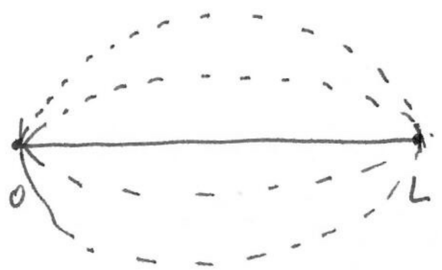
10.7 The Wave Equation. Vibrations of an Elastic String.



elastic
Vibrating string

$u(x,t)$: Vertical displacement at point x at time t.

1) If not damping.
2) Amplitude not too large.

Governing equation:

$$u_{tt} = a^2 u_{xx}, \qquad 0<x<L,\ t>0.$$

Wave Equation (one-dimensional).

$a^2 = T/\rho$

T: tension in string (force).

ρ: lineal density $= M/L$

Then $[a^2] = \dfrac{\cancel{M}\cdot L/t^2}{\cancel{M}/L} = \dfrac{L^2}{t^2} \Rightarrow [a] = \dfrac{L}{t}$. Velocity.

Initial Boundary Value Problem:

If ends $x=0$, $x=L$ are fixed (don't move)

BCs: $u(0,t)=0$, $u(L,t)=0$, $t>0$

Initial position of string and initial Velocity:

Similar to mass-Vibration problem. what's the difference?

$$u(x,0)=f(x), \quad u_t(x,0)=g(x), \quad 0\le x\le L$$

In order to be consistent, it is also required

$$f(0)=f(L)=0, \quad g(0)=g(L)=0$$

Complete IBVP:

$$\begin{cases} u_{tt}=a^2 u_{xx}, & 0<x<L,\ t>0 & (2.1)\\ u(0,t)=0, \quad u(L,t)=0, & t>0 & (2.2)\\ u(x,0)=f(x), \quad u_t(x,0)=g(x), & 0\le x\le L & (2.3)\end{cases}$$

Solution. Separation of Variables.

If the string is disturbed from its equil. position and then released at time $t=0$ with zero velocity to vibrate freely, then

$$u(x,0) = f(x), \quad u_t(x,0) = 0. \quad 0 \le x \le L.$$

Assume

$$u(x,t) = X(x)\,T(t).$$

Substitution into (2.1)

$$X T'' = a^2 X'' T \Rightarrow \frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T}$$

$$\underbrace{\frac{X''(x)}{X(x)}}_{\text{function of } x} = \frac{1}{a^2}\underbrace{\frac{T''(t)}{T(t)}}_{\text{function of } t} = -\lambda$$

true for any x and t.

only possibility equal to a constant.

Then,

$$X'' + \lambda X = 0$$

$$T'' + a^2 \lambda T = 0$$

Eigenvalue Problem:

$$X''(x) + \lambda X(x) = 0$$
$$X(0) = 0, \quad X(L) = 0$$

Same as for Heat Conduction problem:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \sin \frac{n\pi}{L} x$$
$$n = 1, 2, \ldots$$

Then, the equation for $T(t)$

$$T''(t) + \frac{a^2 n^2 \pi^2}{L^2} T(t) = 0$$

General Soln. $\Rightarrow$ $T_n(t) = K_1 \cos\left(\frac{n\pi}{L} at\right) + K_2 \sin\left(\frac{n\pi}{L} at\right)$

We can satisfy the I.C.: $u_t(x,0) = 0$

if $T_n'(t)\Big|_{t=0} = -K_1 \frac{n\pi}{L} a \sin\left(\frac{n\pi}{L} at\right) + K_2 \frac{n\pi}{L} a \cos\left(\frac{n\pi}{L} at\right)\Big|_{t=0} = 0$

or $0 = K_2 \frac{n\pi}{L} a \Rightarrow \boxed{K_2 = 0}$

So $T_n(t) = K_1 \cos\left(\frac{n\pi}{L} at\right)$

Product Solns: $\boxed{u_n(t) = C_n \cos\left(\frac{n\pi}{L} at\right) \sin\left(\frac{n\pi}{L} x\right)}$

Formal Soln. for IBVP:

$$\boxed{u(x,t) = \underbrace{\sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{L} a t\right) \sin\left(\frac{n\pi}{L} x\right)}} \tag{5.1}$$

The coefficients c_n are determined from

I.C. $u(x,0) = f(x)$.

In fact,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \overbrace{\cos(0)}^{=1} \sin\left(\frac{n\pi}{L} x\right)$$

$$= \sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{L} x$$

Hence,

$$\boxed{c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx} \tag{5.2}$$

The infinite series (5.1) with the coefficients (5.2) is the Soln. of our IBVP with I.Cs: $\begin{cases} u(x,0) = f(x). \\ u_t(x,0) = 0. \end{cases}$

Some important definitions for wave problems:

The product solns:

$$\boxed{u_n(x,t) = \sin\frac{n\pi}{L}x \cos\frac{n\pi a}{L}t, \quad n=1,2\ldots} \quad (6.1)$$

are periodic in time with period

$\boxed{T = \frac{2L}{na}}$, obtained from $\frac{n\pi a}{L} = \frac{2\pi}{T}$

or $T = \frac{2\pi}{\frac{n\pi a}{L}}$

Show MATLAB Simulation.

frequency of vibrations $= \frac{n\pi a}{L}$, $n=1,2,\ldots$

Also called natural frequencies.

$\sin\frac{n\pi}{L}x$: displacement patterns
or natural modes. of vibrations.

periodic in space. Period $T_{\lambda} = \frac{2\pi}{\frac{n\pi}{L}} = \frac{2L}{n}$.

λ is also called wavelength.

and $k = \frac{n\pi}{L}$ is called frequency (spatial).

So eigenvalues are proportional to k^2 wave freq.
and eigenfunctions are the natural modes of vibrations.

The period for the total motion

$$u(x,t) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L} a t\right) \sin\left(\frac{n\pi}{L} x\right)$$

is $T = 2L/a$. $(n=1)$