5.2
5.3 Series Solutions near Ordinary points.

Consider

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \qquad (1)$$

P, Q, R are polynomials in the neighborhood of a point x_0, with no common factors.

We will look for solns. of (1) as power series about x_0,

$$y(x) = \phi(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \qquad (2)$$

Questions:

① What conditions should equ. (1) satisfy in order to have a soln. as (2)?

② How can we determine the radius of convergence ρ of a soln. of (1) given by a power series as (2)? Is there an alternative to the ratio test if we do not know the general term of the power series?

Exercise # 2

Find the general Solution of

$y'' - xy' - y = 0, \quad x_0 = 0.$

using power series.

Answer: $$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{\substack{n=0 \\ \text{or} \\ n=2}}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \overset{m=n-2}{=\!=} \sum_{\substack{m=-2 \text{ or} \\ m=0}}^{\infty} (m+2)(m+1) a_{m+2} x^m$$

$$xy' = x \sum_{\substack{n=0 \\ \text{or} \\ n=1}}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^n$$

Therefore,
$$\sum_{n=0}^{\infty} \left[(m+2)(n+1) a_{m+2} - (n+1) a_n \right] x^n = 0$$

Then,
$$a_{n+2} = \frac{\cancel{n+1}}{(n+2)\cancel{(n+1)}} a_n = \frac{a_n}{n+2}, \quad n = 0, 1, 2 \ldots$$

<u>n even:</u>

$k=1$ $\quad n=0: \quad a_2 = \frac{a_0}{2}$

$k=2$ $\quad n=2: \quad a_4 = \frac{a_2}{4} = \frac{a_0}{2\cdot4}$

$k=3$ $\quad n=4 \quad a_6 = \frac{a_4}{6} = \frac{a_0}{2\cdot4\cdot6}$

$\vdots \quad \vdots$

k $\quad n=2k-2: \quad a_{2k} = \frac{a_{2k-2}}{2k} = \frac{a_0}{2\cdot4\cdot6\cdots(2k-2)(2k)}$

$$= \frac{a_0}{\underbrace{2}_{1}\cdot\underbrace{(2)(2)}_{2}\cdot\underbrace{(2)(3)}_{3}\cdots\underbrace{(2)k}_{k}}$$

$$= \frac{a_0}{2^k(1\cdot2\cdot3\cdots k)}$$

Summarizing:

$$\boxed{a_{2k} = \frac{a_0}{2^k k!}}, \quad k=1,2,\dots$$

<u>n odd:</u> $\quad a_{n+2} = \dfrac{a_n}{n+2}, \quad n=0,1,2\ldots$

$k=1 \quad n=1: \quad a_3 = \dfrac{a_1}{3}$

$k=2 \quad n=3 \quad a_5 = \dfrac{a_3}{5} = \dfrac{a_1}{3\cdot 5}$

$k=3 \quad n=5 \quad a_7 = \dfrac{a_5}{7} = \dfrac{a_1}{3.5.7}$

$\vdots \qquad \vdots$

Key

$k \quad \boxed{n = 2k-1} \quad a_{2k+1} = \dfrac{a_{2k-1}}{2k+1} = \dfrac{a_1}{3.5.7.\ldots(2k+1)}$

$$= \frac{a_1\,(2)(4)\ldots(2k)}{(2.3).(4.5).(6.7)\ldots(2k)(2k+1)}$$

$$= \frac{2^k(1\cdot 2\ldots k)\,a_1}{(2k+1)!}$$

$$= \frac{2^k\,k!\,a_1}{(2k+1)!}$$

Summarizing,

$$\boxed{a_{2k+1} = \frac{k!\,2^k\,a_1}{(2k+1)!},} \quad k=1,2,\ldots$$

Finally,

$$y(x) = a_0 + a_1 x + a_0 \sum_{k=1}^{\infty} \frac{1}{2^k k!} x^{2k}$$

$$+ a_1 \sum_{k=1}^{\infty} \frac{2^k k!}{(2k+1)!} x^{2k+1}$$

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k}$$

$$+ a_1 \sum_{k=0}^{\infty} \frac{2^k k!}{(2k+1)!} x^{2k+1}$$

$$= a_0 S_0(x) + a_1 S_1(x).$$

Now, we will find out the radius of convergence for $S_1(x)$ and $S_2(x)$

For S_1: $$\lim_{\to\infty} \frac{\frac{1}{2^{k+1}(k+1)!}}{\frac{1}{2^k k!}} \frac{|x|^{2k+2}}{|x|^{2k}} = |x|^2 \lim_{k\to\infty} \frac{2^k k!}{2^{k+1}(k+1)!}$$

$$|x|^2 \lim_{k\to\infty} \frac{1}{2(k+1)} = 0 < 1, \text{ for all } x$$

So radius of convergence $\rho = +\infty$.

For S_2:
$$\lim_{k\to\infty} \frac{2^{k+1}(k+1)!\,(2k+1)!}{(2k+3)!\,2^k k!} \frac{|x|^{2k+3}}{|x|^{2k+1}}$$

$$|x|^2 \lim_{k\to\infty} \frac{2(k+1)}{(2k+2)(2k+3)} = 0 < 1 \text{ for all } x$$

Therefore, the General Soln:

$$y(x) = a_0 \underbrace{\sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k}}_{=y_1(x)} + a_1 \underbrace{\sum_{k=0}^{\infty} \frac{2^k k!}{(2k+1)!} x^{2k+1}}_{=y_2(x)}$$

Converges on $(-\infty, \infty)$ with $\rho = \infty$

On the other hand,

$$y_1'(x) = \left(1 + \frac{1}{2}x^2 + \cdots\right)' = x + \cdots$$

$$y_2'(x) = \left(x + \frac{2}{3!}x^3 + \cdots\right)' = 1 + \cdots$$

Then,

$$y_1'(0) = 0$$

$$y_2'(0) = 1$$

Finally,

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

$\{y_1, y_2\}$ is a fund. set of solutions.

Sect. 5.2 : Problem 19

$$y'' + (x-1)^2 y' + (x^2-1)\, y = 0 \longrightarrow y'' + (x-1)^2 y' + [(x-1)^2 + 2(x-1)]\, y = 0 \quad (1)$$

Find two linearly independent solutions.

Change variables : $t = x-1$. Assume $\tilde{y}(t) = \sum_{n=0}^{\infty} a_n t^n$ (2)

At least discuss until here.

$$x - 1 = t \Rightarrow x = t+1 \Rightarrow x^2 - 1 = (t+1)^2 - 1 = t^2 + 2t$$

Also, $\tilde{y}(t) = y(x(t)) = y(t+1)$

$$\Rightarrow \frac{d\tilde{y}}{dt} = \frac{dy}{dt}\frac{dx}{dt} = \frac{dy}{dt}\, 1 = \frac{dy}{dt}.$$

Equ. (1) in terms of t transforms into

$$\boxed{\tilde{y}''(t) + t^2 \tilde{y}' + (t^2 + 2t)\tilde{y} = 0} \qquad (3)$$

Subst. of (2) into (3)

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + t^2 \sum_{n=1}^{\infty} n a_n t^{n-1} + (t^2 + 2t) \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} t^m + \sum_{n=1}^{\infty} n a_n t^{n+1} + \sum_{n=0}^{\infty} a_n t^{n+2} + \sum_{n=0}^{\infty} 2 a_n t^{n+1} = 0 \qquad (m = n-2)$$

$m = n+1 \qquad m = n+2$

$$\Rightarrow \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} t^m + \sum_{m=2}^{\infty} (m-1) a_{m-1} t^m + \sum_{m=2}^{\infty} a_{m-2} t^m +$$

$m = n+1$

$$+ \sum_{m=1}^{\infty} 2 a_{m-1} t^m = 0$$

$$\Rightarrow \boxed{\sum_{m=2}^{\infty} \left[(m+2)(m+1) a_{m+2} + \overbrace{(m-1+2)}^{m+1} a_{m-1} + a_{m-2} \right] t^m + 2(1) a_2 + \left[(3)(2) a_3 + 2 a_0 \right] t}$$

$\therefore$ Next page

interesting because coefficients are coupled!

$$2a_2 = 0 \Rightarrow \underline{\underline{a_2 = 0}}$$

$$a_3 = \frac{-2a_0}{6} = -\frac{1}{3}a_0$$

$$a_{m+2} = \frac{-(m+1)a_{m-1} - a_{m-2}}{(m+2)(m+1)}, \qquad n = 2, 3, \dots$$

$m = 2$: $$a_4 = \frac{-3a_1 - a_0}{(4)(3)} = -\frac{a_1}{4} - \frac{a_0}{12}$$

$m = 3$: $$a_5 = \frac{-4a_2 - a_1}{(5)(4)} = -\cancel{\frac{a_2}{5}}^{\,0} - \frac{a_1}{20} = -\frac{a_1}{20}$$

$m = 4$: $$a_6 = \frac{-5a_3 - a_2}{(6)(5)} = -\frac{a_3}{6} - \cancel{\frac{a_2}{30}}^{\,0} = -\frac{a_3}{6} = \frac{a_0}{18}$$

$m = 5$: $$a_7 = \frac{-6a_4 - a_3}{(7)(6)} = -\frac{a_4}{7} - \frac{a_3}{42} = \frac{a_1}{28} - \frac{a_0}{84} + \frac{a_0}{126}$$
$$= \frac{a_1}{28} - \frac{a_0}{252}$$

$$\therefore \quad y(t) = a_0\left[1 - \frac{t^3}{3} - \frac{t^4}{12} + \frac{t^6}{18} - \frac{t^7}{252} + \dots\right]$$
$$+ a_1\left[t - \frac{t^4}{4} - \frac{t^5}{20} + \frac{t^7}{28} + \dots\right]$$

Finally,
$$y(x) = a_0\left[1 - \frac{(x-1)^3}{3} - \frac{(x-1)^4}{12} + \frac{(x-1)^6}{18} - \frac{(x-1)^7}{252} + \dots\right]$$
$$+ a_1\left[x - 1 - \frac{(x-1)^4}{4} - \frac{(x-1)^5}{20} + \frac{(x-1)^7}{28} + \dots\right]$$

Two important cases:

A) $P(x)$, $Q(x)$, and $R(x)$ are polynomials with no common factors $(x-x_0)$.

B) $$p(x) = \frac{Q(x)}{P(x)}, \quad q(x) = \frac{R(x)}{P(x)}$$

are analytic functions at $x = x_0$.

CASE A: Polynomials

Def. i) If x_0 is such that $P(x_0) \neq 0$ then, $x = x_0$ is called an ordinary point of

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0 \qquad (*)$$

ii) If $P(x_0) = 0$ and at least $Q(x_0) \neq 0$ or $R(x_0) \neq 0$, then $x = x_0$ is called a singular point of $(*)$.

Determine the Singular points for

5.2

10) $(4-x^2)y''+2y=0$

Here, $P(x)=4-x^2 \Rightarrow x^2=4$ or $x=\pm 2$ are Singular points. Thus, $x_0=0$ is an ordinary point and we can try to find a soln. as

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

5.3

7) $(1+x^3)y''+4xy'+y=0$

Here, $P(x)=1+x^3$ the $x=-1$ is a singular point. We can also look for a soln as

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

5.4

14) $4x^2y''+8xy'+17y=0, \quad x>0$

$x_0=0$ is a singular point, because

$P(x)=4x^2\big|_{x_0=0}=0.$

As a consequence, we do not expect a power Series Solution as

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

Soln. discussed in Sect 5.4

Procedure: If x_0 is an ordinary point,

① Propose as Solution a power Series

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

② Subst. into the equation (*)

and find

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0\, y_1(x) + a_1\, y_2(x).$$

where a_0 and a_1 are arbitrary constants and $y_1(x)$ and $y_2(x)$ form a fundamental set of of Solutions of (*).

③ Determine the radius of Convergence ρ of the power series $y_1(x)$ and $y_2(x)$.

This Can be done if the general term of the power Series is known.

If not Compute the distance from x_0 to the nearest zero of $P(x)$ (including Complex roots).

The radius of convergence of y_1 and y_2 is at least as large as the distance from x_0 to the nearest zero of $P(x)$.

Why?

Two Important theorems supporting this procedure

Weaker Version of Theorem 5.3.1

Theorem. i) If $P(x)$, $Q(x)$, and $R(x)$ are polynomials and $(x-x_0)$ is not a common factor to all of them.

ii) If $p(x)=\frac{Q(x)}{P(x)}$, $q(x)=\frac{R(x)}{P(x)}$, and $P(x_0)\neq 0$.

It means x_0 is an ordinary point of $(*)$.

Then,

A) The general solution of $(*)$ is given by

$$y(x)=\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 y_1(x) + a_1 y_2(x)$$

where a_0, a_1 are arbitrary and y_1 and y_2 are two power series solutions that converges at $x=x_0$.

B) The solutions y_1 and y_2 form a fundamental set.

C) The radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radius of convergence of the series for $p(x)=\frac{Q(x)}{P(x)}$ and $q(x)=\frac{R(x)}{P(x)}$.

<u>Theorem</u> i) If $P(x)$ and $Q(x)$ are polynomials without common factors

ii) $P(x_0) \neq 0$

Then,

A) $\dfrac{Q(x)}{P(x)}$ has a convergent power series expansion about $x = x_0$.

B) The radius of convergence of this power series about $x = x_0$ is the distance from x_0 to the nearest zero of $P(x)$.

Back to problems

5.2 # 10) $(4-x^2)y''+2y=0 \Rightarrow x=\pm 2$ Sing. points

According to thm 3.5.1,

$y(x) = a_0 y_1(x) + a_1 y_2(x)$ is

where y_1 and y_2 are power series soln. (convergent)

about any $x_0 \neq \pm 2$. In particular,

y_1 and y_2 can be power series solution

about $x_0 = 0$.

Also, the radius of convergence of y_1 and y_2

is at least $\rho = 2$.

5.3 #7) $(1+x^3)y''+4xy'+y=0$

$x=-1$ only sing. point

Using thm 3.5.1

the general soln. of this equation can be

represented as

a) $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \underbrace{y_1(x)}_{\text{power series}} + a_2 \underbrace{y_2(x)}_{\text{power series}}$.

at $x_0 = 0$

or

b) $y(x) = a_0 \tilde{y}_1(x) + a_2 \tilde{y}_2(x)$

where $\tilde{y}_1 = \sum_{n=0}^{\infty} b_n (x-2)^n$, $\tilde{y}_2 = \sum_{n=0}^{\infty} c_n (x-2)^n$

and the radius of convergence $\rho_{a,b}$ of y_1 and y_2 and $\tilde{y}_1, \tilde{y}_2$ will be at least as large as the distance from $x_0 = 0, 2$ to the roots of

$x^3 + 1 = 0$ or

$$x = (-1)^{1/3} = \left(e^{i(\pi + 2m\pi)}\right)^{1/3} \quad m = 0, 1, 2$$

$$m=0: \; x_1 = e^{i\pi/3} = \cos \pi/3 + i \sin \pi/3 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$m=1: \; x_2 = e^{i(\pi/3 + 2\pi/3)} = e^{i\pi} = -1$$

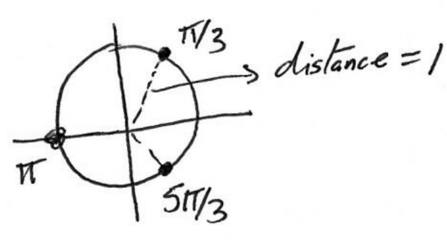
$$m=2: \; x_3 = e^{i\left(\pi/3 + \frac{4\pi}{3}\right)} = e^{i(5\pi/3)}$$

$$= \cos(5\pi/3) + i\sin(5\pi/3) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Then,

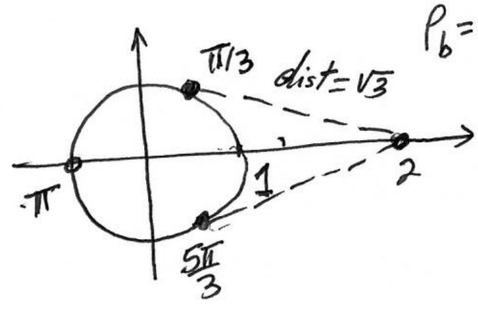
a) For y_1 and y_2 power series about $x_0 = 0$.

$\rho_a \geq 1$, because



b) for $\tilde{y}_1$ and $\tilde{y}_2$ P.S. about $x_0 = 2$

$\rho_b \geq \sqrt{3}$, because



$$\rho_b = \sqrt{\left(2-\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2} = \sqrt{\left(\tfrac{3}{2}\right)^2 + \tfrac{3}{4}} = \sqrt{\tfrac{12}{4}} = \sqrt{3}.$$

Consider now,

5.3 #12) $\quad e^x y'' + xy = 0 \qquad (**)$

$P(x) = e^x$ not a polynomial

$Q(x) = 0, \quad R(x) = x.$

What can be said abut power series solutions?

First, we need to extend the notion of ordinary and singular points for equations as (**) where $P(x)$, $Q(x)$, or $R(x)$ are not polynomials.

For this, we also need the definition of analytic functions.

Def.- Consider

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

and $y'' + p(x)y' + q(x)y = 0$, where $p = \frac{Q}{P}$, $q = \frac{R}{P}$.

a) The functions p, q are analytic at $x = x_0$ if they have Taylor series expansions that converge to them in some interval about $x = x_0$.

$$p(x) = p_0 + p_1(x - x_0) + \cdots + p_n(x - x_0)^n + \cdots$$

$$q(x) = q_0 + q_1(x - x_0) + \cdots + q_n(x - x_0)^n + \cdots$$

b) If the functions $p(x)$ and $q(x)$ are analytic at $x = x_0$, the point $x = x_0$ is an ordinary point. otherwise, it is a singular point

Thm 5.3.1

i) If x_0 is an ordinary point of $(*)$

ii) If $p = \frac{Q}{P}$ and $q = \frac{R}{P}$ are analytic at x_0

Then,

A) The general soln. of $(*)$ is given by

$$y(x) = a_0\, y_1(x) + a_1\, y_2(x)$$

where a_0, a_1 arbitrary and y_1, y_2 are two power series solution that converges at $x = x_0$.

B) The solns. y_1 and y_2 form a fundamental set.

C) The radius of convergence for y_1 and y_2 is at least as large as the minimum of the radius of convergence of the series p and q.

$\underline{\underline{5.3}}$ #12) $\quad e^x y'' + xy = 0.$ (Very good!)

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0.$$

or

$$\sum_{2} n(n-1) a_n x^{n-2} + \sum_{n=2} n(n-1) a_n x^{n-1} + \sum_{n=2} \frac{n(n-1)}{2} a_n x^n + \sum_{2} \frac{n(n-1)}{6} a_n x^{n+1} +$$

$$+ \sum_{2} \frac{n(n-1)}{12} a_n x^{n+2} + \sum_{2} \frac{n(n-1)}{120} a_n x^{n+3} + \sum_{n=0} a_n x^{n+1} = 0$$

or

$$\sum_{0} (n+2)(n+1) a_{n+2} x^n + \sum_{1} (n+1) n a_{n+1} x^n + \sum_{2} \frac{n(n-1)}{2} a_n x^n + \sum_{3} \frac{(n-1)(n-2)}{6} a_{n-1} x^n$$

$$+ \sum_{4} \frac{(n-2)(n-3)}{12} a_{n-2} x^n + \sum_{5} \frac{(n-3)(n-4)}{120} a_{n-3} x^n + \sum_{n=1} a_{n-1} x^n = 0.$$

x^0: $\quad 2(1) a_2 = 0 \Rightarrow \boxed{a_2 = 0}$

x^1: $\quad 3(2) a_3 + 2(1) \cancel{a_2}^{0} + a_0 = 0$

$\Rightarrow \boxed{a_3 = -\frac{a_0}{6}}$

x^2: $\quad 4(3) a_4 + 3(2) a_3 + \frac{2(1)}{2} \cancel{a_2}^{0} + a_1 = 0$

$12 a_4 + 6 a_3 + a_1 = 0 \Rightarrow a_4 = \frac{1}{12}(-a_1 - 6a_3)$

$\Rightarrow \boxed{a_4 = \frac{1}{12}(a_0 - a_1)}$

$$x^3: \quad 5(4)a_5 + 4(3)a_4 + \frac{3(2)}{2}a_3 + \left(\frac{2(1)}{6}+1\right)a_2$$

$$\Rightarrow a_5 = \frac{-1}{20}\left[12a_4 + 3a_3 + \frac{4}{3}\cancel{a_2}^{0}\right]$$

$$\Rightarrow a_5 = -\frac{1}{20}\left[a_0 - a_1 - \frac{a_0}{2}\right] = -\frac{1}{20}\left[\frac{a_0}{2} - a_1\right]$$

so $\boxed{a_5 = -\frac{1}{20}\left[\frac{a_0}{2} - a_1\right]}$

Therefore,

$$y(x) = a_0 + a_1 x - \frac{a_0}{6}x^3 + \frac{1}{12}(a_0 - a_1)x^4 - \frac{1}{20}\left(\frac{a_0}{2} - a_1\right)x^5 + \cdots$$

or $$y(x) = a_0\left[1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \cdots\right] +$$

$$+ a_1\left[x - \frac{x^4}{12} + \frac{x^5}{20} \cdots\right]$$

$$x^4: \quad 6(5)a_6 + 5(4)a_5 + \frac{4(3)}{2}a_4 + \left[\frac{3(2)}{6}+1\right]a_3 +$$

$$+ \frac{2(1)}{12}\cancel{a_2}^{0} = 0.$$

$$a_6 = \frac{-1}{30}\left[20a_5 + 6a_4 + 2a_3\right] = -\frac{1}{30}\left[a_1 - \cancel{\frac{a_0}{2}} + \cancel{\frac{1}{2}a_0} - \frac{1}{2}a_1 - \frac{a_0}{3}\right] = -\frac{a_1}{60} + \frac{a_0}{90}$$

$$\Rightarrow \boxed{a_6 = -\frac{a_1}{60} + \frac{a_0}{90}}$$

$$\boxed{a_6 = -\frac{a_1}{60} + \frac{a_0}{90}}$$

Therefore,

$$y(x) = a_0\left[1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \frac{x^6}{90} + \cdots\right]$$

$$+ a_1\left[x - \frac{x^4}{12} + \frac{x^5}{20} - \frac{x^6}{60} + \cdots\right]$$