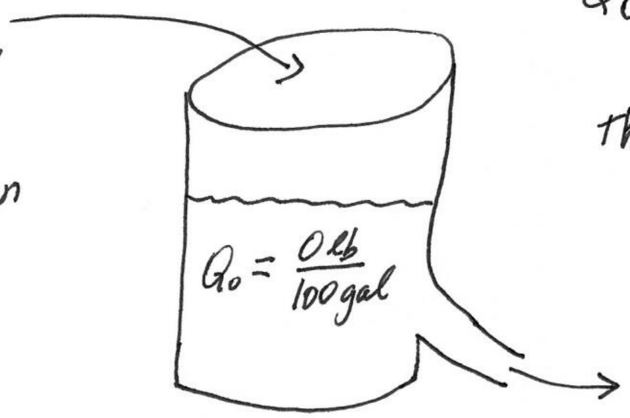
2.3 Modeling with 1^{st} order Equations.

i) Concentration problems:

Exercise 3

$C_{in} = \frac{1}{2}$ lb/gal

$r_{in} = 2$ gal/min



$Q(t)$: lbs of Salt at time t.

Then,

$$\frac{dQ}{dt} = \text{rate of change of Salt} = \frac{lb}{min}$$

$r_{out} = 2$ gal/min

$$C_{out} = \frac{Q(t) \text{ lb}}{100 \text{ gal}}$$

Additional Assumption: a) Salt is not introduced into the tank using other sources

b) Salt is not taking out in another way.

Conservation Law:

rate of Change of Salt into the tank = rate in − rate out or

$$\frac{dQ}{dt}(t) = \frac{1}{2}\frac{lb}{\cancel{gal}} \cdot 2\frac{\cancel{gal}}{min} - 2\frac{\cancel{gal}}{min} \cdot \frac{Q(t)\ lb}{100\cancel{gal}}$$

or

$$\frac{dQ}{dt}(t) = 1\frac{lb}{min} - \frac{Q(t)}{50}\frac{lb}{min}.$$

So IVP:

$$\begin{cases} Q'(t) = 1 - \dfrac{Q(t)}{50} = \dfrac{1}{50}\left(50 - Q(t)\right) \\ Q(0) = 0 \end{cases}$$

The soln. is easily found separating variables

$$\frac{Q'(t)}{50 - Q(t)} = \frac{1}{50} \quad \text{or} \quad \frac{Q'(t)}{Q(t) - 50} = -\frac{1}{50}$$

$\int dt$: $$\ln|Q(t)-50| = \frac{-1}{50}t + C_1$$

then, $$\ln|Q(t)-50| = -\frac{t}{50} + \tilde{C}$$

$e^{\bigcirc}$: $$|Q(t)-50| = C_1 e^{-t/50}$$

or $$Q(t) - 50 = \pm C_1 e^{-t/50} = C_2 e^{-t/50}$$

or $$\boxed{Q(t) = 50 + C_2 e^{-t/50}}$$ Amount of Salt at any time t.

Using I.C. $Q(0)=0$

$$0 = Q(0) = 50 + C_2 e^0 = 50 + C_2$$

$$C_2 = -50$$

and the soln. is given by

$$\boxed{Q(t) = 50 - 50e^{-t/50}}$$

Then

$$Q(10) = 50 - 50\, e^{-1/5} \simeq 9.1 \text{ lb}$$

After 10 mins. process stops

and new conditions: $r_{in} = 2 \text{ gal/min}$

$C_{in} = 0 \text{ lb/gal}$

while: $r_{out} = 2 \text{ gal/min}$, $C_{out} = \dfrac{Q_2(t)}{100}$

So

New IVP:
$$\begin{cases} Q_2'(t) = 0 \text{ lb/gal} \cdot 2 \text{ gal/min} - 2 \text{ gal/min} \cdot \dfrac{Q_2(t)}{100} \\ Q_2(0) = Q(10) = 50 - 50e^{-1/5} \end{cases}$$

Therefore,
$$\begin{cases} Q_2'(t) = -\dfrac{Q_2(t)}{50} \\ Q_2(0) = 50\left(1 - e^{-1/5}\right) \end{cases}$$

Then soln, $Q_2(t) = C e^{-t/50}$

$$50\left(1 - e^{-1/5}\right) = C e^{0} = C$$

thus, $\boxed{Q_2(t) = 50\left(1 - e^{-1/5}\right) e^{-t/50}}$

and $Q_2(10) = 50\left(1 - e^{-1/5}\right) e^{-1/5} \simeq 7.4 \text{ lb.}$

In general, we will have (for r_{in} and r_{out} constants)

$$\frac{dQ}{dt} = \overset{\text{known}}{r_{in}}\, \overset{\text{known}}{C_{in}} - \overset{\text{known}}{r_{out}}\, \overset{\text{unknown}}{C_{out}}$$

or

$$\boxed{\frac{dQ}{dt} = r_{in} C_{in} - r_{out} \frac{Q(t)}{(r_{in} - r_{out})\, t + V_0}} \qquad (5.1)$$

Attempt problem #4.

Ask for possibilities of (5.1)

When $r_{in}(t)$, $C_{in}(t)$, $r_{out}(t)$.

Problem # 4 (10th Edition)

$$\begin{cases} \dfrac{dq}{dt} = 3 - \dfrac{q(t)}{v(t)} \cdot 2 = 3 - 2\,\dfrac{q(t)}{200+t} \\ q(0) = 100 \end{cases}$$

Non-separable equation

Use integrating factor for

$$q'(t) + \frac{2}{200+t}\, q(t) = 3$$

then,

$$\mu(t) = e^{2\int \frac{1}{t+200} dt} = e^{2\ln(t+200)} = (t+200)^2.$$

Thus

$$\frac{d}{dt}\left[q(t)\,(t+200)^2 \right] = 3\,(t+200)^2$$

and $\int dt$:

$$q(t)(t+200)^2 = (t+200)^3 + C$$

or

$$\boxed{q(t) = t + 200 + \frac{C}{(t+200)^2}}$$

Using I.C.

$$100 = q(0) = 200 + \frac{C}{(200)^2}$$

Then,

$$C = -100\,(200)^2.$$

and Soln. for IVP:

$$q(t) = t + 200 - \frac{100(200)^2}{(t+200)^2}$$

If Capacity of tank is 500 gal and initially it has 200 gal and 1 gal/min are entering the tank then tank will overflow when $t = 300$ min.

At this time amount of salt in tank is

$$q(300) = 500 - \frac{100\,(200)^2}{(500)^2}$$

or $$q(300) = 500 - \frac{100 \cdot 4\times\cancel{10^4}}{25\times\cancel{10^4}} = 500 - \frac{400}{25}$$

or $$q(300) = 500 - 16 = 484.$$

then Concentration when $t = 300$

$$C(300) = \frac{q(300)}{500} = \frac{484}{500} = \frac{121}{125}\ \frac{lb}{gal}.$$

Example 3 (Book). Chemical – Pond

$r_{in} = 5$ mill gal/year

$C_{in} = \gamma(t)$ g/gal

$\gamma(t) = 2 + \sin(2t)$ (Discuss Equill Soln.)

$r_{out} = 5$ mill gal/year

$C_{out} = \dfrac{Q(t)}{10 \text{ mill gal}}$ g

$$Q'(t) = r_{in} C_{in} - r_{out} C_{out}$$

or
$$\boxed{Q'(t) = 5 \times 10^6 (2 + \sin 2t) - \frac{Q(t)}{10^7} 5 \times 10^6} \quad (8.1)$$

Changing Variables:

$$q(t) = Q(t)/10^6 \Rightarrow Q(t) = q(t)\, 10^6$$

Subst. into (8.1)

$$10^6 \left[q'(t) = 5(2 + \sin(2t)) - \frac{q(t)}{2} \right]$$

or
$$\boxed{q'(t) + \frac{q(t)}{2} = 10 + 5 \sin(2t).}$$

Soln:
$$\boxed{q(t) = 20 - \frac{40}{17} \cos(2t) + \frac{10}{17} \sin 2t - \frac{300}{17} e^{-t/2}}$$

Savings, Loan and Mortgage Problems and Compound interest.

$S(t)$: Capital or Debt.

r: interest rate.

Simple Case: Continuous compounding

Initial investment: $S(0) = S_0$

$$\frac{ds}{dt} = r\,S(t)$$ (Derivation) (Explain why?).

(Compare with discrete compounding in book)

If deposits and withdrawals takes place at a constant rate "k" $\frac{\$}{\text{time}}$

Then,

$$\frac{ds}{dt} = r\,S(t) + k. = r\left(S(t) + \frac{k}{r}\right)$$

Gen. Soln: $$\boxed{S(t) = C e^{rt} - \frac{k}{r}}$$

If $S(t)$: Mortgage

M_p: Mortgage payments every month.

A_p: Annual payment $= M_p \times 12$

Then

$$\begin{cases} \dfrac{dS}{dt} = rS(t) - A_p = r\left(S(t) - \dfrac{A_p}{r}\right) \\ S(0) = S_0 \quad (\text{Initial debt}). \end{cases}$$

$$S(t) = Ce^{rt} + \frac{A_p}{r}$$

Using I.C.

$$S_0 = S(0) = C + \frac{A_p}{r}$$

or

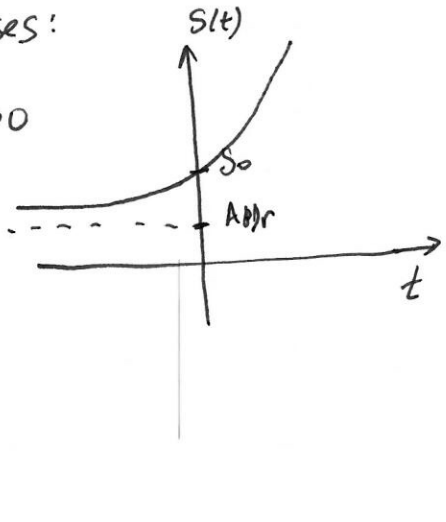
$$C = S_0 - \frac{A_p}{r}$$

Thus,

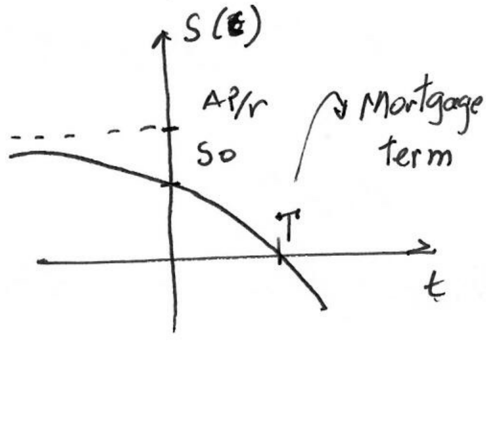
$$S(t) = \left(S_0 - \frac{A_p}{r}\right)e^{rt} + \frac{A_p}{r}$$

Two Cases:

a) $S_0 - \frac{A_p}{r} > 0$



b) $S_0 - \frac{A_p}{r} < 0$



$$\boxed{S(t) = \left(S_0 - \frac{A_p}{r}\right)e^{rt} + \frac{A_p}{r}} \qquad (9.3)$$

We want to know; Assuming A_p, r, S_0 Known

a) How long $(t = T)$ will it take to pay off the mortgage?

$$0 = S(T) = \left(S_0 - \frac{A_p}{r}\right)e^{rT} + \frac{A_p}{r}$$

or

$$e^{rT} = \frac{-A_p}{r(S_0 - A_p/r)} = \frac{A_p}{r(A_p/r - S_0)}$$

Thus,

$$rT = \ln(A_p) - \ln\left(r(A_p/r - S_0)\right)$$

$$= \ln(A_p) - \ln(r) - \ln(A_p/r - S_0)$$

Finally,

$$\boxed{T = \frac{1}{r}\ln\left(\frac{A_p}{r(A_p/r - S_0)}\right)} \qquad (9.4)$$

b) We may want to know: Assuming "r" and "S_0" known, What should be the monthly payment to pay off the mortgage in 30 years?

$$0 = S(30) = \left(S_0 - \frac{A_p}{r}\right)e^{30r} + \frac{A_p}{r} = S_0 e^{30r} + \frac{A_p}{r}\left(1 - e^{30r}\right)$$

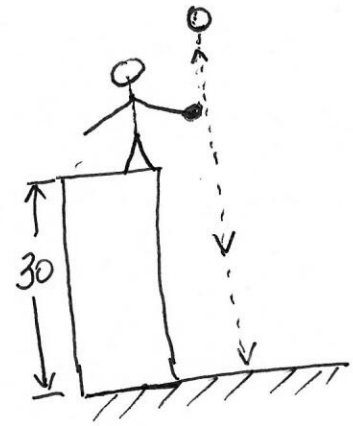
Then,

$$\boxed{A_p = r\left(\frac{S_0 e^{30r}}{e^{30r} - 1}\right)} \qquad (9.5)$$

What should be the formula to get "r" knowing A_p, S_0, and T?

Go to MAPLE worksheets for other scenarios.

#20 Ball thrown upward without air resistance



$$\cancel{m}\frac{dv}{dt} = -\cancel{m}g$$

or
IVP
$$\begin{cases} \dfrac{dv}{dt} = -g, \\ v(0) = 20\ m/s \end{cases}$$

Soln: $v(t) = -gt + C$

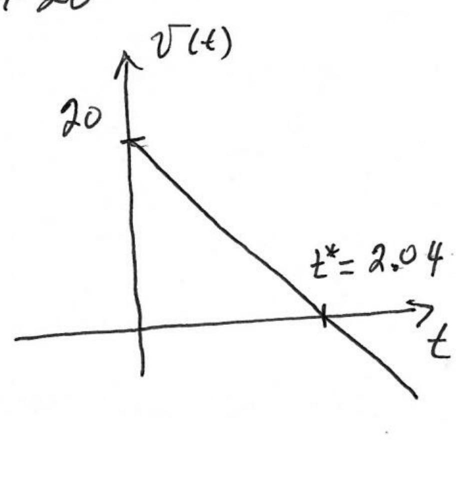
Using I.C. $20 = v(0) = C$.

Then, $\boxed{v(t) = -9.8t + 20}$

a) Max height occurs at time $t = t^*$ when $v(t^*) = 0$

Therefore
$$0 = v(t^*) = -9.8t^* + 20$$

or
$$t^* = \frac{20}{9.8} = 2.04$$



To find max height, we need to find the position $x(t)$ of the ball at any time t and then evaluate $x(t^*)$.

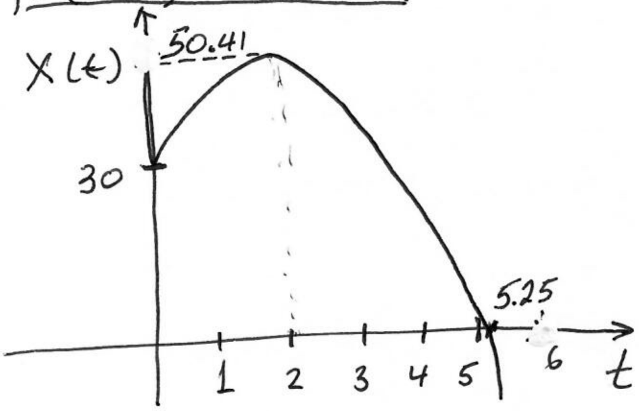
So,

$$x(t) = \int v(t)\,dt = -9.8\frac{t^2}{2} + 20t + C$$

I.C. : $30 = x(0) = C$

then $\boxed{x(t) = -9.8\frac{t^2}{2} + 20t + 30}$

Max height: $\boxed{x(2.04) \simeq 50.41}$



b) Time $\bar{t}$ when it hits the ground.

$\bar{t} = ?$ $\quad x(\bar{t}) = 0$

$$0 = -9.8\frac{\bar{t}^2}{2} + 20\bar{t} + 30$$

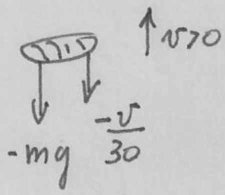
then, $\boxed{\bar{t} \simeq 5.25}$

Problem 2~~1~~ Section 2.3

Ball thrown up with air resistance $= \frac{|v|}{30}$
magnitude

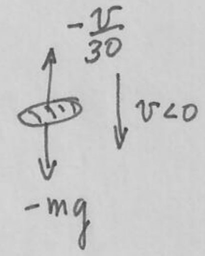
and initial velocity $v(0) = 20$.

<u>Up.</u>:



$$m\frac{dv}{dt} = -mg - \frac{v}{30}$$

<u>Down:</u>



$$m\frac{dv}{dt} = -mg - \frac{v}{30}$$

Air resistance always opposite to direction of motion.

Equations modelling phenomenon happens to be the same in both directions (this is not the case in probl. 23).

For $m = 0.15\,kg$, $g = 9.8\ m/s^2$

<u>IVP</u>:

$$\begin{cases} \frac{dv}{dt} = -9.8 - \frac{v(t)}{4.5} \\ v(0) = 20 \end{cases}$$

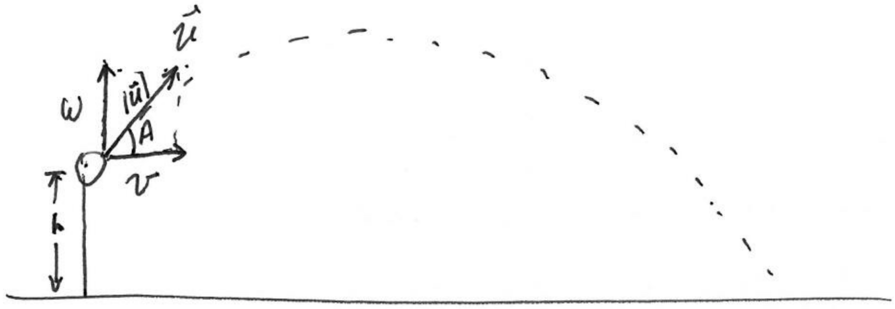
Int. factor: $\mu(t) = e^{+\frac{t}{4.5}}$

$$\therefore \quad \frac{d}{dt}\left[v(t)\, e^{+t/4.5}\right] = -9.8\, e^{+t/4.5}$$

$$\Rightarrow \quad v(t) =$$

30 Thrown Baseball without air resistance.

$\boxed{v'(t) = 0}$, $\boxed{w'(t) = -g}$



$|\vec{u}| = u$

$v(0) = u\cos A$ $\quad$ $w(0) = u\sin A$

then

$v(t) \equiv C_1$, $\quad$ $w(t) = -gt + C_2$

Using IC's:

a) $\boxed{v(t) = u\cos A}$, $\boxed{w(t) = -gt + u\sin A}$

b) $x(t) = \int v(t)\,dt = (u\cos A)t + C_1$

$y(t) = \int w(t)\,dt = -g\frac{t^2}{2} + u\sin A\, t + C_2$

$x(0) = 0$, $\quad y(0) = h$

$\Rightarrow \boxed{x(t) = (u\cos A)t}$,

$h = y(0) = C_2 \Rightarrow \boxed{y(t) = -g\frac{t^2}{2} + u\sin A\, t + h}$