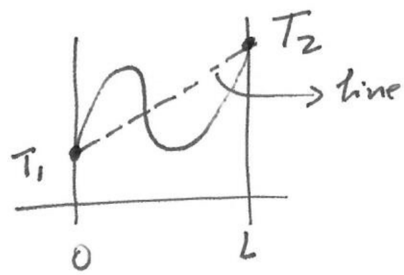
10.6 Other Heat Conduction Problems.

A) Nonhomogeneous BCs.

$$\begin{cases} u_t = \alpha^2 u_{xx}, & 0<x<L, \quad t>0. & (1.1) \\ u(0,t) = T_1, \quad u(L,t) = T_2, & t>0 & (1.2) \\ u(x,0) = f(x), & 0 \le x \le L & (1.3) \end{cases}$$

Physical expectation:

$$u(x,t) \longrightarrow v(x) =$$



$$= \frac{T_2 - T_1}{L} x + T_1$$

Proof.

Since $v(x)$ is indep. of time

$$\begin{cases} v_{xx} = 0 \\ v(0) = T_1, \quad v(L) = T_2 \end{cases}$$

Then
$$v(x) = C_1 x + C_2$$
$$T_1 = v(0) = C_2 \Rightarrow \boxed{C_2 = T_1}$$
$$T_2 = v(L) = C_1 L + T_1 \Rightarrow \boxed{C_1 = \frac{T_2 - T_1}{L}}$$

then,
$$\boxed{v(x) = \frac{T_2 - T_1}{L} x + T_1}$$

Strategy to Solve (1.1) - (1.3).

Use linearity of IBVP and decompose the Soln. $u(x,t)$ into a Sum of two functions one of them $v(x)$ and the other one $w(x)$ Satisfying homogeneous BCs.
It means the original nonhomogeneous problem is reduced to a homogeneous IBVP. that we have already solved.

So define, $\quad w(x,t) = u(x,t) - v(x)$

then $w(x)$ Satisfies

$$w_t = u_t - \cancel{v_t}^{\,0} = \alpha^2 u_{xx} = \alpha^2 w_{xx}$$

because $\quad w_{xx} = u_{xx} - \cancel{v_{xx}}^{\,0} = u_{xx}.$

Therefore, $w(x,t)$ Satisfies the following IBVP:

$$\begin{cases} w_t = \alpha^2 w_{xx} \\ w(0,t) = u(0,t) - v(0) = T_1 - T_1 = 0 \\ w(L,t) = u(L,t) - v(L) = T_2 - T_2 = 0 \\ w(x,0) = u(x,0) - v(x) = f(x) - \left(\frac{T_2 - T_1}{L} x + T_1\right) \end{cases}$$

10.6 #9) Aluminum rod $\Rightarrow$ $\alpha^2 = 0.86$

(1) $u_t = \alpha^2 u_{xx}$, $0 < x < 20$, $t > 0$

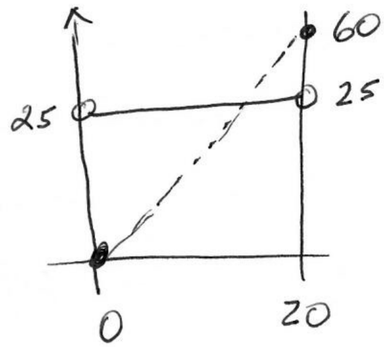
(2) $u(x,0) = 25$, $0 < x < 20$

(3) $u(0,t) = 0$, $u(20,t) = 60$, $t > 0$.

Answ:

Equilibrium temp or steady state:

$$u_t \xrightarrow[t \to \infty]{} 0$$



$$v(x) = \frac{60}{20} x = 3x$$

You can obtain this from

$$\begin{cases} v''(x) = 0 \\ v(0) = 0, \quad v(20) = 60 \end{cases}$$

$v(x) = c_1 x + c_2$

$0 = v(0) = c_2 \Rightarrow c_2 = 0,$ | $60 = v(20) = 20c_1$ $\Rightarrow c_1 = \frac{60}{20} = 3.$

and $\boxed{v(x) = 3x}$

$u(x,t) = w(x,t) + v(x)$ or $w(x,t) = u(x,t) - v(x)$.

It can be easily shown that $w(x,t)$ satisfies the IBVP:

$$\begin{cases} w_t = \alpha^2 w_{xx}, \quad 0 < x < 20, \ t > 0 & (4) \\ w(x,0) = 25 - 3x & (5) \\ w(0,t) = 0 - 0 = 0, \quad w(20,t) = 60 - 60 = 0 & (6) \end{cases}$$

Since,

$$W_t = U_t - V_t = \alpha^2 U_{xx} - \overset{=0}{(3x)_t} = \alpha^2 U_{xx}$$

$$= \alpha^2 \left(W_{xx} - \overset{=0}{V_{xx}} \right)$$

$$= \alpha^2 W_{xx}.$$

and $\quad W(x,0) = U(x,0) - V(x) = 25 - 3x.$

$$W(0,t) = U(0,t) - V(0) = 0 - 0 = 0$$

$$W(20,t) = U(20,t) - V(20) = 60 - 60 = 0$$

Solving (4)–(6)

$$W(x,t) = \sum_{n=1}^{\infty} C_n \, e^{-\left(\frac{n\pi}{20}\right)^2 0.86\, t} \sin \frac{n\pi}{20} x$$

Where $\quad C_n = \frac{2}{20} \int_0^{20} (25 - 3x) \sin \frac{n\pi}{L} x \, dx$

or $\quad C_n = \frac{25}{10} \int_0^{20} \sin \frac{n\pi}{20} x \, dx - \frac{6}{20} \int_0^{20} x \sin \frac{n\pi}{20} x \, dx$

$$= \frac{-25}{\cancel{10}} \frac{2\cancel{0}}{n\pi} \cos \frac{n\pi}{20} x \Big|_0^{20}$$

$$- \frac{6}{20} \left[-x \frac{20}{n\pi} \cos \frac{n\pi}{20} x \Big|_0^{20} + \int_0^{20} \frac{20}{n\pi} \cos \frac{n\pi}{20} x \, dx \right]$$

Then,

$$\boxed{\begin{aligned} &= -\frac{50}{n\pi}(\cos(n\pi) - 1) - \frac{3}{1\cancel{0}}\left[-\frac{400}{n\pi}(\cos(n\pi) - 1) + 0 \right] \\ C_n &= (\cos n\pi - 1)\left(-\frac{50}{n\pi} + \frac{120}{n\pi} \right) = \left((-1)^n - 1 \right)\left(+70 + \frac{12}{n\pi} \right) \end{aligned}}$$

$$= -\frac{50}{n\pi}\left(\cos(n\pi)-1\right)$$

$$-\frac{6}{\cancel{20}}\,\frac{\cancel{20}}{n\pi}\left[-20\cos(n\pi)-0+0\right]$$

$$= \frac{-50}{n\pi}\left(\cos n\pi\right)+\frac{50}{n\pi}+\frac{120}{n\pi}\cos n\pi$$

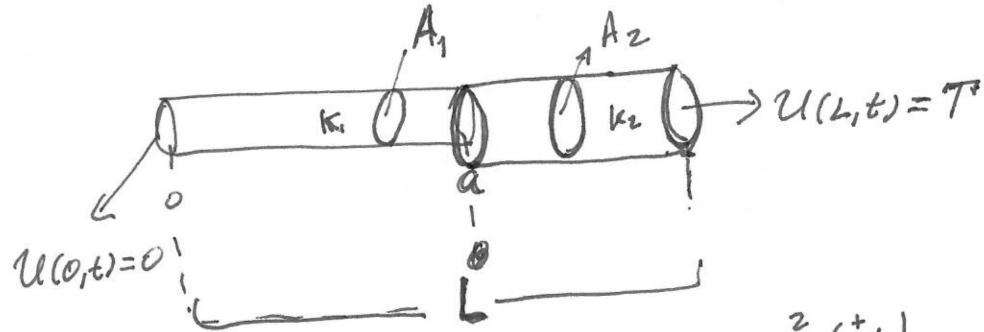
$$= \frac{70}{n\pi}\cos n\pi+\frac{50}{n\pi} = \frac{70\cos(n\pi)+50}{n\pi}$$

Or

$$u(x,t) = 3x + \sum_{n=1}^{\infty} \frac{70\cos(n\pi)+50}{n\pi}\, e^{-\frac{0.8\,n^2\pi^2}{400}t}\,\sin\frac{n\pi}{20}x$$

10.6

\# 19)



$$\begin{cases} K_1 A_1 u^1_x(a^-,t) = K_2 A_2 u^2_x(a^+,t) & (1) \\ u^1(a^-,t) = u^2(a^+,t) & (2) \\ \Rightarrow (\hat{u}^1)''(x) = 0, \quad 0 < x < a \\ (\hat{u}^2)''(x) = 0, \quad a < x < L \\ u^1(0,t) = 0, \quad u^2(L,t) = T. \end{cases}$$

$$\hat{u}^1(x) = c_1 x + c_2, \qquad \hat{u}^2(x) = d_1 x + d_2$$

$$0 = \hat{u}^1(0) = c_2 \Rightarrow \boxed{c_2 = 0} \qquad \hat{u}^2(L) = d_1 L + d_2 = T$$

So $u^1(x) = c_1 x$

From (2) $\quad u^1(a^-,t) = u^2(a^+,t)$

$$u^1(a^-) = u^2(a^+) \Rightarrow c_1 a = d_1 a + d_2$$

From (1) $\quad K_1 A_1 c_1 = K_2 A_2 d_1$

$$\begin{cases} -ac_1 + ad_1 + d_2 = 0 \\ c_1 - \xi d_1 = 0 \\ L d_1 + d_2 = T \end{cases} \qquad \xi = K_2A_2/K_1A_1$$

$$c_1 = \xi d_1$$

$$\Rightarrow \quad (-a\xi + a) d_1 + d_2 = 0$$
$$L \; d_1 + d_2 = T$$

$$\big(a(1-\xi)\, d_1 + d_2 = 0\big) -$$
$$L d_1 + d_2 = T$$

$$[L + a(\xi - 1)]\, d_1 = T$$

$$\Rightarrow \quad \boxed{d_1 = \frac{T}{L + a(\xi-1)} = \frac{1}{a}\,\frac{T}{L/a + \xi - 1}}$$

and $\boxed{c_1 = \dfrac{T}{a}\,\dfrac{\xi}{L/a - 1 + \xi}}$

and $d_2 = T - L d_1 = T - \dfrac{L}{a}\,\dfrac{T}{L/a + \xi - 1}$

or $\boxed{d_2 = T\left(1 - \dfrac{L}{a}\,\dfrac{1}{L/a + \xi - 1}\right)}$