10.2 Fourier Series. 10.3 Fourier Convergence Theorem.

$$\frac{a_0}{2} + \sum a_m \cos \frac{m\pi}{L} x + b_m \sin \frac{m\pi}{L} x \quad (4.1)$$

Questions:

1) Is this series converging for some x.
2) Can we use series (4.1) to represent some functions?
   (Similar to Taylor's series).
3) If (2) is true for certain $f(x)$, How do we find the coefficients a_m, b_m, a_0, $m = 1, 2, \dots$?

Periodicity: T is a period of f if

$$f(x+T) = f(x), \quad \text{for every } x.$$

Notice that $f(x+2T) = f((x+T)+T) = f(x+T) = f(x)$.

So $2T$ is also a period.

The smallest T such that $f(x+T) = f(x)$ is called the fundamental period of f.

The functions $\sin\left(\frac{m\pi}{L}x\right)$ and $\cos\left(\frac{m\pi}{L}x\right)$

$m = 1, 2, \ldots$

are periodic.

In fact, $\sin x$ and $\cos x$ are periodic with period 2π.

Also, $\sin(\alpha x)$ and $\cos(\alpha x)$ are periodic

To find this period: $\alpha x = 2\pi \Rightarrow x = \frac{2\pi}{\alpha}$

or $T = \frac{2\pi}{\alpha}$.

In particular,

for $\sin\left(\frac{m\pi}{L}x\right)$ and $\cos\left(\frac{m\pi}{L}x\right)$

$$\frac{m\pi}{L}x = 2\pi \Rightarrow \boxed{x = \frac{2L}{m} = T} \text{ period.}$$

Orthogonality:

Consider u, v two real valued functions defined in (α, β).

Inner Product:

$$(u, v) = \int_\alpha^\beta u(x)\, v(x)\, dx.$$

Recall inner product of two vectors:

$$\vec{u} = (u_1, u_2, u_3), \quad \vec{v} = (v_1, v_2, v_3)$$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

We say $\vec{u}$ and $\vec{v}$ are orthogonal if

$$\vec{u} \cdot \vec{v} = 0.$$

Similarly, u, v are orthogonal in (α, β) if

$$\int_\alpha^\beta u(x) v(x)\, dx = 0.$$

The Set of trigometric function

$$\begin{cases} \sin \frac{m\pi}{L} x, \\ \cos \frac{m\pi}{L} x, \end{cases} \quad m = 1, 2, \dots$$

form a mutually orthogonal set of functions. in the interval $[-L, L]$.

In fact,

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = 0.$$

<u>Proof</u>:-

$$\cos(\alpha+\beta)x = \cos\alpha x \cos\beta x - \sin\alpha x \sin\beta x$$

$$\cos(\alpha-\beta)x = \cos\alpha x \cos\beta x + \sin\alpha x \sin\beta x$$

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then $\cos(\alpha-\beta)x - \cos(\alpha+\beta)x = 2\sin\alpha x \sin\beta x.$

$$\Rightarrow \quad \sin\alpha x \sin\beta x = \frac{1}{2}\left[\cos(\alpha-\beta)x - \cos(\alpha+\beta)x\right]$$

Thus,

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = \frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(m-n)\pi}{L}x - \cos\left(\frac{(m+n)\pi}{L}x\right)\right] dx$$

$$= \frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(m-n)\pi}{L}x - \cos\frac{(m+n)\pi}{L}x\right] dx$$

$$= \frac{1}{2}\left[\frac{L}{\pi(m-n)} \sin \frac{(m-n)\pi}{L} x - \frac{L}{\pi(m-n)} \sin\left(\frac{(m+n)\pi}{L} x\right)\right]\Bigg|_{-L}^{L}$$

$$\frac{1}{m-n}\cdot\frac{1}{2}\frac{L}{\pi}\Bigg[\sin\left[\frac{(m-n)\pi}{\cancel{L}}\right]\cdot \sin\left[\frac{(m+n)\pi}{\cancel{L}}\cancel{L}\right]$$

$$- \sin\left[\frac{(m-n)\pi}{\cancel{L}}(-\cancel{L})\right]\cdot \sin\left[\frac{(m+n)\pi}{\cancel{L}}(-\cancel{L})\right]\Bigg]$$

$$= 0, \qquad \text{if } m \neq \pm n.$$

Similarly, we can easily prove that

$$\int_{-L}^{L} \cos\frac{m\pi}{L}x \cos\frac{n\pi}{L}x\, dx = \begin{cases} 0, & m \neq n \\ L, & m = n. \end{cases}$$

$$\int_{-L}^{L} \cos\frac{m\pi}{L}x \sin\frac{n\pi}{L}x\, dx = 0, \qquad \text{for any } m \text{ and } n$$

$$\int_{-L}^{L} \sin\frac{m\pi}{L}x \sin\frac{n\pi}{L}x\, dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$

Euler - Fourier formula

Assume

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \frac{m\pi}{L}x + b_m \sin \frac{m\pi}{L}x$$

$$\overset{\text{converges}}{=} f(x), \quad \text{on } [-L,L]. \qquad (9.1)$$

The coefficients a_m and b_m can be obtained in terms of $f(x)$ as follows:

i) Multiply (9.1) by $\cos(\frac{n\pi}{L}x)$, n fixed, $n>0$

ii) Assuming that we can integrate term by term.

$$\int_{-L}^{L} f(x) \cos \frac{n\pi}{L}x \, dx = \frac{a_0}{2}\int_{-L}^{L} \cos \frac{n\pi}{L}x \, dx$$

$$+ \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \frac{m\pi}{L}x \cos \frac{n\pi}{L}x \, dx$$

$$+ \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{m\pi}{L}x \cos \frac{n\pi}{L}x \, dx$$

or $\qquad$ n fixed and $n>0$

$$\int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx = \frac{a_0}{2} \int_{-L}^{L} \cos \frac{n\pi}{L} x \, dx$$

$$+ a_1 \int_{-L}^{L} \cos \frac{\pi}{L} x \cos \frac{n\pi}{L} x \, dx + a_2 \int_{-L}^{L} \cos \frac{2\pi}{L} x \cos \frac{n\pi}{L} x \, dx$$

$$+ \cdots + a_n \int_{-L}^{L} \cos \frac{n\pi}{L} x \cos \frac{n\pi}{L} x \, dx + \cdots$$

$$+ b_1 \int_{-L}^{L} \sin \frac{\pi}{L} x \cos \frac{n\pi}{L} x \, dx + b_2 \int_{-L}^{L} \sin \frac{2\pi}{L} x \cos \frac{n\pi}{L} x \, dx$$

$$+ \cdots + b_n \int_{-L}^{L} \sin \frac{n\pi}{L} x \cos \frac{n\pi}{L} x \, dx + \cdots$$

$$= 0 + \cdots + 0 + 0 + \cdots + a_n L + \cdots + 0 + 0 + 0 + \cdots$$

$$= a_n L$$

Therefore,

$$\boxed{a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx} \quad (10.1)$$

Similarly,

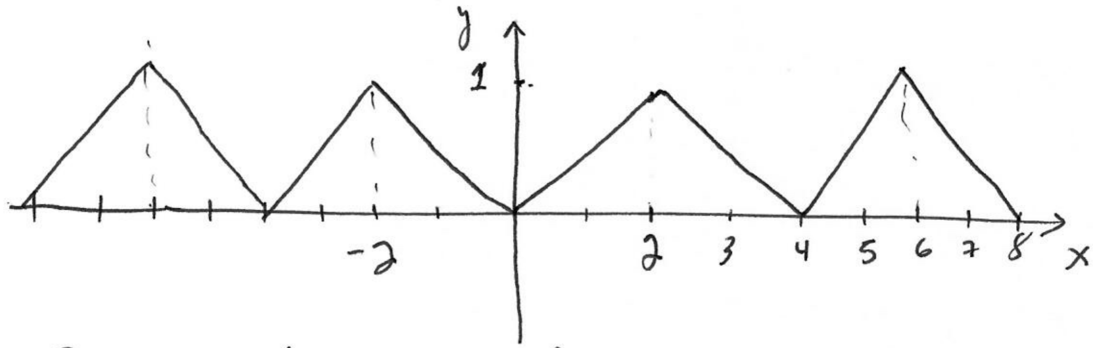
$$\boxed{a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx, \quad (10.2)} \qquad \boxed{b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx \quad (10.3)}$$

Example 1

Assume there is a Fourier Series Converging to the function

$$f(x) = \begin{cases} -x, & -2 \le x < 0 \\ x, & 0 \le x \le 2 \end{cases}$$

$$f(x+4) = f(x)$$



Periodic function with period 4

So the interval $[-L, L] = [-2, 2]$, $\underline{L = 2}$.

The Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \frac{m\pi}{2} x + b_m \sin \frac{m\pi}{2} x$$

we can compute the coefficients using (10.1) – (10.3)

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\,dx = \frac{1}{2} \int_{-2}^{0} (-x)\,dx + \frac{1}{2} \int_{0}^{2} x\,dx$$

$$= \frac{-1}{2} \frac{x^2}{2}\Big|_{-2}^{0} + \frac{1}{2} \frac{x^2}{2}\Big|_{0}^{2} = -\frac{1}{2}(-2) + \frac{1}{2}(2) = 2.$$

for $m \geq 1$

$$a_m = \frac{1}{2}\int_{-2}^{0} -x \cos\frac{m\pi}{2}x\,dx + \frac{1}{2}\int_{0}^{2} x \cos\frac{m\pi}{2}x\,dx$$

$\alpha = \frac{m\pi}{2}$

$\frac{1}{\alpha} = \frac{2}{m\pi}$

$$= \frac{1}{2}\left[\frac{2}{m\pi}(-x)\sin\frac{m\pi}{2}x - \left(\frac{2}{m\pi}\right)^2 \cos\frac{m\pi}{2}x\right]\Bigg|_{-2}^{0}$$

$$+ \frac{1}{2}\left[\frac{2}{m\pi}x\sin\frac{m\pi}{2}x + \left(\frac{2}{m\pi}\right)^2 \cos\frac{m\pi}{2}x\right]\Bigg|_{0}^{2}$$

$$= \frac{1}{2}\left[-\left(\frac{2}{m\pi}\right)^2 + \left(\frac{2}{m\pi}\right)^2 \cos m\pi\right] + \frac{1}{2}\left[\left(\frac{2}{m\pi}\right)^2 \cos m\pi - \left(\frac{2}{m\pi}\right)^2\right]$$

$$= -\left(\frac{2}{m\pi}\right)^2 + \left(\frac{2}{m\pi}\right)^2 \cos m\pi = \frac{4}{(m\pi)^2}\left[\cos m\pi - 1\right]$$

$$= \begin{cases} -\dfrac{8}{(m\pi)^2}, & m \text{ odd} \\ 0, & m \text{ even.} \end{cases}$$

<u>Remark:</u>

$$\int \underset{\uparrow f}{x}\, \underset{\uparrow g'}{\sin\alpha x}\,dx = -x\,\frac{\cos\alpha x}{\alpha} + \int \frac{\cos\alpha x}{\alpha}\,dx = -\frac{x}{\alpha}\cos\alpha x + \frac{\sin\alpha x}{\alpha^2}$$

$$\int \underset{\uparrow f}{x}\, \underset{\uparrow g'}{\cos\alpha x}\,dx = x\,\frac{1}{\alpha}\sin\alpha x - \int \frac{\sin\alpha x}{\alpha}\,dx = \frac{x}{\alpha}\sin\alpha x + \frac{\cos\alpha x}{\alpha^2}$$

Using (10.3), we arrive to

$$b_m = 0, \qquad m = 1, 2, \ldots$$

Therefore,

$$f(x) = \frac{2}{2} - \frac{8}{\pi^2} \cos \frac{\pi}{2} x \quad \frac{8}{3^2 \pi^2} \cos \frac{3\pi}{2} x$$

$$- \frac{8}{5^2 \pi^2} \cos \frac{5\pi}{2} x + \cdots$$

$$= 1 - \frac{8}{\pi^2} \left[\cos \frac{\pi x}{2} + \frac{1}{3^2} \cos \frac{3\pi}{2} x + \cdots \right]$$

$$= 1 - \frac{8}{\pi^2} \sum_{m=1,3,5\ldots} \frac{1}{m^2} \cos \frac{m\pi}{2} x$$

$$= 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \cos \frac{(2n-1)x}{2} \left(\frac{1}{(2n-1)^2} \right)$$

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Consider again the function in Example 1 and its Fourier series (20). Investigate the speed with which the series converges. In particular, determine how many terms are needed so that the error is no greater than 0.01 for all x.

The mth partial sum in this series

$$s_m(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{m} \frac{\cos(2n-1)\pi x/2}{(2n-1)^2} \tag{27}$$

can be used to approximate the function f. The coefficients diminish as $(2n-1)^{-2}$, so the series converges fairly rapidly. This is borne out by Figure 10.2.4, where the partial sums for $m = 1$ and $m = 2$ are plotted. To investigate the convergence in more detail, we can consider the error $e_m(x) = f(x) - s_m(x)$. Figure 10.2.5 shows a plot of $|e_6(x)|$ versus x for $0 \le x \le 2$. Observe that $|e_6(x)|$ is greatest at the points $x = 0$ and $x = 2$, where the graph of $f(x)$ has corners. It is more difficult for the series to approximate the function near these points, resulting in a larger error there for a given m. Similar graphs are obtained for other values of m.

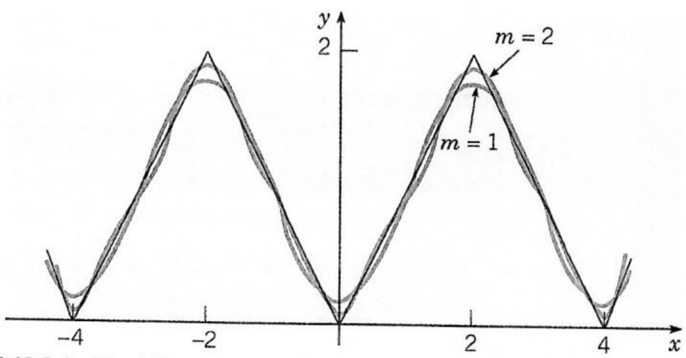

FIGURE 10.2.4 Partial sums in the Fourier series, Eq. (20), for the triangular wave.

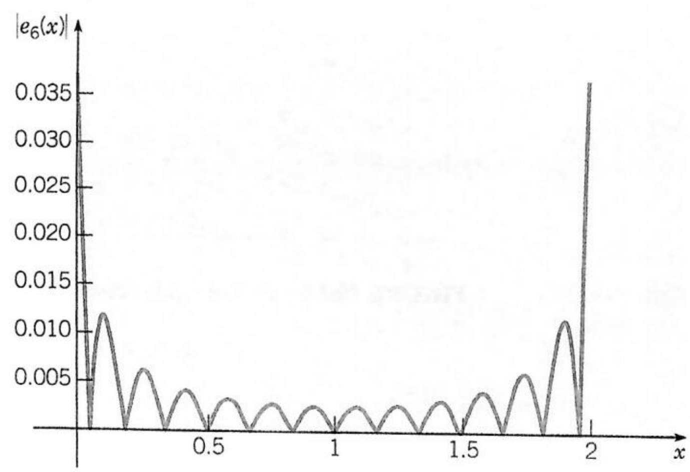

FIGURE 10.2.5 Plot of $|e_6(x)|$ versus x for the triangular wave.

Once you realize that the maximum error always occurs at $x = 0$ or $x = 2$, you can obtain a *uniform* error bound for each m simply by evaluating $|e_m(x)|$ at one of these points. For example, for $m = 6$ we have $e_6(2) = 0.03370$, so $|e_6(x)| < 0.034$ for $0 \le x \le 2$ and consequently of x and, if so, whether its sum is $f(x)$. Examples have been discovered showing that the Fourier series corresponding to a function f may not converge to $f(x)$ and may even diverge. Functions whose Fourier series do not converge to the value of the function at isolated points are easily constructed, and examples will be presented later in this section. Functions whose Fourier series diverge at one or more points are more pathological, and we will not consider them in this book.

To guarantee convergence of a Fourier series to the function from which its coefficients were computed, it is essential to place additional conditions on the function. From a practical point of view, such conditions should be broad enough to cover all situations of interest, yet simple enough to be easily checked for particular functions. Through the years, several sets of conditions have been devised to serve this purpose.

Before stating a convergence theorem for Fourier series, we define a term that appears in the theorem. A function f is said to be **piecewise continuous** on an interval $a \le x \le b$ if the interval can be partitioned by a finite number of points $a = x_0 < x_1 < \cdots < x_n = b$ so that

1. f is continuous on each open subinterval $x_{i-1} < x < x_i$.
2. f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

The graph of a piecewise continuous function is shown in Figure 10.3.1.

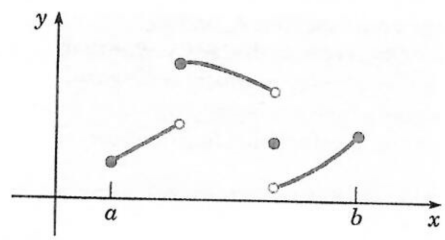

FIGURE 10.3.1 A piecewise continuous function.

The notation $f(c+)$ is used to denote the limit of $f(x)$ as $x \to c$ from the right; similarly, $f(c-)$ denotes the limit of $f(x)$ as x approaches c from the left.

Note that it is not essential that the function even be defined at the partition points x_i. For example, in the following theorem we assume that f' is piecewise continuous; but certainly f' does not exist at those points where f itself is discontinuous. It is also not essential that the interval be closed; it may also be open, or open at one end and closed at the other.

Theorem 10.3.1 Suppose that f and f' are piecewise continuous on the interval $-L \le x < L$. Further, suppose that f is defined outside the interval $-L \le x < L$ so that it is periodic with period $2L$. Then f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right), \tag{4}$$

whose coefficients are given by Eqs. (2) and (3). The Fourier series converges to $f(x)$ at all points where f is continuous, and it converges to $[f(x+) + f(x-)]/2$ at all points where f is discontinuous.

value. Instead, they tend to overshoot the mark at each end of the jump, as though they cannot quite accommodate themselves to the sharp turn required at this point. This behavior is typical of Fourier series at points of discontinuity and is known as the Gibbs[5] phenomenon.

Additional insight is attained by considering the error $e_n(x) = f(x) - s_n(x)$. Figure 10.3.4 shows a plot of $|e_n(x)|$ versus x for $n = 8$ and for $L = 1$. The least upper bound of $|e_8(x)|$ is 0.5 and is approached as $x \to 0$ and as $x \to 1$. As n increases, the error decreases in the interior of the interval [where $f(x)$ is continuous], but the least upper bound does not diminish with increasing n. Thus we cannot uniformly reduce the error throughout the interval by increasing the number of terms.

Figures 10.3.3 and 10.3.4 also show that the series in this example converges more slowly than the one in Example 1 in Section 10.2. This is due to the fact that the coefficients in the series (6) are proportional only to $1/(2n-1)$.

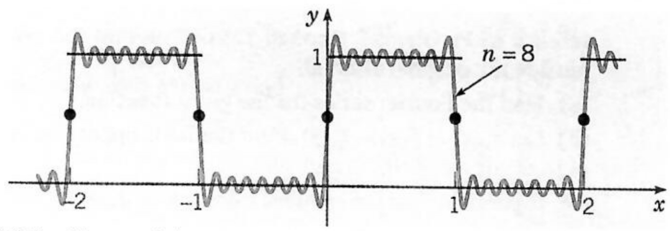

FIGURE 10.3.3 The partial sum $s_8(x)$ in the Fourier series, Eq. (6), for the square wave.

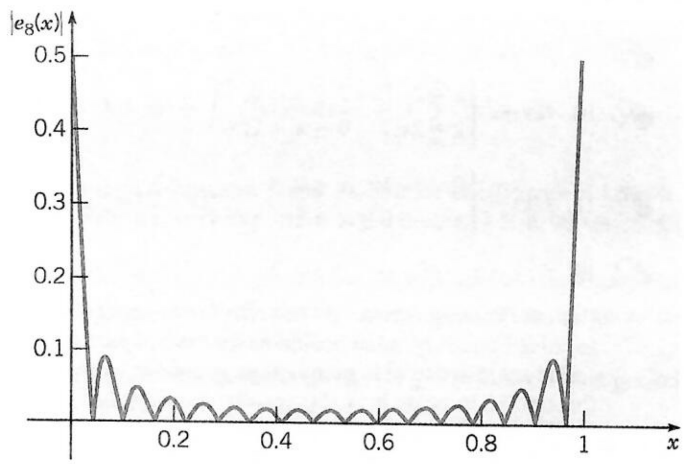

FIGURE 10.3.4 A plot of the error $|e_8(x)|$ versus x for the square wave.

[5]The Gibbs phenomenon is named after Josiah Willard Gibbs (1839–1903), who is better known for his work on vector analysis and statistical mechanics. Gibbs was professor of mathematical physics at Yale and one of the first American scientists to achieve an international reputation. The Gibbs phenomenon is discussed in more detail by Carslaw (Chapter 9).