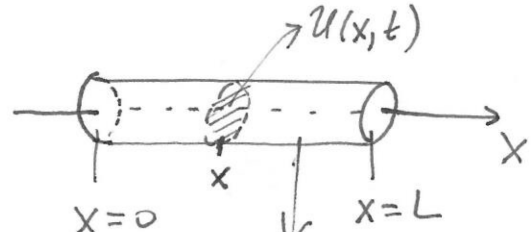
10.5 Separation of Variables. Heat Conduction.



insulated sides.

$u(x,t)$: Temperature at cross section x at time t.

Heat conduction equation:

$$\alpha^2 u_{xx} = u_t, \quad 0<x<L, \; t>0$$

α^2: thermal diffusivity (depends on the material)

$$\alpha^2 = \kappa/\rho s$$

κ: Thermal conductivity, ρ: density, s: Specific heat.

Initial temperature distribution:

$$u(x,0) = f(x), \qquad 0 \le x \le L.$$

Boundary conditions:

$$u(0,t) = T_1, \qquad u(L,t) = T_2, \quad t>0.$$

We will assume $T_1 = T_2 = 0$

So $u(0,t) = 0, \quad u(L,t) = 0., \quad t>0.$

Initial boundary Value Problem (IVBVP) or (IBVP).

$$\begin{cases} u_t = \alpha^2 u_{xx}, & 0<x<L,\ t>0 \quad (1) \\ u(0,t) = T_1,\ u(L,t) = T_2, & t>0. \quad (2) \\ u(x,0) = f(x), & 0 \le x \le L \quad (3) \end{cases}$$

Linear problem.

Strategy to find solns:

① Find Solns. of the BVP (1)-(2)

② Superimpose these conditions to satisfy the ICs.

Separation of Variables:

Assumption: $u(x,t) = X(x)T(t)$

Subst. into (1):

$$X(x)\,T'(t) = \alpha^2 X''(x)\,T(t)$$

Then,

$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

Since x and t are two independent variables

$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \text{(constant)}.$$

From where

$$\begin{cases} T'(t) + \alpha^2 \lambda T(t) = 0 & (3.1) \\ X''(x) + \lambda X(x) = 0 & (3.2) \end{cases}$$

These two ODEs can be solved for any value of λ

However, we want solns. that also satisfy the BCs.

and the initial condition.

To satisfy B.Cs.

$$u(0,t) = 0 \Leftrightarrow X(0)T(t) = 0, \quad t>0$$

Then $X(0) = 0$ or $T(t) = 0$, for all $t>0$

To avoid trivial solns of (1)

$$\boxed{X(0) = 0}$$

Similarly,

$$u(L,t) = X(L)T(t) = 0, \quad t>0$$

Then, $\boxed{X(L) = 0}$

Therefore, $X(x)$ Satisfies the BVP:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, \quad X(L) = 0 \end{cases}$$

This is <u>an eigenvalue problem</u> that we have Solved before

We found that there are infinitely many solns.

$$\underbrace{X_n(x) = \sin\left(\frac{n\pi}{L}x\right)}_{\text{eigenfunctions}}, \quad n = 1, 2, \ldots \quad (4.1)$$

Corresponding to

$$\underbrace{\lambda_n = \frac{n^2\pi^2}{L^2}}_{\text{eigenvalues.}}, \quad n = 1, 2, \ldots$$

Subst. into (3.1) leads to

$$T'(t) + \left(n^2\pi^2\alpha^2/L^2\right)T = 0.$$

With solns.

$$T_n(t) = C_n \, e^{-\left(n^2\pi^2\alpha^2/L^2\right)t} \quad (4.2)$$

Then, Solns. of (1) can be obtained as a product of (4.1) and (4.2)

$$\boxed{u_n(x,t) = e^{-(n^2\pi^2\alpha^2/L^2)t} \sin\left(\frac{n\pi}{L}x\right)} \quad (5.1)$$

$n=1,2,\ldots$

These solns. also satisfied the B.C.s..

$u(0,t) = 0, \quad u(L,t) = 0, \quad n=1,2,\ldots$

However, the I.C.: $u(x,0) = f(x)$

is not satisfied unless

$$f(x) = \sin\frac{n\pi}{L}x..$$

The partial solns. (5.1) can be considered as a fundamental set of solns., so a linear combination of them with unknown coefficients is also a solution. If we consider an "infinite linear combination" assuming that the series converges

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n^2\pi^2\alpha^2/L^2)t} \sin\left(\frac{n\pi}{L}x\right)$$

and that satisfies (1)-(2).

Then, the only condition not satisfied yet is the I. C. $u(x,0) = f(x)$

To satisfy it, the following equ. should hold

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) = f(x). \quad (6.1)$$

Then, $u(x,0)$ is the Fourier sine series of $f(x)$. Then, if $f(x)$ is p.w.c., we can find coefficients C_n such that the series converges to $f(x)$. More precisely,

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-(n^2\pi^2\alpha^2/L^2)t} \sin\left(\frac{n\pi}{L}x\right) \quad (6.2)$$

where

$$C_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L}x \, dx \qquad (6.3)$$

is the soln. of our original problem (1)-(3) if the series (6.1) converges.

Initial Boundary Value Problem.

$$u_t = u_{xx}, \qquad 0<x<1, \; t>0 \qquad (1.1)$$

$$u(0,t) = 0, \qquad u(L,t) = 0 \qquad (1.2)$$

$$u(x,0) = f(x) = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2(1-x), & 1/2 \le x \le 1 \end{cases} \qquad (1.3)$$

$$\boxed{u(x,t) = X(x)\,T(t)} \qquad (1.4)$$

Separation of Variables

Substitution into (1.1)

$$X(x)\,T'(t) = X''(x)\,T(t)$$

$$\Rightarrow \quad \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}, \quad \text{for all } x \text{ and } t$$

x and t are independent Variables.

Then,

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} \equiv -\lambda$$

$\Rightarrow$ Two ODEs are obtained:

$$\boxed{T'(t) + \lambda T(t) = 0} \qquad (1.5)$$

$$\boxed{X''(x) + \lambda X(x) = 0} \qquad (1.6)$$

To satisfy BCs (1.2)

$$u(0,t) = 0 \Leftrightarrow X(0)\,T(t) = 0, \quad \text{for all } t.$$

$\Rightarrow$ $\boxed{X(0) = 0}$ to avoid trivial solns.

Similarly,

$$u(L,t) = 0 \Leftrightarrow X(L)\,T(t) = 0, \quad \text{for all } t.$$

$$\Rightarrow \boxed{X(L) = 0}$$

As a result, we have obtained the following <u>eigenvalue problem</u> for $X(x)$.

$$\begin{cases} X''(x) + \lambda X(x) = 0 & (2.1) \\ X(0) = 0, \quad X(L) = 0 & (2.2) \end{cases}$$

<u>Soln</u>: $r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{\lambda}\, i$

$\lambda > 0$
only possibility
for nontrivial soln.

$$\boxed{X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x}$$

$$0 = X(0) = C_1 \Rightarrow \boxed{C_1 = 0}$$

$$X(x) = C_2 \sin\sqrt{\lambda}\, x$$

$$0 = X(L) = \overset{\neq 0}{C_2} \sin\sqrt{\lambda} L \Rightarrow \sin\sqrt{\lambda} L = 0$$

$$\Rightarrow \sqrt{\lambda} L = n\pi, \quad n = 1, 2 \ldots$$

$$\Rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2 \ldots}$$ .. Eigenvalues. (2.3)

$$\boxed{X_n(x) = \sin \frac{n\pi}{L} x, \quad n = 1, 2, \ldots} \tag{3.1}$$

eigenfunctions.

On the other hand,

$$T'(t) + \left(\frac{n\pi}{L}\right)^2 T(t) = 0$$

or

$$T'(t) = -\left(\frac{n\pi}{L}\right)^2 T(t)$$

$$\Rightarrow \boxed{T_n(t) = C_n \, e^{-\left(\frac{n\pi}{L}\right)^2 t}} \tag{3.2}$$

<u>Product soln:</u>

$$u_n(x,t) = T_n(t)\, X_n(t)$$

$$\boxed{u_n(x,t) = e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin \frac{n\pi}{L} x} \tag{3.3}$$

<u>Principle of Superposition:</u>

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin \frac{n\pi}{L} x} \tag{3.4}$$

To Satisfy I.C. (1.3):

$u(x,0) = f(x)$ is equivalent to

$$\sum_{n=1}^{\infty} c_n \overbrace{e^{0}}^{=1} \sin \frac{n\pi}{L} x = f(x).$$

Fourier Sine Series of $f(x)$.

$$\sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{L} x = f(x)$$

(Make a stop here and explain even and odd extension again.)

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

In our case, $L = 1$ and $f(x) = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2(1-x), & 1/2 \le x \le 1 \end{cases}$

Then

$$c_n = 2 \int_0^{1/2} 2x \sin n\pi x \, dx + 2 \int_{1/2}^{1} 2(1-x) \sin n\pi x \, dx$$

or $$c_n = 4 \left[-\frac{x}{n\pi} \cos n\pi x + \frac{\sin n\pi x}{(n\pi)^2} \right]_0^{1/2} + \cdots$$

$$\boxed{\begin{aligned} C_N &= 4\left[-\frac{x}{n\pi}\cos(n\pi x)+\frac{\sin n\pi x}{(n\pi)^2}\Bigg|_0^{1/2}\right] \\ &+4\int_{1/2}^{1}\sin n\pi x-2\int_{1/2}^{1}x\sin(n\pi x)\,dx \end{aligned}}$$

$$= 4\left[-\frac{1}{2n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{\sin n\pi/2}{(n\pi)^2}-0\right]$$

$$+\ 4\frac{1}{n\pi}(-\cos n\pi x)\Bigg|_{1/2}^{1}-4\left[-\frac{x}{n\pi}\cos n\pi x+\frac{\sin n\pi x}{(n\pi)^2}\Bigg|_{1/2}^{1}\right]$$

$$= 4\left[-\frac{1}{2n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{\sin(n\pi/2)}{(n\pi)^2}\right]$$

$$+4\left[\frac{-\cos(n\pi)}{n\pi}+\frac{\cos\left(\frac{n\pi}{2}\right)}{n\pi}\right]$$

$$-4\left[\frac{-\cos(n\pi)}{n\pi}+\frac{\cancel{\sin n\pi}^{\,0}}{(n\pi)^2}+\frac{1/2}{n\pi}\cos\left(\frac{n\pi}{2}\right)-\frac{\sin(n\pi/2)}{(n\pi)^2}\right]$$

$$= \cos\left(\frac{n\pi}{2}\right)\left[\overbrace{-\frac{2}{n\pi}+\frac{4}{n\pi}-\frac{2}{n\pi}}^{=0}\right]$$

$$+\ \sin(n\pi/2)\left[\underbrace{\frac{4}{(n\pi)^2}+\frac{4}{(n\pi)^2}}_{=8/(n\pi)^2}\right]+\cos(n\pi)\left[\overbrace{-\frac{4}{n\pi}+\frac{4}{n\pi}}^{=0}\right]$$

$$= \frac{8}{(n\pi)^2}\sin(n\pi/2). \qquad (5.1)$$

Finally, Substituting (5.1) into (3.4).

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}{}' \frac{8}{(n\pi)^2} \sin(n\pi/2) \sin\left(\frac{n\pi}{L}x\right)} \quad (6.1).$$

Show Animation as $t \nearrow$ in MAPLE.