POSSIBILITIES OF SOLUTIONS FOR LINEAR SYSTEMS.

Consider

$$A_1 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Find the solutions of

a) $A_1 \vec{x} = 0$, b) $A_1 \vec{x} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

c) $A_2 \vec{x} = \vec{0}$, d) $A_2 \vec{x} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, e) $A_2 \vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

Answers:

a) $\begin{pmatrix} 3 & 1 & 0 \\ 2 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & 0 \\ 0 & 4/3 & 0 \end{pmatrix}$ $(-2/3)$ $\sim$ $\frac{4}{3}x_2 = 0 \Rightarrow \boxed{x_2 = 0}$

$3x_1 = -x_2 \Rightarrow \boxed{x_1 = 0}$

So, $\boxed{\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$ unique solution.

b) $\begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & -1 \\ 0 & 4/3 & 11/3 \end{pmatrix} \Rightarrow 4x_2 = 11$

$\boxed{x_2 = 11/4}$

$3x_1 = -1 - x_2 = -\frac{4}{4} - \frac{11}{4} = \frac{-15}{4}$

$x_1 = \frac{-15}{(3)(4)} = \frac{-5}{4}$ $\boxed{x_1 = -\frac{5}{4}}$

So, $\boxed{\vec{x} = \begin{pmatrix} -5/4 \\ 11/4 \end{pmatrix}}$ unique solution.

c) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim$ $x_1 = -2x_2$

$\vec{x} = \begin{pmatrix} -2x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Infinitely many Solns. Any multiple of $\vec{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is a soln. For example, $\begin{pmatrix} -4 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}$

d) $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \end{pmatrix} \Rightarrow$ $0 = 5$!

No Solution.

e) Consider $A_2 \vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = 3 - 2x_2$

or $\vec{x} = \begin{pmatrix} 3 - 2x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Infinitely many Solutions: $\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ +1 \end{pmatrix}, \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}, \ldots$

Matrix A_1 is invertible because

$$\det A_1 = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} = 6-2 = 4 \neq 0.$$

Also its columns are linearly independent.

Matrix A_2 is noninvertible because

$$\det A_2 = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4-4=0$$

Clearly, its columns are linearly dependent.

In general, we have the following theorems:

Thm 1 If the matrix $A_{n \times n}$ is invertible

then i) $A\vec{x} = \vec{0}$ has only the trivial soln.

ii) $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$ and this solution is unique.

Thm 2 If the matrix A is noninvertible then,

i) $A\vec{x}=\vec{0}$ has infinitely many solutions.

ii) $A\vec{x}=\vec{b}$ has infinitely many solns. if $\vec{b}$ is a linear combination of the columns of A.
Otherwise, $A\vec{x}=\vec{b}$ has no solutions.

Eigenvalues and Eigenvectors

Given a matrix $A_{n\times n}$, if

$$A\vec{x}=\lambda\vec{x}, \qquad \vec{x}\neq 0, \quad \lambda \text{ scalar } \left(\begin{matrix}\text{real}\\ \text{or}\\ \text{complex}\end{matrix}\right)$$

λ is called an eigenvalue and $\vec{x}$ is called an eigenvector.

Example: For $A=\begin{bmatrix}3&1\\2&2\end{bmatrix}$

$$A\begin{pmatrix}1\\1\end{pmatrix}=\begin{bmatrix}3&1\\2&2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix}=4\begin{bmatrix}1\\1\end{bmatrix}$$

Then $\lambda_1=4$ is an eigenvalue and $\vec{v}_1=\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector corresponding to $\lambda_1=4$.

Also, $A \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

So, $\lambda_2 = 1$ is an eigenvalue

and $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector corresp. to $\lambda_2 = 1$.

i) Why do we care about eigenvectors and eigenvalues?

ii) How do we compute them?

Talk about applications in ODEs. and PDEs.

Computation of Eigenvalues and Eigenvectors

Given a matrix $A_{n \times n}$, if $\vec{x} \neq \vec{0}$

Satisfies
$$A\vec{x} = \lambda \vec{x}, \qquad (5.1)$$

then λ is an eigenvalue with corresponding eigenvector $\vec{x}$.

In order to find λ's and corresp. $\vec{x}$s.

we rewrite (5.1) as

$$A\vec{x} - \lambda\vec{x} = \vec{0} \iff (A - \lambda I)\vec{x} = \vec{0} \qquad (5.2)$$

$$\Updownarrow$$

$$A\vec{x} - \lambda I \vec{x} = 0$$

Therefore, according to theorems 1 and 2 in previous pages, the linear syst. (5.2) will have a nontrivial solution (actually infinitely many) if

$$|A - \lambda I| = 0. \qquad (5.3)$$

Equation (5.3) allows us to compute the eigenvalues of A. In fact, for our example we have

$$\begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 2$$
$$= (\lambda-3)(\lambda-2) - 2 = 0$$

$$\Leftrightarrow \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1) = 0$$

or $\lambda_1 = 4$, $\lambda_2 = 1$ are the eigenvalues of A.

Now, we can use λ_1 and λ_2 in (5.2) and compute the eigenvectors.

$$(A-\lambda I)\vec{x} = \vec{0}$$

For $\lambda_1 = 4$ $\quad \left(\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $\quad \begin{bmatrix} 3-4 & 1 \\ 2 & 2-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or $\quad \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2$

$$\vec{x} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So for $\lambda_1 = 4$ the corresp. eigenvectors are the multiples of $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda_2 = 1$ $\quad (A - \lambda_2 I)\vec{x} = 0 \Leftrightarrow \begin{bmatrix} 3-1 & 1 \\ 2 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = -2x_1 \quad \vec{x} = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

So for $\lambda_2 = 1$ Corresp. eigenvectors are multiples of $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Algebraic System

Eigenvalue Problem:

Find $\vec{x} \neq \vec{0}$ and λ such that

$$\boxed{A\vec{x} = \lambda\vec{x}} \text{ or } \qquad (A.1)$$

$$A\vec{x} - \lambda\vec{x} = \vec{0} \text{ or}$$

$$A\vec{x} - \lambda I\vec{x} = \vec{0} \text{ or}$$

$$\boxed{(A - \lambda I)\vec{x} = \vec{0}} \qquad (A.2)$$

Ex. 1. $(A - 3I)\vec{x} = \vec{0}$

$$\left[\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Two-point BVP

Eigenvalue Problem:

$$\begin{cases} y''(x) + \lambda y(x) = 0, & a < x < b \\ y(a) = 0, \quad y(b) = 0 \end{cases} \qquad (B.1)$$

Find $y(x) \not\equiv 0$ and λ real or complex such that (B.1) is satisfied.

Ex. 1 $\begin{cases} y'' + 2y = 0, & 0 < x < \pi \\ y(0) = 0, \quad y(\pi) = 0 \end{cases}$

$$r^2 + 2 = 0 \Rightarrow r = \pm\sqrt{2}\,i$$

General soln

$$y(x) = C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x$$

$$\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x_1 = 0, \quad 2x_1 = x_2.$

$\Rightarrow \quad x_2 = 0$

Only Soln is $\boxed{\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$

Conclusion:

$\boxed{\lambda = 3}$ is not eigenvalue of A.

Using BCs:

$y(0) = 0$

$y(0) = C_1 \overset{=1}{\cos(0)} + C_2 \overset{=0}{\sin(0)} = 0$

$\Rightarrow \boxed{C_1 = 0}$

$\Rightarrow \; y(x) = C_2 \sin(\sqrt{2}\,x)$

$y(\pi) = 0$

$y(\pi) = C_2 \overset{\neq 0}{\sin(\sqrt{2}\,\pi)} = 0$

$\Rightarrow \boxed{C_2 = 0}$

Only Soln is $\boxed{y(x) \equiv 0}$

Conclusion:

$\boxed{\lambda = 2}$ is not an eigenvalue of

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0, \quad y(\pi) = 0 \end{cases}$$

$\underline{\text{Ex. 2}}$ $(A-4I)\vec{x} = \vec{0}$

$$\left[\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x_1 = x_2$, or $\vec{x} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix}$

or $\boxed{\vec{x} = \underbrace{x^2}_{\text{arbitrary}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

Infinitely many solns.

$\underline{\text{Conclusion:}}$

$\lambda_1 = 4$ is an eigenvalue of A

and $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a corresponding eigenvector.

$\underline{\text{Ex 2.}}$ $y'' + y = 0$, $y(0) = 0$, $y(\pi) = 0$

$r^2 + 1 = 0 \Rightarrow r = \pm i$

$\underline{\text{Gen. soln:}}$ $y(x) = C_1 \cos x + C_2 \sin x$

$\underline{\text{Using BCs}}$

$y(0) = 0$

$y(0) = C_1 \overset{=1}{\cos 0} + C_2 \overset{=0}{\sin 0} = 0$

$\Rightarrow \boxed{C_1 = 0}$

$y(x) = C_2 \sin x$

$y(\pi) = 0$

$y(\pi) = C_2 \overset{=0}{\sin \pi} = 0$

$\Rightarrow \boxed{C_2 \text{ is arbitrary}}$

$\Rightarrow \boxed{y(x) = C_2 \sin x}$

Infinitely many solns.

$\underline{\text{Conclusion:}}$ $\boxed{\lambda_1 = 1}$ is an eigenvalue

and $\boxed{y_1(x) = \sin x}$ is a corresp. eigenfunction.