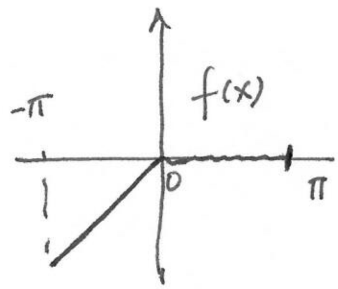
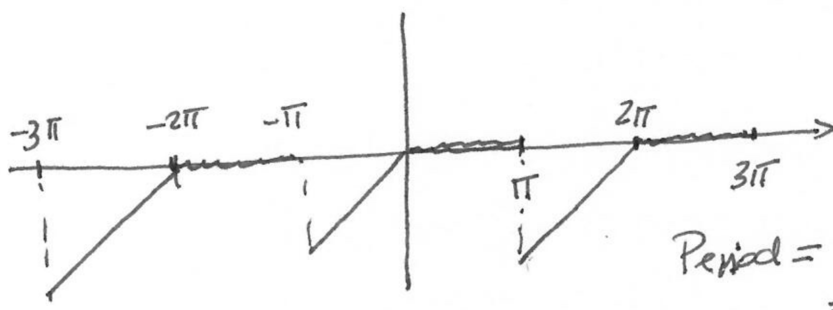
10.2 #15) Find Fourier Series for

$$f(x) = \begin{cases} x, & -\pi \le x < 0 \\ 0, & 0 \le x < \pi \end{cases}$$



Periodic extension of $f(x)$



Period $= 2L = 2\pi$
$\Rightarrow L = \pi.$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx = \frac{1}{\pi}\frac{x^2}{2}\Big|_{-\pi}^{0} = \frac{1}{\pi}\left(-\frac{(-\pi)^2}{2}\right)$$

$$\Rightarrow \boxed{a_0 = \frac{-1}{\pi}\frac{\pi^2}{2} = \frac{-\pi}{2}}$$

$n = 1, 2 \ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx$$

IPP form. 10.2b

$$= \frac{1}{\pi}\left[-\frac{x}{n}\cos nx + \frac{\sin nx}{n^2}\right]\Big|_{-\pi}^{0} = -\frac{1}{\pi}\left[\frac{\pi}{n}\cos(-n\pi) + \cancel{\frac{\sin(-n\pi)}{n^2}}^{\,0}\right]$$

$$= -\frac{1}{\pi}\frac{\pi}{n}\cos(-n\pi) = -\frac{1}{n}(-1)^n = \frac{(-1)^{n+1}}{n}$$

or $\boxed{b_n = \frac{(-1)^{n+1}}{n}}$, $n = 1, 2 \ldots$

$n \neq 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx =$$

$$\frac{1}{\pi}\left[\frac{x}{n}\sin nx + \frac{\cos nx}{n^2}\right]\Bigg|_{-\pi}^{0} = \frac{1}{\pi}\left[\frac{1}{n^2} - \left(\frac{\pi}{n}\overset{0}{\cancel{\sin(n\pi)}} + \frac{\cos(n\pi)}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{1}{n^2} - \left(\frac{(-1)^n}{n^2}\right)\right) = \frac{1}{\pi}\left(\frac{1}{n^2} + \frac{(-1)^{n+1}}{n^2}\right)$$

$$= \frac{1}{\pi}(\quad) = \begin{cases} 0, & n = \ 2, \ldots \\ \frac{1}{\pi}\frac{2}{n^2}, & n = 1, 3, 5, \ldots \end{cases}$$

Thus,

$$a_{2n-1} = \frac{1}{\pi}\frac{2}{(2n-1)^2}, \quad n = 1, 2, \ldots$$

$$a_{2n} = 0, \quad n = 1, 2, \ldots$$

$$\therefore \quad f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)^2}\cos(2n-1)x + \frac{(-1)^{n+1}}{n}\sin nx\right] + \frac{\overbrace{-\frac{\pi}{2}}^{=a_0}}{2}$$

or

$$\boxed{f(x) = -\frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{\cos(2n-1)x}{(2n-1)^2} + \frac{(-1)^{n+1}}{n}\sin nx\right]}$$

10.3 # 14) Find the formal Soln. of the IVP:

$$\begin{cases} y'' + \omega^2 y = \sum_{n=1}^{\infty} b_n \sin nt & (1) \\ y(0) = 0, \quad y'(0) = 0 & (2) \end{cases}$$

The Soln. of (1) Consists of the general Soln. of the homogeneous equ. plus a particular Soln.

$$y(x) = y_h(t) + y_p(t) \qquad (3)$$

$y_h(x)$ Satisfies

$$y_h'' + \omega^2 y_h = 0, \quad y(0) = 0, \ y'(0) = 0$$

Thus, $\quad y(x) = C_1 \cos \omega.. + C_2 \sin \omega$

To obtain $y_p(x)$ Satisfying

$$y_p'' + \omega^2 y_p = \sum_{n=1}^{\infty} b_n \sin nt \qquad (4)$$

We use thm 10.3.1 and assume

$$y_p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(d_m \cos nt + e_m \sin nt \right)$$

Subst. into (4) (assuming derivation of Series can be done term by term)

leads to

$$y_p'' + \omega^2 y_p =$$

$$= -\sum_{n=1}^{\infty} \left(n^2 d_m \cos nt - n^2 e_m \sin nt \right)$$

$$+ \omega^2 \sum_{n=1}^{\infty} \left(d_n \cos nt + e_n \sin nt \right) = \sum_{n=1}^{\infty} b_n \sin nt$$

or

$$\sum_{n=1}^{\infty} (\omega^2 - n^2) d_m \cos nt + \sum_{n=1}^{\infty} (\omega^2 - m^2) e_n \sin nt$$

$$= \sum_{n=1}^{\infty} b_n \sin (nt)$$

Thus, $(\omega^2 - n^2) d_n = 0$, $n = 1, 2 \ldots \Rightarrow \boxed{d_n = 0}$ $n = 1, 2 \ldots$

and $(\omega^2 - n^2) e_n = b_n \Rightarrow e_n = \dfrac{b_n}{\omega^2 - n^2}$, $n = 1, 2 \ldots$

Therefore,

$$\boxed{y_p(t) = \sum_{n=1}^{\infty} \frac{b_n}{\omega^2 - n^2} \sin nt}$$

and the general Soln. of (1)

$$y(t) = C_1 \cos \omega t + C_2 \sin \omega t + \sum_{n=1}^{\infty} \frac{b_n}{\omega^2 - n^2} \sin(nt)$$

Using I.C.

$$0 = y(0) = C_1 \Rightarrow \boxed{C_1 = 0}$$

and
$$\boxed{y(t) = C_2 \sin \omega t + \sum_{n=1}^{\infty} \frac{b_n}{\omega^2 - n^2} \sin(nt)}$$

then

$$y'(t) = \omega C_2 \cos \omega t + \sum_{n=1}^{\infty} \frac{n\, b_n}{\omega^2 - n^2} \cos nt$$

Using the 2nd I.C.

$$0 = y'(0) = \omega C_2 \overset{=1}{\cos 0} + \sum_{n=1}^{\infty} \frac{n\, b_n}{\omega^2 - n^2} \overset{=1}{\cos 0}$$

$$\Rightarrow \boxed{C_2 = \frac{-1}{\omega} \sum_{n=1}^{\infty} \frac{n\, b_n}{\omega^2 - n^2}}$$

and finally

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{\omega^2 - n^2} \left(b_n \sin nt - \frac{n b_n}{\omega} \sin \omega t \right)$$

or

$$\boxed{y(t) = \sum_{n=1}^{\infty} \frac{b_n}{(\omega^2 - n^2)\omega} \left(\omega \sin nt - n \sin \omega t \right)}$$