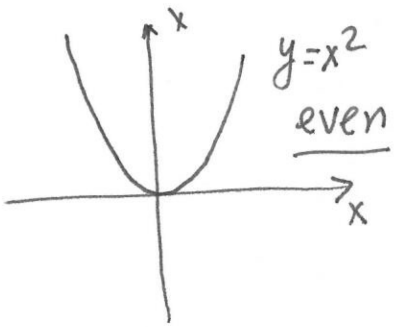
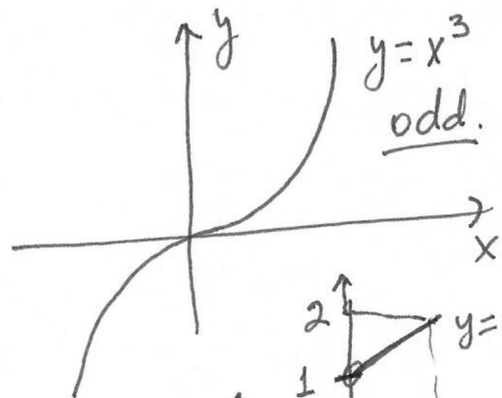
10.4 Even and odd functions.

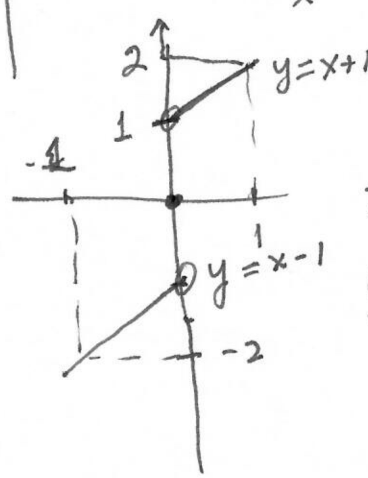
Definitions.

even function: $f(-x) = f(x)$

odd function: $f(-x) = -f(x)$







should be $f(0)=0$

odd

Important properties:

1) Sum and product of even fns. is an even fn.

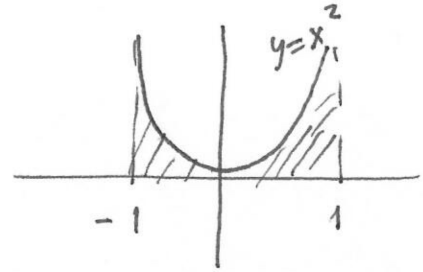
2) Sum of odds is odd
product of odds is even.

3) Sum of odd and even ?
product of odd an even = odd.

Very easy to prove.

4) If $f(x)$ is even in $[-L, L]$.

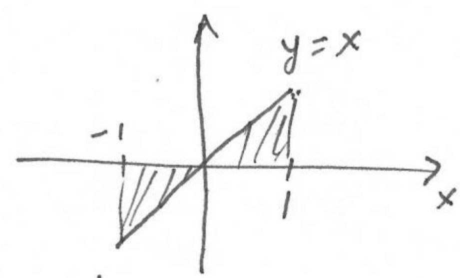
$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$



$$\int_{-1}^{1} x^2\,dx = 2\int_{0}^{1} x^2\,dx.$$

5) If $f(x)$ is odd in $[-L, L]$

$$\int_{-L}^{L} f(x)\,dx = 0$$

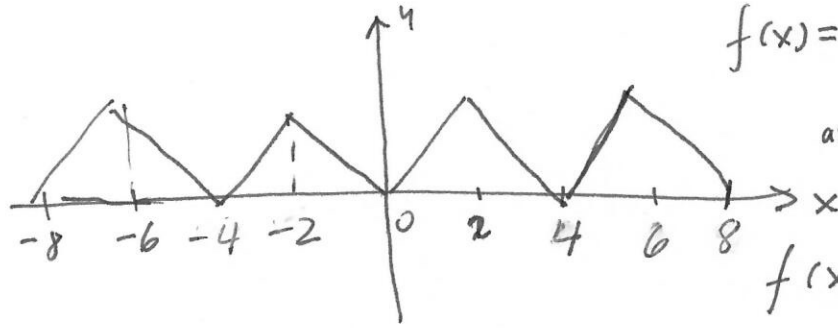


$$\int_{-1}^{1} x\,dx = 0.$$

Cosine Series

f, f' p.w.c. on $[-L, L]$

and f is an even periodic of period $2L$.



$$f(x) = \begin{cases} x, & 0 \le x \le 2 \\ -x, & -2 \le x \le 0 \end{cases}$$

and $f(x) = f(x+4)$

$f(x) = f(x+4)$

$2L = 4 \Rightarrow L = 2$

then $f(x) \sin \frac{n\pi}{2} x$ is odd

in $[-L, L]$.

$f(x) \cos \frac{n\pi}{2} x$ is even

Then
$$\int_{-L}^{L} f(x) \sin \frac{n\pi}{2} x \, dx = 0. \Rightarrow b_n = 0, \quad n = 1, 2 \ldots$$

and $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx$

$= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, dx$

and the Fourier Series reduces to

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{2} x}$$

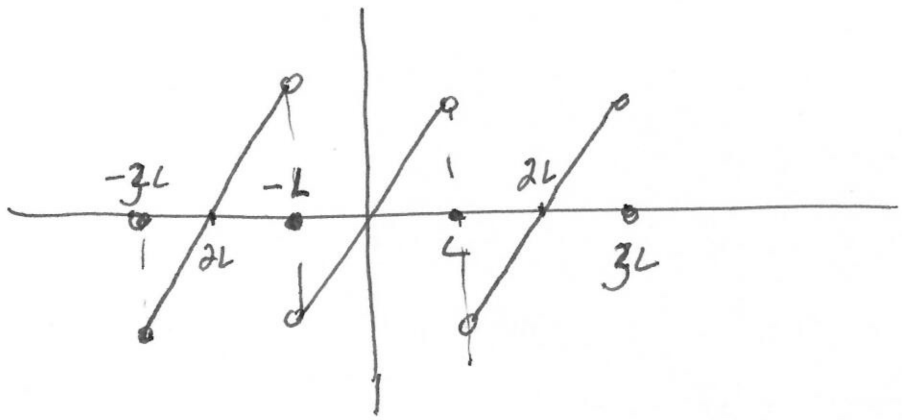
After evaluating the coefficients a_n, we found

$$f(x) = 1 - \frac{8}{\pi^2} \left(\cos \frac{\pi}{2} x + \frac{1}{3^2} \cos \frac{3\pi}{2} x + \frac{1}{5^2} \sin \frac{5\pi}{2} x + \ldots \right)$$

$$= 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(n-1)^2} \cos \frac{(2n-1)\pi}{2} x$$

Sine Series

$$\begin{cases} f(x) = x, & -L \le x \le L \\ f(x) = f(x+2L). \end{cases}$$



$f(x)$ is odd on $[-L, L]$.

Then,
$$\int_{-L}^{L} \underbrace{f(x) \cos \frac{n\pi}{L} x}_{\text{odd}} \, dx = 0 \Rightarrow a_n = 0$$

and
$$b_n = \frac{1}{L} \int_{-L}^{L} \underbrace{f(x) \sin \frac{n\pi}{L} x}_{\text{even}} \, dx = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

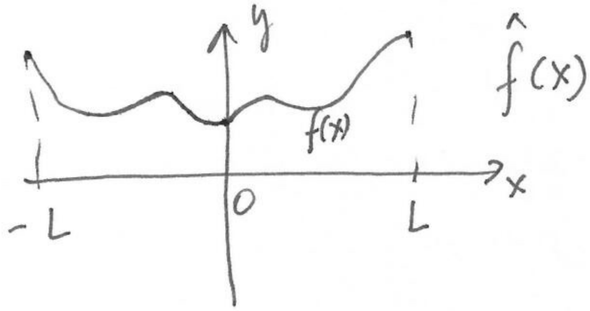
In our case,
$$b_n = \frac{2}{L} \int x \sin \frac{n\pi}{L} x \, dx = \frac{2L}{n\pi} (-1)^{n+1}, \quad n = 1, 2 \ldots$$

and
$$\boxed{f(x) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{L} x.}$$

If $f(x)$ is defined on $[0,L]$ only

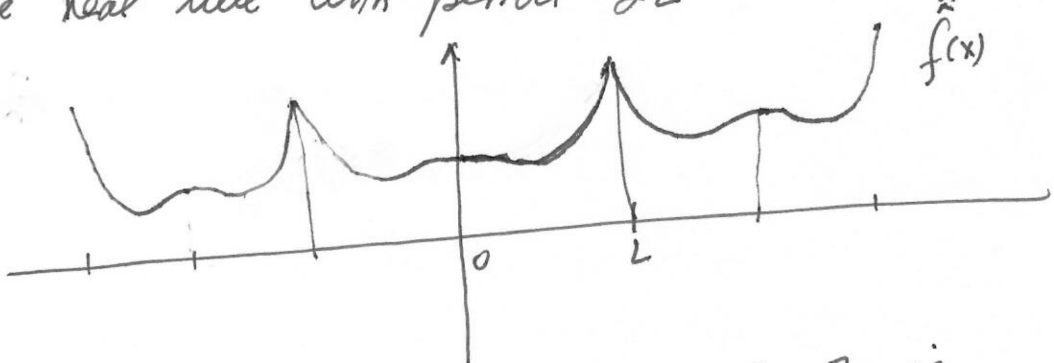
(1) then, this function can be extended as an even function to the interval $[-L,L]$. In fact

$$\hat{f}(x) = \begin{cases} f(x), & [0,L] \\ f(-x), & x \in [-L,0). \end{cases}$$



We can do even more, extend $\hat{f}$ periodically to the real line with period $2L$

This is Called "Even periodic extension of $f(x)$"



Cosine Fourier Series representing $f(x)$ on $[0,L]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx$$