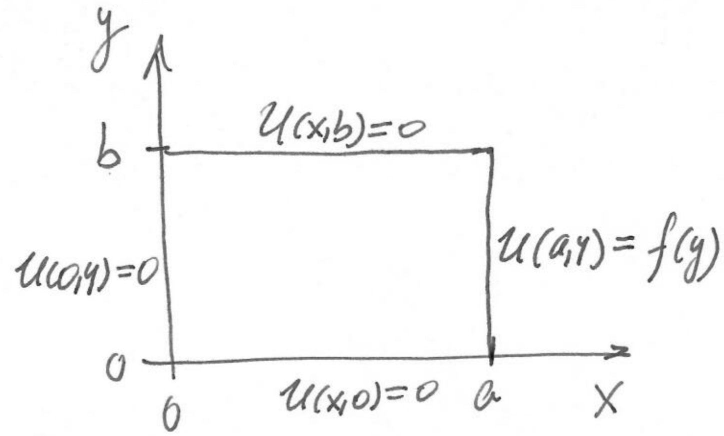
10.8 Laplace Equation.

It can be obtained as the steady state equation for a heat condition equation.

$$\boxed{u_t = \alpha^2 (u_{xx} + u_{yy})} \xrightarrow[t\to\infty]{} \quad u_t \to 0 \quad \boxed{u_{xx} + u_{yy} = 0.}$$

Dirichlet Problem for a Rectangle.

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a, \quad 0 < y < b. \\ u(x,0) = 0, \quad u(x,b) = 0, & 0 < x < a \\ u(0,y) = 0, \quad u(a,y) = f(y) & 0 < y < b \end{cases}$$



<u>Sep. Variables:</u>

$$u(x,y) = X(x)\,Y(y)$$

$$X''(x)\,Y(y) + X(x)\,Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

Then, two ODEs:

$$X''(x) - \lambda X(x) = 0 \qquad (2.1)$$

$$Y''(y) + \lambda Y(y) = 0 \qquad (2.2)$$

<u>From BCs:</u>

To avoid Trivial Soln.

$$u(x,0) = Y(0)\,X(x) = 0 \Rightarrow \boxed{Y(0)=0}$$

$$u(x,b) = Y(b)\,X(x) = 0 \Rightarrow \boxed{Y(b)=0} \qquad (2.3)$$

$$u(0,y) = X(0)\,Y(y) = 0 \Rightarrow \boxed{X(0)=0} \qquad (2.4)$$

Thus, we have an eigenvalue problem for $Y(y)$.

$$\begin{cases} Y''(y) + \lambda Y(y) = 0, & 0 < y < b \\ Y(0) = 0, \quad Y(b) = 0 \end{cases}$$

$$\lambda_n = \left(\frac{n\pi}{b}\right)^2, \quad Y_n(x) = \sin\frac{n\pi}{b}y, \quad n = 1, 2 \ldots$$

The other ODE is

$$X''(x) - \lambda_n X(x) = 0 \qquad (3.1)$$

From the boundary condition

$u(0,y) = 0$, we obtain $0 = u(0,y) = X(0)\,Y(y)$

$\Rightarrow$ $\boxed{X(0) = 0}$ to avoid a trivial solution

The other B.C.

$u(a,y) = f(y)$ is not satisfied for an arbitrary $f(y)$ by specifying $X(a)$.

In fact, the general solution of (3.1) for $\lambda_n = \left(\frac{n\pi}{b}\right)^2$ is

$$X''(x) - \left(\frac{n\pi}{b}\right)^2 X(x) = 0$$

$$X_n(x) = C_1 e^{\frac{n\pi}{b}x} + C_2 e^{-\frac{n\pi}{b}x}$$

or $X_n(x) = C_1 \cosh\left(\frac{n\pi}{b}x\right) + C_2 \sinh\left(\frac{n\pi}{b}y\right)$

Using the boundary condition

$$X_n(0) = 0$$

we get $$0 = X_n(0) = C_1 \overbrace{\cosh 0}^{=1} + C_2 \overbrace{\sinh 0}^{=0}$$

$$\Rightarrow \boxed{C_1 = 0}$$

Therefore, $$\boxed{X_n(x) = C_n \sinh\left(\frac{n\pi}{b} x\right)}.$$

Now, we can form the product solutions

$$\boxed{u_n(x,y) = \sinh\left(\frac{n\pi}{b} x\right) \sin\left(\frac{n\pi}{b} y\right)} \quad n = 1, 2, \ldots$$

None of these product solutions individually satisfy the boundary condition

$$u(a,y) = f(y)$$

because for arbitrary $f(y)$,

$$u_n(a,y) = \sinh\left(\frac{n\pi}{b} a\right) \sin\left(\frac{n\pi}{b} y\right) \neq f(y)$$

However, if we consider a soln $u(x,y)$ as

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi}{b} x \, \sin \frac{n\pi}{b} y} \qquad (5.1)$$

then, we can satisfy : $u(a,y) = f(y)$

if

$$u(a,y) = \boxed{\sum_{n=1}^{\infty} C_n \, \mathrm{Si} \frac{n\pi}{b} a \, \sin \frac{n\pi}{b} y = f(y)}$$

Fourier
Sine series in y
$\Rightarrow$

$$\sin \frac{n\pi}{b} a \, C_n = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi}{b} y \, dy$$

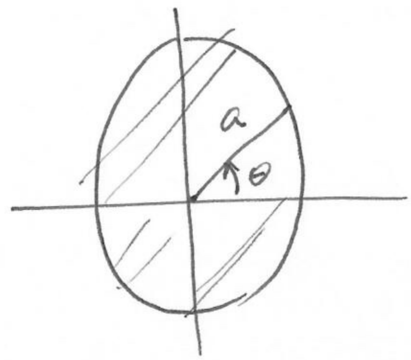
or

$$\boxed{C_n = \frac{2}{b \sin \frac{n\pi}{b} a} \int_0^b f(y) \sin \frac{n\pi}{b} y \, dy} \qquad (5.2)$$

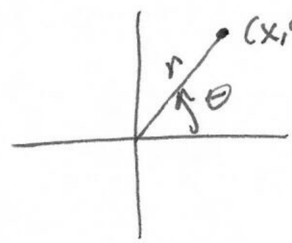
SOLUTION IS FORMED BY (5.1)-(5.2).

Dirichlet Problem for a Circle:

Consider a circular plate:



$x = r\cos\theta$
$y = r\sin\theta$ polar coords.



Laplace Equ:

$$u_{xx} + u_{yy} = 0 \xrightarrow[\text{Multivariable Calculus problem}]{\text{Polar coordinates}} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$\underline{\underline{\text{Not}}}$

$$u_{rr} + u_{\theta\theta} = 0.$$

<u>BVP:</u>

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & 0 < r < a \\ & 0 \le \theta \le 2\pi \\ u(a,\theta) = f(\theta), & 0 \le \theta \le 2\pi \end{cases}$$

Additional conditions:

$|u(0,\theta)| \le M$ bounded.

$u(a,0) = u(a, 2\pi)$

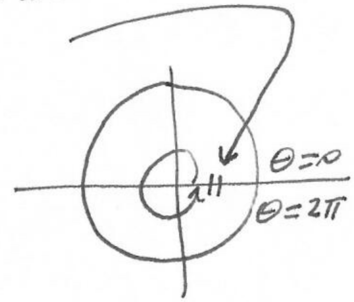
So after Separation of Variables, we have arrived to

$$\begin{cases} r^2 R''(r) + r R'(r) - \lambda R(r) = 0, & \text{(A.1)} \\ \Theta''(\theta) + \lambda \Theta(\theta) = 0, & \text{(A.2)} \end{cases}$$

Equation (A.2) also has two BCs. (Periodic Conditions)

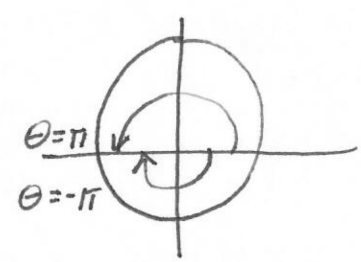
In fact,

$$\begin{cases} u(r, 0) = u(r, 2\pi), & 0 \le r \le a \\ \text{and} \quad u_\theta(r, 0) = u_\theta(r, 2\pi) \end{cases}$$



We can also state these conditions as

$$\begin{cases} u(r, \pi) = u(r, -\pi) & \text{(A.3)} \\ u_\theta(r, \pi) = u_\theta(r, -\pi) & \text{(A.4)} \end{cases}$$



This last representation is more Convenient for the formulation of the eigenvalue problem for $\Theta(\theta)$.

From (A.3) and (A.4)

$$\text{and} \quad \begin{cases} R(r)\,\Theta(\pi) = R(r)\,\Theta(-\pi), & 0 \le r \le a \\ R(r)\,\Theta'(\pi) = R(r)\,\Theta'(-\pi), \end{cases}$$

Because $R(r) \equiv 0$ (otherwise $u \equiv 0$, trivial soln.)

then, $\quad \Theta(\pi) = \Theta(-\pi)$ and $\Theta'(\pi) = \Theta'(-\pi)$.

And we have arrived to the <u>eigenvalue problem</u>:

$$\begin{cases} \Theta''(\theta) + \lambda\,\Theta(\theta) = 0 & \text{(B.1)} \\ \Theta(\pi) = \Theta(-\pi) & \text{(B.2)} \\ \Theta'(\pi) = \Theta'(-\pi) & \text{(B.3)} \end{cases}$$

If $\underline{\underline{\lambda > 0}}$, $\quad \boxed{\Theta(\theta) = C_1 \sin\sqrt{\lambda}\,\theta + C_2 \cos\sqrt{\lambda}\,\theta}$

Using (B.2),

$$C_1 \sin(\sqrt{\lambda}\pi) + \cancel{C_2 \cos(\sqrt{\lambda}\pi)} = C_1 \sin(-\sqrt{\lambda}\pi) + C_2 \cos(-\sqrt{\lambda}\pi)$$

$$= -C_1 \sin(\sqrt{\lambda}\pi) + \cancel{C_2 \cos(\sqrt{\lambda}\pi)}$$

$$\Rightarrow \boxed{2C_1 \sin(\sqrt{\lambda}\pi) = 0} \qquad \text{(B.4)}$$

Also,

$$\Theta'(\pi) = \Theta'(-\pi)$$

or

$$\sqrt{\lambda}\, C_1 \cancel{\cos}(\sqrt{\lambda}\pi) - \sqrt{\lambda}\, C_2 \sin(\sqrt{\lambda}\pi)$$

$$= \sqrt{\lambda}\, C_1 \cos(-\sqrt{\lambda}\pi) - \sqrt{\lambda}\, C_2 \sin(-\sqrt{\lambda}\pi)$$

$$= \sqrt{\lambda}\, C_1 \cancel{\cos}(\sqrt{\lambda}\pi) + \sqrt{\lambda}\, C_2 \sin(\sqrt{\lambda}\pi)$$

$$\Rightarrow \quad \boxed{2\sqrt{\lambda}\, C_2 \sin(\sqrt{\lambda}\pi) = 0} \qquad (B.5).$$

If $\sin(\sqrt{\lambda}\pi) \neq 0 \Rightarrow C_1 = 0, \; C_2 = 0$

Trivial soln.

Therefore, for nontrivial soln. to exist

$$\sin(\sqrt{\lambda}\pi) = 0 \Rightarrow \sqrt{\lambda}\pi = n\pi$$

$$\Rightarrow \boxed{\lambda = n^2}$$

and the associated eigenfunctions are both

for $\underline{\lambda_n = n^2}$

$$\begin{cases} \Theta_n^1(\theta) = \overset{\sin(\sqrt{\lambda_n}\theta)}{\sin(n\theta)} \\ \Theta_n^2(\theta) = \cos(n\theta), \end{cases} \qquad n = 1, 2, \ldots$$

D

If $\underline{\underline{\lambda = 0}}$, (B.1) reduces to

$$\Theta''(\theta) = 0 \Rightarrow \Theta(\theta) = C_1\theta + C_2$$

Using the periodic conditions:

$\Theta(\pi) = \Theta(-\pi)$: $C_1\pi + \cancel{C_2} = -C_1\pi + \cancel{C_2}$

$\Theta'(\pi) = \Theta'(-\pi)$: $C_1 = C_1$

Then, $$2C_1\pi = 0 \Rightarrow C_1 = 0$$

and $$\Theta(\theta) = C_2 = C_2\,\Theta_0(\theta) = C_2(1).$$

$$\boxed{\Theta_0(\theta) = 1} \quad \text{for } \underline{\lambda_0 = 0}$$

If $\lambda < 0$, It can be easily prove that $\Theta(\theta) \equiv 0$. Trivial soln.
then $\lambda < 0$ is not an eigenvalue.

Now, consider (A.1)

$$r^2 R''(r) + r R'(r) - \lambda_n R(r) = 0$$

$\underline{\underline{\text{For } \lambda_n = n, \quad n = 1, 2 \ldots}}$

Assuming,

$$R(r) = r^p$$

$$r^2 p(p-1) r^{p-2} + r p r^{p-1} - \lambda_n r^p = 0$$

$$\left(p(p-1) + p - \lambda_n\right) r^p = 0$$

$$\Rightarrow \quad p^2 - \cancel{p} + \cancel{p} - \lambda_n = 0 \Rightarrow p^2 = \lambda_n = n^2 \Rightarrow \boxed{p = \pm n}$$

Two fund. Solutions are

$$R_1(r) = r^n, \qquad R_2(r) = r^{-n}.$$

and the general soln is given by

$$R_n(r) = d_n^1 r^n + d_n^2 r^{-n}$$

now $|u(0,\theta)| \leq M$,

then $d_n^2 = 0$, otherwise $R(r) \xrightarrow[r \to 0]{} \infty$.

and $\boxed{R_n(r) = d_n r^n, \quad n = 1, 2, \cdots}$.

For $\underline{\underline{\lambda_0 = 0}}$, (A.1) reduces to

$$r^2 R''(r) + r R'(r) = 0$$

Assuming, $R(r) = r^p$

$$p(p-1)\, r^2 r^{p-2} + p\, r\, r^{p-1} \stackrel{?}{=} 0$$

or $\left(p(p-1)+p\right) r^p = 0 \Rightarrow p^2 - \cancel{p} + \cancel{p} = 0 \Rightarrow \underline{\underline{p=0}}$

repeated twice.

Fund. Soln: $\boxed{R_1(r) = r^0 = 1}$.

Another fund. Soln can be obtained directly integrating

$$r^2 R''(r) + r R'(r) = 0$$

or $r \dfrac{d}{dr}\left(r \dfrac{dR}{dr}\right) = 0 \Rightarrow \dfrac{d}{dr}\left(r \dfrac{dR}{dr}\right) = 0$

$\int dr$: $\quad r \dfrac{dR}{dr} = d_1 \Rightarrow \dfrac{dR}{dr} = \dfrac{d_1}{r}$

for $d_2 = 0$

$R(r) = d_1 \ln r + d_2 \Rightarrow \boxed{R_2(r) = \ln r.}$

another fund. Soln.

Using $|u(0,\theta)| \le M$

Implies $C_1 = 0$, because $\ln r \xrightarrow[r \to 0]{} \infty$

$\begin{vmatrix} \ln r & 1 \\ \frac{1}{r} & 0 \end{vmatrix} = -\frac{1}{r} \ne 0$ if $r \ne 0$.

Therefore, for $\boxed{\begin{array}{l} \lambda_0 = 0 \\ R_0(r) = 1 \end{array}}$

and the product Solns are

$$u_0(r,\theta) = R_0(r)\,\Theta_0(\theta) = 1.$$

$$u_n(r,\theta) = R_n(r)\,\Theta_n(\theta) = d_n r^n \left(c_n^1 \cos n\theta + c_n^2 \sin n\theta\right)$$

and using Superposition, we obtain the general solution

$$\boxed{u(r,\theta) = \frac{c_0}{\underset{\text{Convenience}}{2}} + \sum_{n=1}^{\infty} r^n \left(c_n \cos(n\theta) + k_n \sin(n\theta)\right)}$$

and to determine the coefficients c_0, c_n, k_n. $n=1,2,\ldots$

we use the BC. $u(a,\theta) = f(\theta)$.

$$u(a,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n \left(c_n \cos(n\theta) + k_n \sin(n\theta)\right) = f(\theta)$$

Fourier series of sin's and cos's of $f(\theta)$

$$c_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

$$a^n c_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta, \qquad k_n a^n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta.$$

$n=1,2,\ldots$