Chapter 2: First Order Differential Equations

Consider:

a) $v'(t) = \frac{1}{5}\left(49 - v(t)\right) = -\frac{v(t)}{5} + 9.8$ (1.1)

or $\boxed{v'(t) + \frac{1}{5}v(t) = 9.8}$ (1.2)

b) $t\,y'(t) + 2y(t) = \sin t, \quad t > 0$ (2.1)

or

$$\boxed{y'(t) + \frac{2}{t}y(t) = \frac{\sin t}{t}} \qquad (2.2)$$

or

$$y'(t) = -\frac{2}{t}y(t) + \frac{\sin t}{t} \qquad (2.3)$$

c) $\sin x\, y'(x) - \cos x\, y(x) = \sin(2x), \quad 0 < x < \pi$ (3.1)

or

$$\boxed{y'(x) - \frac{\cos x}{\sin x}y(x) = \frac{\sin(2x)}{\sin x} = 2\cos x} \qquad (3.2)$$

or

$$y'(x) = \frac{\cos x}{\sin x}y(x) + 2\cos x. \qquad (3.3)$$

d) $$\left(3y^2(x)-4\right)y'(x) = 3x^2$$

or

$$y'(x) = \frac{3x^2}{3y^2(x)-4}$$

Discussion: Common form ? $\left\{\begin{array}{l}\text{First order O.D.E.} \\ \text{and} \\ \text{First order linear} \\ \text{ODE.}\end{array}\right.$

Classification ?

1.1 Solutions for First order linear ODE

a) $$v'(t) + \frac{1}{5}v(t) = 9.8$$

A more direct way to obtain Solution: $\boxed{v(t) = C_2 e^{-t/5} + 49}$

Multiply both sides by $\boxed{e^{t/5}}$ (magic function)

$$e^{t/5}v'(t) + \frac{1}{5}e^{t/5}v(t) = 9.8\, e^{t/5}$$

Then,

$$\left[e^{t/5}v(t)\right]' = 9.8\, e^{t/5}$$

Then $\quad \int dt$

$$e^{t/5}v(t) + C_0 = 9.8\left(5e^{t/5}\right) = 49e^{t/5} + C'$$

then,

$$e^{t/5}v(t) = 49e^{t/5} + C_1 - C_0 = 49e^{t/5} + C_2$$

or

$$\boxed{v(t) = 49 + C_2 e^{-t/5}}$$

b) $\quad y'(t) + \frac{2}{t}y(t) = \frac{\sin t}{t}, \quad t > 0.$

Magic function: $\boxed{t^2}$

$$t^2 y'(t) + 2t\,y(t) = t\sin t.$$

Then, $\quad \left(t^2 y(t)\right)' = t\sin t$ $\qquad \int fg' = fg - \int f'g$

$\int dt$: $\quad t^2 y(t) = \int \overset{f}{t}\, \overset{g'}{\sin t}\, dt + C$

or $\quad t^2 y(t) = -t\cos t + \int \cos t\, dt + C = -t\cos t + \sin t + C$

Thus, $\quad \boxed{y(t) = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{C}{t^2}}$

How can we obtain the "magic function"?

A general first order linear equation in standard form is written as:

$$y'(t) + p(t)y(t) = g(t) \qquad (4.1)$$

If we multiply (4.1) by $\mu(t) \to$ magic function?

$$\mu(t)y'(t) + \mu(t)p(t)\,y(t) = \mu(t)\,g(t).$$

We want

$$lhs = [\mu(t)\,y(t)]'$$

So what should $\mu(t)$ be?

$$\mu(t)\,y'(t) + \mu'(t)\,y(t) = \mu(t)\,y'(t) + \mu(t)p(t)\,y(t)$$

Which is equivalent to

$$\mu'(t)\,y(t) = \mu(t)\,p(t)\,y(t)$$

or

$$\mu'(t) = \mu(t)\,p(t)$$

Assuming $\mu(t) \neq 0$

$$\frac{\mu'(t)}{\mu(t)} = p(t)$$

Then, $\int dt$ $\quad \ln|\mu(t)| = \int p(t)\,dt + C$

Thus, $e^{\bigcirc}$: $\quad |\mu(t)| = C_1 e^{\int p(t)dt}$

or $\quad \mu(t) = \pm C_1 e^{\int p(t)dt} = C_2 e^{\int p(t)dt}$

Choosing $C_2 = 1$; $\boxed{\mu(t) = e^{\int p(t)dt}}$ $\quad (5.1)$

Note $\mu(t) > 0$. always.

General form 1st order ODE and Standard form 1st order linear ODE.

1) $v'(t) = 9.8 - \frac{1}{5} v(t)$

2) $t\, y'(t) + 2y(t) = \sin t, \quad t > 0$

3) $\sin x\, y'(x) - \cos x\, y(x) = \sin(2x), \quad 0 < x < \pi$

4) $(3y^2(x) - 4)\, y'(x) = 3x^2$

Ask: i) Linear or not.?

ii) General form

iii) Standard form if 1st order linear.

(1) - (3) Linear

(1) - (4) Can be written as:

$$y'(t) = f(t, y(t)) \quad \text{or} \quad y'(x) = f(x, y(x)).$$

We can rewrite (1) - (3) in Standard form:

1) $v'(t) - \frac{1}{5} v(t) = 9.8$

2) $y'(t) + \frac{2}{t} y(t) = \frac{\sin t}{t}, \quad t > 0.$

3) $y'(x) - \frac{\cos x}{\sin x} y(x) = \frac{\sin(2x)}{\sin x} = \frac{2\cos x \cancel{\sin x}}{\cancel{\sin x}}$

In general, Standard form: $= 2\cos x$

$$\boxed{y'(t) + p(t) y(t) = g(t).}$$

we found integrating factors for (1) and (2)

1) $\mu(t) = e^{t/5}$

2) $\mu(t) = t^2$

In fact,

2) $y'(t) t^2 + t^2 \frac{2}{t} y(t) = t^2 \frac{\sin t}{t}$

$y'(t) t^2 + 2t\, y(t) = t \sin t$

or $\frac{d}{dt}\left[t^2 y(t) \right] = t \sin t$

Thus, $\int dt$

$$t^2 y(t) = \int t \sin t \, dt = -t\cos t + \sin t + C$$

or

$$\boxed{y(t) = -\frac{\cos t}{t} + \frac{\sin t}{t} + \frac{C}{t^2}}.$$

We also showed that in general

$$\boxed{\mu(t) = e^{\int p(t)dt}}$$

Find integrating factor for (3)
and obtain soln.

$$\mu(x) = e^{\int p(x)dx} = e^{\int -\frac{\cos x}{\sin x}dx} = e^{-\ln(\sin x)}$$

$\sin x > 0, \quad 0 < x < \pi$

$$= \frac{1}{e^{\ln(\sin x)}} = \frac{1}{\sin x}$$

multiplying (3) by $\frac{1}{\sin x}$.

$$\frac{1}{\sin x} y'(x) - \frac{\cos x}{\sin^2 x} y(x) = \frac{2\cos x}{\sin x} = \frac{2\sin x \cos x}{\sin x}$$

$$= 2\cos x.$$

or

$$\frac{d}{dx}\left[y(x)/\sin x \right] = 2\,\frac{\cos x}{\sin x}$$

$\int dx$:

$$\frac{y(x)}{\sin x} = 2\ln(\sin x) + C$$

Thus,

$$\boxed{y(x) = 2\sin x \ln(\sin x) + C\sin x.}$$