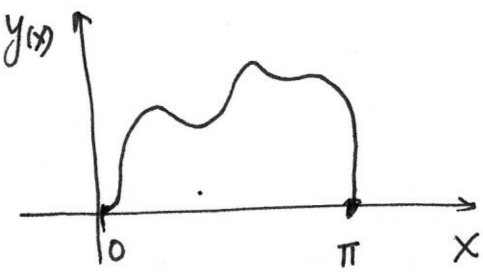
Solutions of BVPs

I) Homogeneous:

a) $y'' + 2y = 0, \quad y(0) = 0, \quad y(\pi) = 0$

looking for



We found $y(x) = C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x$

$$0 = y(0) = C_1 \cos 0 + C_2 \sin 0 = C_1 \Rightarrow \boxed{C_1 = 0}$$

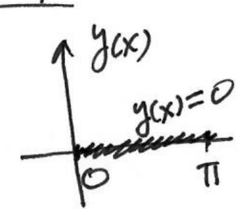
Then, $\quad y(x) = C_2 \sin\sqrt{2}x$

using 2nd B.C. : $y(\pi) = 0$

$$0 = y(\pi) = C_2 \underbrace{\sin\sqrt{2}\pi}_{\neq 0} \Rightarrow \boxed{C_2 = 0}$$

Then,

Soln: $\boxed{y(x) = 0}$ only soln.



Similar to linear System:

$A\vec{x} = 0$ Where $\underbrace{\begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}}_{\text{invertible matrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

which has as soln

$x_1 = 0$

$2x_1 = 3x_2 \Rightarrow x_2 = 0$

$\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ only soln.

b) $y'' + y = 0, \quad y(0) = 0, \quad y(\pi) = 0$

We find

$$y(x) = C_1 \cos x + C_2 \sin x$$

$$0 = y(0) = C_1 \overset{=1}{\cos 0} + C_2 \overset{=0}{\sin 0} = C_1 \Rightarrow \boxed{C_1 = 0}$$

So, $\quad y(x) = C_2 \sin x$

Using another B.C.:

$$0 = y(\pi) = C_2 \overset{=0}{\sin \pi} \Rightarrow C_2 \text{ is arbitrary.}$$

and we have infinitely many Solns:

$$\boxed{y(x) = C_2 \sin x, \quad C_2 \text{ in } \mathbb{R}}$$

Similar to linear System:

$$A\vec{x} = 0, \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \quad x_1 = -2x_2, \quad \vec{x} = c \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

non invertible matrix

infinitely many Solns.

II) Nonhomogeneous:

a) $y'' + 2y = 0, \quad y(0) = 0, \quad y(\pi) = 2$

$$y(x) = C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x$$

$$0 = y(0) = C_1 \overset{=1}{\cos 0} + C_2 \overset{=0}{\sin 0} = C_1 \Rightarrow \boxed{C_1 = 0}$$

$$y(x) = C_2 \sin\sqrt{2}x$$

Second B.C.:

$$2 = C_2 \sin\sqrt{2}\pi \Longrightarrow \boxed{C_2 = \frac{2}{\sin\sqrt{2}\pi}}$$

and soln:

$$\boxed{y(x) = \frac{2}{\sin\sqrt{2}\pi} \sin\sqrt{2}x}$$

unique soln.

Similar to linear algebraic system:

$$A\vec{x} = \vec{b} \qquad \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\boxed{x_1 = 3}$

$2x_1 - 3x_2 = 1$

$3x_2 = 6 - 1 \Rightarrow \boxed{x_2 = 5/3}$

unique soln.

$$\vec{x} = \begin{bmatrix} 3 \\ 5/3 \end{bmatrix}$$

b) $y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = \begin{cases} 2 \\ -1 \end{cases}$

$$y(x) = C_1 \cos x + C_2 \sin x$$

$$= y(0) = C_1 \Rightarrow \boxed{C_1 = 1}$$

$$2 = y(\pi) = C_2 \overset{=0}{\sin\pi} = 0$$

$2 = 0$!
no Solution.

or

$$-1 = y(\pi) = \cos\pi + C_2 \overset{=0}{\sin\pi} = -1 \checkmark$$

for any C_2.

infinitely many solns.

Similar to linear algebraic system:

$A\vec{x} = \hat{b} \qquad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \hat{b}$

If $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ then $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -5 \end{bmatrix}$

$0 = -5$!
no soln.

or If $b = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = 4 - 2x_2$, x_2 free

inf. many solns. $\begin{bmatrix} -2 \\ 3 \end{bmatrix}, \underset{x_2=1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$

Eigenvalue Problem:

$$A\hat{x} = \lambda \vec{x} \qquad \text{Algebraic System} \qquad (1.1)$$

$$y'' + \lambda y = 0, \quad y(0) = 0, \; y(\pi) = 0 \qquad \text{BVP.} \; (1.2)$$

Previous results:

a) $\lambda = 2$, only trivial Soln. $\Rightarrow \lambda = 2$ is not an eigenvalue.

b) $\lambda = 1$, infinitely many Solns $\Rightarrow \lambda = 1$ is an eigenvalue.

$y(x) = C_2 \sin x$, C_2 arbitrary.

The goal now is to find all eigenvalues and their corresponding eigenfunctions for (1.2)

$\Downarrow$ or

nontrivial Solns.

Computation of eigenvalues and eigenfunctions.

A) For $\boxed{\lambda > 0}$ $\qquad y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$

Gen. Soln.: $\quad y(x) = C_1 \cos\sqrt{\lambda}\, x + C_2 \sin\sqrt{\lambda}\, x$

1st B.C.: $\quad 0 = C_1 \overset{=1}{\cos 0} + C_2 \overset{=0}{\sin 0} = C_1 \Rightarrow C_1 = 0.$

Then, $\quad y(x) = C_2 \sin\sqrt{\lambda}\, x.$

2nd B.C.: $\quad 0 = C_2 \sin\sqrt{\lambda}\, \pi$

For nontrivial Soln $C_2 \neq 0$.

Therefore, $\sin\sqrt{\lambda}\,\pi = 0 \Rightarrow \sqrt{\lambda}\,\pi = \pm n\pi, \quad n = 0, 1, 2 \ldots$

$$\Rightarrow \sqrt{\lambda} = \pm \frac{n\pi}{\pi} = \pm n$$

$$\Rightarrow \boxed{\lambda_n = n^2; \quad n = 1, 2, \ldots}$$

Since $\lambda > 0$, $n = 0$ is not included.

For each, $\lambda_n = n^2$ the corresponding eigenfunction

is $\boxed{y_n(x) = \sin(nx), \quad n = 1, 2, \ldots}$

B) For $\boxed{\lambda = 0}$, $y'' = 0, \quad y(0) = 0, \quad y(\pi) = 0$

$$y(x) = c_1 + c_2 x$$

B.Cs: $\left.\begin{array}{l} 0 = y(0) = c_1 \Rightarrow c_1 = 0. \\ 0 = c_2\pi \Rightarrow c_2 = 0. \end{array}\right\} \Rightarrow y(x) = 0$ only soln.

C) For $\boxed{\lambda < 0}$, $r^2 = -\lambda > 0 \Rightarrow \begin{array}{l} r_1 = \sqrt{-\lambda} \text{ real} \\ r_2 = -\sqrt{-\lambda} \text{ real.} \end{array}$

$$y(x) = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x}$$

B.C.s: $\left.\begin{array}{l} c_1 + c_2 = 0 \\ c_1 e^{\sqrt{-\lambda}\,\pi} + c_2 e^{-\sqrt{-\lambda}\,\pi} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 0 \end{array}$

d) There is still a possibility to have λ complex eigenvalue. But, this is not possible (#23).

<u>Summarizing:</u>

<u>Eigenvalue Problem</u>: $y'' + \lambda y = 0$
$y(0) = 0, \quad y(\pi) = 0$

<u>Eigenvalues:</u> <u>Eigenfunctions:</u>

$\lambda_n = n^2, \ n = 1, 2 \ldots, \quad y_n = \sin(nx), \quad n = 1, 2, \ldots$

Based on the algebraic eigenvalue problem a natural expectation is

i) Eigenfunctions : $y_n(x) = \sin(nx)$ are orthogonal.

ii) Any soln. of

$y'' - \mu y = g(x), \ y(0) = 0, \quad y(\pi) = 0, \quad \mu \neq \lambda_n$ for all n.

Can be written as

$$y(x) = \sum_{n=0}^{\infty} C_n \sin(nx).$$

These observations lead us to study <u>Fourier Series</u>.

10.2 Fourier Series

$$\frac{a_0}{2} + \sum a_m \cos \frac{m\pi}{L} x + b_m \sin \frac{m\pi}{L} x \quad (4.1)$$

Questions:

1) Is this series converging for some x.

2) Can we use series (4.1) to represent some functions?
(Similar to Taylor's series).

3) If (2) is true for certain $f(x)$,
How do we find the coefficients
$a_m, b_m, a_0, \quad m = 1, 2, \ldots$?

Periodicity: T is a period of f if
$f(x+T) = f(x)$, for every x.

Notice that $f(x+2T) = f((x+T)+T) = f(x+T) = f(x)$.

So $2T$ is also a period.
The smallest T such that $f(x+T) = f(x)$
is called the fundamental period of f.

The functions $\sin\left(\frac{m\pi}{L}x\right)$ and $\cos\left(\frac{m\pi}{L}x\right)$

$m=1,2,\ldots$

are periodic.

In fact, $\sin x$ and $\cos x$ are periodic with period $\underline{2\pi}$.

Also, $\sin(\alpha x)$ and $\cos(\alpha x)$ are periodic

To find this period: $\alpha x = 2\pi \Rightarrow x = \frac{2\pi}{\alpha}$

or $T = \frac{2\pi}{\alpha}$.

In particular,

for $\sin\left(\frac{m\pi}{L}x\right)$ and $\cos\left(\frac{m\pi}{L}x\right)$

$$\frac{m\pi}{L}x = 2\pi \Rightarrow \boxed{x = \frac{2L}{m} = T}$$ period.

Orthogonality:

Consider u, v two real valued functions defined in (α, β).

Inner Product:

$$(u,v) = \int_{\alpha}^{\beta} u(x)\, v(x)\, dx.$$

Recall inner product of two vectors:

$$\vec{u} = (u_1, u_2, u_3), \quad \vec{v} = (v_1, v_2, v_3)$$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

We say $\vec{u}$ and $\vec{v}$ are orthogonal if

$$\vec{u} \cdot \vec{v} = 0.$$

Similarly, u, v are orthogonal in (α, β) if

$$\int_{\alpha}^{\beta} u(x) v(x)\, dx = 0.$$

The set of trigometric function

$$\begin{cases} \sin \frac{m\pi}{L} x, \\ \cos \frac{m\pi}{L} x, \end{cases} \quad m = 1, 2, \dots$$

form a mutually orthogonal set of functions on the interval $[-L, L]$.

In fact,

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = 0.$$

Proof:-

$$\cos(\alpha+\beta)x = \cos\alpha x \cos\beta x - \sin\alpha x \sin\beta x$$

$$\cos(\alpha-\beta)x = \cos\alpha x \cos\beta x + \sin\alpha x \sin\beta x$$

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then $\cos(\alpha-\beta)x - \cos(\alpha+\beta)x = 2\sin\alpha x \sin\beta x.$

$$\Rightarrow \quad \sin\alpha x \sin\beta x = \frac{1}{2}\left[\cos(\alpha-\beta)x - \cos(\alpha+\beta)x\right]$$

Thus,

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = \frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(m-n)\pi}{L}x - \cos\left(\frac{(m+n)\pi}{L}x\right)\right] dx$$

$$= \frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(m-n)\pi}{L}x - \cos\frac{(m+n)\pi}{L}x\right] dx$$

$$= \frac{1}{2}\left[\frac{L}{\pi(m-n)} \sin \frac{(m-n)\pi}{L}x - \frac{L}{\pi(m-n)} \sin\left(\frac{(m+n)\pi}{L}x\right)\right]\Bigg|_{-L}^{L}$$

$$\frac{1}{m-n}\cdot\frac{1}{2}\frac{L}{\pi}\left[\sin\left[\frac{(m-n)\pi}{\cancel{L}}\right]\cdot \sin\left[\frac{(m+n)\pi}{\cancel{L}}\cancel{L}\right] - \sin\left[\frac{(m-n)\pi}{\cancel{L}}(-\cancel{L})\right]\circ \sin\left[\frac{(m+n)\pi}{\cancel{L}}(-\cancel{L})\right]\right]$$

$$= 0, \quad \text{if } m \neq \pm n.$$

Similarly, we can easily prove that

$$\int_{-L}^{L} \cos\frac{m\pi}{L}x \cos\frac{n\pi}{L}x\, dx = \begin{cases} 0, & m\neq n \\ L, & m=n. \end{cases}$$

$$\int_{-L}^{L} \cos\frac{m\pi}{L}x \sin\frac{n\pi}{L}x\, dx = 0, \quad m\neq n.$$

$$\int_{-L}^{L} \sin\frac{m\pi}{L}x \sin\frac{n\pi}{L}x\, dx = \begin{cases} 0, & m\neq n \\ L, & m=n \end{cases}$$

Euler - Fourier formula

Assume

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \frac{m\pi}{L} x + b_m \sin \frac{m\pi}{L} x$$

$$\xrightarrow{\text{converges}} f(x), \text{ on } [-L, L]. \quad (9.1)$$

The coefficients a_m and b_m can be obtained in terms of $f(x)$ as follows:

i) Multiply (9.1) by $\cos(\frac{n\pi}{L} x)$, n fixed, $n > 0$

ii) Assuming that we can integrate term by term.

$$\int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx = \frac{a_0}{2} \int_{-L}^{L} \cos \frac{n\pi}{L} x \, dx$$

$$+ \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \frac{m\pi}{L} x \cos \frac{n\pi}{L} x \, dx$$

$$+ \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{m\pi}{L} x \cos \frac{n\pi}{L} x \, dx$$

Using the orthogonality

$$\int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx = \frac{a_0}{2} \cdot 0 + \overset{m=n}{a_n L} + \sum_{m=1}^{\infty} b_m \cdot 0$$

$$= a_n L$$

Thus,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx .$$

Similarly, $$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx$$

and $$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx .$$