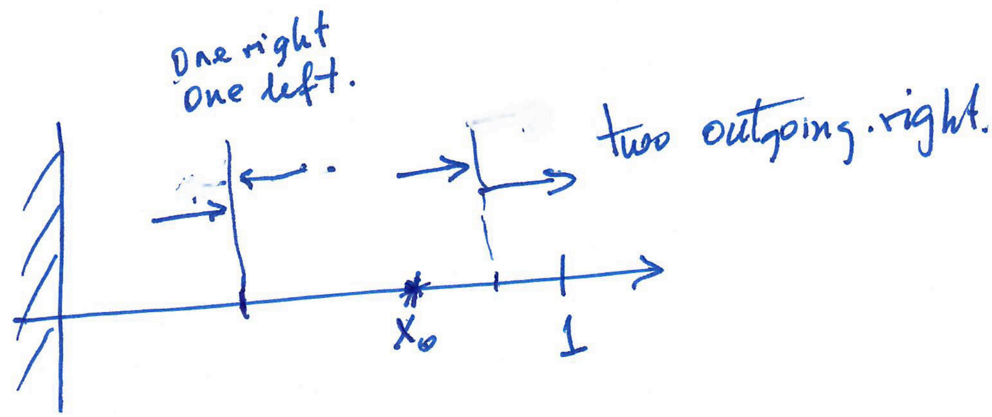
$$= G(x_0,x) \checkmark$$

Therefore, for scattering from a rigid wall at $x=0$. The Green's function is given by

$$G(x,x_0) = \frac{1}{2ik}\begin{cases} e^{ik(x+x_0)} + e^{ik(x-x_0)}, & x > x_0 \\ e^{ik(x+x_0)} + e^{-ik(x-x_0)}, & x < x_0 \end{cases}$$

or

$$G(x,x_0) = \frac{1}{2ik}\left(e^{ik|x+x_0|} + e^{ik|x-x_0|}\right).$$



Checking B.C's.

$x < x_0$ $$G'(x,x_0)\Big|_{x=0} = \frac{1}{2ik}\left[ik\, e^{ik(x+x_0)} - ik\, e^{-ik(x-x_0)}\right]\Big|_{x=0} = \frac{1}{2ik}\left(ik e^{ikx_0} - ik e^{ikx_0}\right) = 0. \checkmark$$

$x > x_0$
$$G'(x,x_0) - ik\,G(x,x_0) = \frac{1}{2ik}\left[\cancel{ik\, e^{ik(x+x_0)}} + \cancel{ik\, e^{ik(x-x_0)}} - \cancel{ik\, e^{ik(x+x_0)}} - \cancel{ik\, e^{ik(x-x_0)}}\right] = 0$$
at any x.

In paticular, $G'(1,x_0) - ik\,G(1,x_0) = 0$ ✓

c) To solve the BVP for u, we will decompose
$$u(x) = u_1(x) + u_2(x),$$
where $u_1'' + k^2 u_1 = 0, \quad u_1'(0) = ik, \quad u_1'(1) - ik\,u(1) = 0 \quad (6)$

and $u_2'' + k^2 u_2 = f(x), \quad u_2'(0) = 0, \quad u_2'(1) - ik\,u_2(1) = 0 \quad (7)$

Solving (6) $u_1(x) = Ae^{ikx} + Be^{-ikx}$

$ik = u_1'(0) = ikA - ikB \Rightarrow A - B = 1$

$0 = u_1'(1) - ik\,u(1) = \cancel{ikA e^{ik}} - ikBe^{-ik} - \cancel{ikAe^{ik}} - ikBe^{-ik}$

$= -2ikBe^{-ik} \Rightarrow \boxed{B=0}$, since $e^{-ik} \neq 0$, for all k.

$\Rightarrow \boxed{A=1}$ and $\boxed{u_1(x) = e^{ikx}}$

The Solution for (7) u_2 Can be obtained using Green's formula,

$$\int_0^1 \left(u_2 L[G] - G L[u_2]\right) dv = \left(\cancel{u_2}^{0} \frac{dG}{dx} - \cancel{G}^{0} \frac{du_2}{dx}\right)\Big|_0^1$$

$$\Rightarrow \int_0^1 u_2(x)\, \delta(x-x_0)\, dx - \int_0^1 G(x,x_0) f(x)\, dx = 0$$

$$\Rightarrow \quad u_2(x_0) = \int_0^1 G(x,x_0) f(x)\, dx.$$

reversing role and using symmetry.

$$\boxed{u_2(x) = \int_0^1 G(x,x_0) f(x_0)\, dx_0.}$$

$$\therefore \quad \boxed{u(x) = \int_0^1 G(x,x_0) f(x_0)\, dx_0 + e^{ikx}} \qquad (8)$$

We can also obtain solu. (8) using Green's formula for $u(x)$ and $G(x,x_0)$. In fact,

$$\int_0^1 \left(u L[G] - G L[u]\right) dx = \left(u \frac{dG}{dx} - G \frac{du}{dx}\right)\Big|_0^1$$

$$u(x_0) - \int_0^1 G(x,x_0) f(x)\, dx = \cancel{u(1) \underbrace{\frac{dG}{dx}(1,x_0)}_{= ikG(1,x_0)} - G(1,x_0) \underbrace{\frac{du}{dx}(1)}_{= iku(1)}}$$

$$- \cancel{u(0) \frac{dG}{dx}(0,x_0)}^{0} + G(0,x_0) \frac{du}{dx}(0) =$$

$$= \frac{1}{\cancel{2}ik} \cancel{2} e^{ikx_0}$$

Changing the role of x and x_0 and using symmetry

$$u(x) = \int_0^1 G(x,x_0)\, f(x)\,dx + \frac{e^{ikx}}{\cancel{ik}}\,\cancel{ik}$$

$$= \cancel{\int_0^1 G(x,x_0)\, f(x)\,dx + \frac{1}{2\cancel{ik}}\left(e^{ik(x+0)} + e^{+ik(x-0)}\right)\cancel{ik}}$$

or

$$\boxed{u(x) = \int_0^1 G(x,x_0)\, f(x)\,dx + e^{ikx}}$$ ✓

#3 Gabe Perry

c) Green's function for 1D Scattering problem

$$G(x,x_0) = \begin{cases} C_1 \sin(kx) + C_2 \cos(kx), & x < x_0 \\ C_3 \sin(kx) + C_4 \cos(kx, & x > x_0 \end{cases}$$

$$G'(x,x_0) = \begin{cases} kC_1 \cos(kx) - kC_2 \sin(kx), & x < x_0 \\ kC_3 \cos(kx) - kC_4 \sin(kx), & x > x_0 \end{cases}$$

then,

$x < x_0$: $0 = G'(0,x_0) \Rightarrow C_1 = 0$

$$\Rightarrow \boxed{G(x,x_0) = C_2 \cos(kx), \quad x < x_0}$$

$x > x_0$: $0 = G'(1,x_0) - ik\,G(1,x_0) \Rightarrow$

$$kC_3 \cos(k) - kC_4 \sin(k) - ikC_3 \sin(k) - ikC_4 \cos(k) = 0$$

or $$kC_3 (\cos k - i \sin(k)) \underbrace{- kC_4 (\sin k + i\cos(k))}_{-ikC_4 (\cos(k) + i\sin(k))} = 0$$

$$\Rightarrow C_4 = \frac{C_3 e^{-ik}}{i e^{ik}} \Rightarrow C_4 = -iC_3 e^{-2ik}$$

$$\boxed{C_3 = i C_4 e^{2ik}}$$

Then,

$x > x_0$: $$\boxed{G(x, x_0) = C_4 \left(i e^{2ik} \cdot \sin(kx) + \cos(kx) \right)}$$

Cont: $G(x_0^+, x_0) = G(x_0^-, x_0)$

$$\Rightarrow \quad C_2 \cos(k x_0) = C_4 \left(i e^{2ik} \sin(k x_0) + \cos(kx) \right)$$

$$\Rightarrow \quad C_2 = C_4 \left(i e^{2ik} \tan(k x_0) + 1 \right)$$

Then,

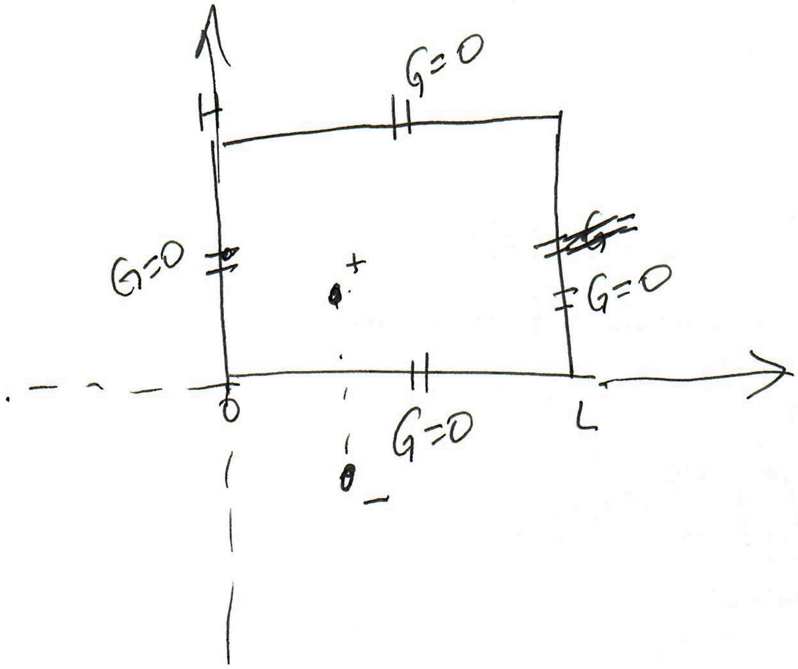
$x < x_0$: $$\boxed{G(x, x_0) = C_4 \left(i e^{2ik} \tan(k x_0) + 1 \right) \cos(kx)}$$

Not a Good Way!

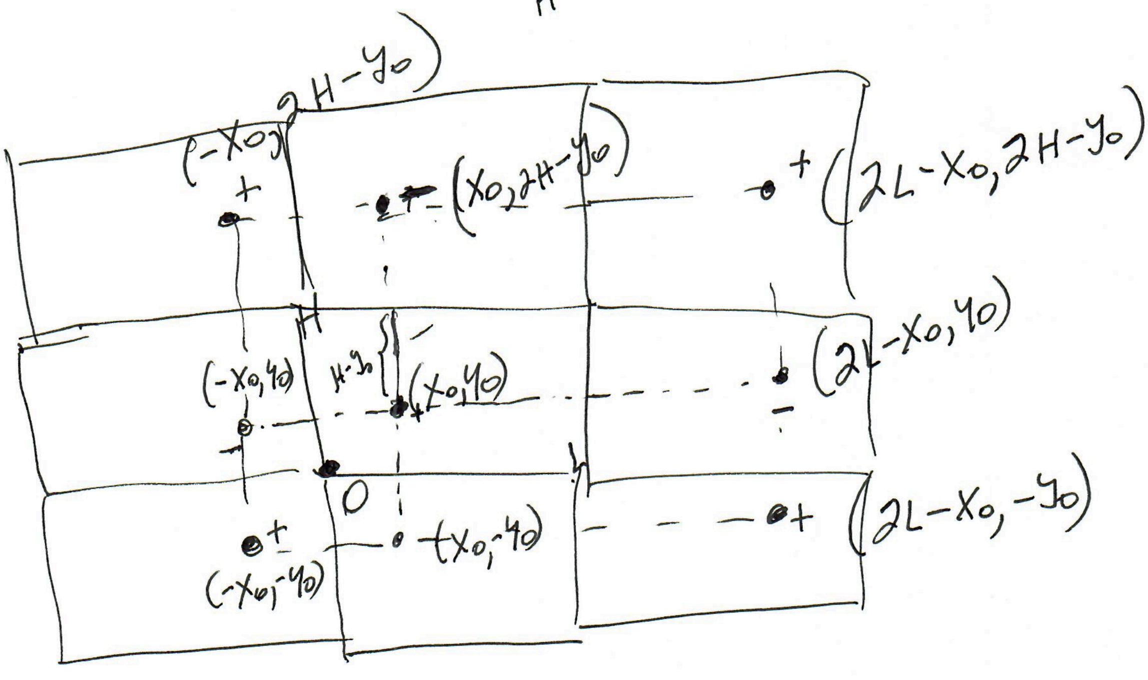
4. (15 points) Use the method of multiple images to obtain the Green's function $G(\mathbf{x}, \mathbf{x_0})$ satisfying

$$\nabla^2 G = \delta(\mathbf{x}, \mathbf{x_0}) \tag{13}$$

inside the infinite strip ($0 < x < L,\ 0 < y < H$) if $G = 0$ at $x = 0$ and $x = L$ and $G = 0$ at $y = 0$ and $y = H$. Try to arrive to a compact formula in form of an infinite series.



$H + H - y_0 = 2H - y_0$



$$\text{up}: \quad (x - x_0)^2 + (y - 2H + y_0)^2$$

5. Consider a thin one-dimensional rod extending from $x = 0$ to $x = L$ with sources of thermal energy given by $Q(x,t)$ (energy per unit of volume generated per unit of time) whose lateral surface are insulated. Assume constant density and specific heat ρ and c, respectively, with variable cross-sectional area $A(x)$ and thermal conductivity $K_0(x)$, which are continuous on $[0, L]$.

   (a) (12 points) From a conservation law obtain a PDE for the temperature $u(x,t)$. Assume that the temperature $u(x,t)$ and the flux $\phi(x,t)$ are continuously differentiable.

   (b) (8 points) Eliminate the source term in the PDE obtained in part (a) and separate variables. Assume that the ends of the bar are insulated. Show that the spatial eigenvalue problem (EVP) obtained after the separation of variables is a regular Sturm-Liouville EVP (be specific, refer to the definition). You do not need to sole this problem List three properties of the eigenfunctions associated to the spatial EVP.

#5

1. Consider a thin one-dimensional rod extending from $x = 0$ to $x = L$ with sources of thermal energy given by $Q(x, t)$ (energy per unit of volume generated per unit of time) whose lateral surface are insulated. Assume constant density and specific heat ρ and c, respectively, with variable cross-sectional area $A(x)$ and thermal conductivity $K_0(x)$.

a) (12 points) From a conservation law obtain a PDE for the temperature $u(x, t)$.

Conservation Law: Consider an arbitrary finite segment $[a,b] \subset [0,L]$

$$\frac{d}{dt}\int_a^b c\rho\, u(x,t) A(x)\,dx = \phi(0,t)A(0) - \phi(L,t)A(L) + \int_a^b Q(x,t)A(x)\,dx$$

$$\Rightarrow \quad c\rho\int_a^b \frac{\partial u}{\partial t}(x,t)A(x)\,dx = -\int_a^b \frac{\partial}{\partial x}\left(A(x)\phi(x,t)\right)dx + \int_a^b Q(x,t)A(x)\,dx$$

$$\Rightarrow \quad \int_a^b \left[c\rho\frac{\partial u}{\partial t}(x,t)A(x) + \frac{\partial}{\partial x}\left(A(x)\phi(x,t)\right) - Q(x,t)A(x)\right]dx = 0$$

Cont of integrand $\Rightarrow$ $c\rho\frac{\partial u}{\partial t}(x,t)A(x) = -\frac{\partial}{\partial x}\left(A(x)\phi(x,t)\right) + Q(x,t)A(x) = 0$

Also $\phi(x,t) \equiv -K_0(x)\frac{\partial u}{\partial x}$ $\Rightarrow$ $\boxed{c\rho A(x)\frac{\partial u}{\partial t}(x,t) = +\frac{\partial}{\partial x}\left(A(x)K_0(x)\frac{\partial u}{\partial x}\right) + Q(x,t)A(x)}$

b) (13 points) Eliminate the source term in the PDE obtained in part (a) and separate variables. Assume that the ends of the bar are insulated. Show that the spatial eigenvalue problem (EVP) obtained after the separation of variables is a regular Sturm-Liouville EVP (be specific, refer to the definition). List three properties of the eigenfunctions associated to the spatial EVP.

$$c\rho\frac{\partial u}{\partial t}(x,t) = \frac{1}{A(x)}\frac{\partial}{\partial x}\left(A(x)K_0(x)\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0$$

Sep. Vabs: $u(x,t) = h(t)\phi(x)$

$$c\rho h'\phi = \frac{h(t)}{A(x)}\frac{d}{dx}(AK_0\phi')$$

Dividing by $c\rho h\phi$

$$\frac{h'}{h} = \frac{1}{c\rho A(x)\phi(x)}\frac{d}{dx}(AK_0\phi') = -\lambda$$

$\Rightarrow$ Time Variable ODE:

$$h'(t) + \lambda h(t) = 0$$

EVP (Spatial Variable)

$$\begin{cases} \frac{d}{dx}(AK_0\phi') + \lambda c\rho A(x)\phi(x) = 0 \\ \frac{d\phi}{dx}(0) = 0, \quad \frac{d\phi}{dx}(L) = 0 \end{cases}$$

This is a regular S-L EVP since

$p(x) \equiv A(x)K_0(x)$ is real a continuous and > 0 for all x.

$q(x) \equiv 0$, $\sigma(x) \equiv c\rho A(x) > 0$ and real continuous.

Also, B.C's are Neumann type

Therefore, eigenfunctions of this EVP satisfy

1) Eigenfns corresponding to different eigenvalues are orthog.

2) They form a complete set (or basis) for the set of piecewise smooth real-valued functions.

3) For each eigenvalue there is only one lin. indep. Eigenfn.