2.4 Differences between linear and nonlinear Equations.

Consider the IVP:

$$t y'(t) + 2 y(t) = \sin t, \quad y(\pi/2) = 1$$

This equation is linear with variable coefficients to solve it, we express it in standard form:

$$\begin{cases} y'(t) + \frac{2}{t} y(t) = \frac{\sin t}{t} & (1.1) \\ y(\pi/2) = 1 & (1.2) \end{cases}$$

From Sect 2.1 Integrating factor: $\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln|t|} = |t|^2 = t^2$.

We use it to multiply both hands of (1.1)

$$y'(t) t^2 + 2t\, y(t) = t \sin t$$

$$\frac{d}{dt}\left[y(t) t^2\right] = t \sin t$$

$\int dt$: $\quad y(t) t^2 = -t\cos t + \sin t + C$

Then general soln: $y(t) = \dfrac{-\cos t}{t} + \dfrac{\sin t}{t^2} + \dfrac{C}{t^2}$

Using I.C.: $1 = y(\pi/2) = \dfrac{1}{(\pi/2)^2} + \dfrac{C}{(\pi/2)^2} \Rightarrow C + 1 = \dfrac{\pi^2}{4}$

$\Rightarrow C = \pi^2/4 - 1$

Finally,
$$\boxed{y(t) = \frac{-\cos t}{t} + \frac{\sin t}{t^2} + \frac{\pi^2/4 - 1}{t^2}} \qquad (2.1)$$

This Solution is valid for $t > 0$

Notice
$$p(t) = \frac{2}{t}, \quad g(t) = \frac{\sin t}{t}$$
are defined on $t > 0$ or $t < 0$
$(0, +\infty)$ $(-\infty, 0)$.

We want to know if this is the unique soln (2.1) $y(t)$ satisfying (1.1) and (1.2).

More general, we would like to know if the IVP:
$$\begin{cases} y'(t) + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

1) has a solution for any pair (t_0, y_0)
2) In case it has, if this soln. is unique.

Theorem 2.4.1 (Book)

Hipothesis:

1) $p(t)$ and $g(t)$ Continuous on (α, β) with $t_0 \in (\alpha, \beta)$.

Then, the IVP:

$$\begin{cases} y'(t) + p(t)y(t) = g(t) \\ y(t_0) = y_0 \end{cases}$$

has a unique soln. for any pair (t_0, y_0) with $t_0 \in (\alpha, \beta)$.

Back to our previous example

(1.1) and (1.2) should have a unique soln. on $(0, \infty)$

Since $y(t) = \dfrac{-\cos t}{t} + \dfrac{\sin t}{t^2} + \dfrac{\pi^2/4 - 1}{t^2}$

is a soln. of (1.1) and (1.2) then this is also the only Soln. according to theorem 2.4.1.

Note: Discuss proff of uniqueness in this case and how it can be extended to the General case.

(3')

$$\frac{dy}{dt} = \frac{\mu(t)\, g(t)\, \mu(t) - \mu'(t)\left[\int \_\_ \, ds + c\right]}{\mu^2(t)}$$

$$\mu(t) = e^{\int p(t)dt}$$

$$\frac{dy}{dt} = g(t) - \frac{\mu'(t)}{\mu(t)}\, y(t) = g(t) - \frac{p(t)\, \cancel{e^{\int p(t)dt}}}{\cancel{e^{\int p(t)dt}}}\, y(t)$$

$$= g(t) - p(t)\, y(t) \checkmark$$

Proof that

$$y(t) = \frac{\int \mu(t)\, g(t)\, dt + c}{\mu(t)}$$ is a soln. of the ODE.

#① Determine where the IVP:

$$\begin{cases} (t-3)\,y' + \ln t \; y = 2t & (4.1) \\ y(1) = 2 & (4.2) \end{cases}$$

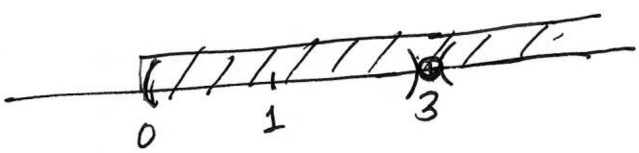
has a soln. and the soln. is unique

Bring to standard form:

$$y' + \frac{\ln t}{(t-3)}\, y = \frac{2t}{t-3}$$

$p(t) = \dfrac{\ln t}{t-3}$ is defined and is cont.

on $(0,3) \cup (3,+\infty)$



$g(t) = \dfrac{2t}{t-3}$ is defined and is cont.

on $(-\infty, 3) \cup (3, +\infty)$

So common interval containing $t_0 = 1$ where both $p(t), g(t)$ are conts. is $(0,3)$.

Using thm 2.4.1, the IVP (4.1)-(4.2) has a unique soln. on $(0,3)$.

Example 2.- $y' = y^{2/3}, \quad y(0) = 0.$

Clearly, $y_1(t) \equiv 0$ is a solution.

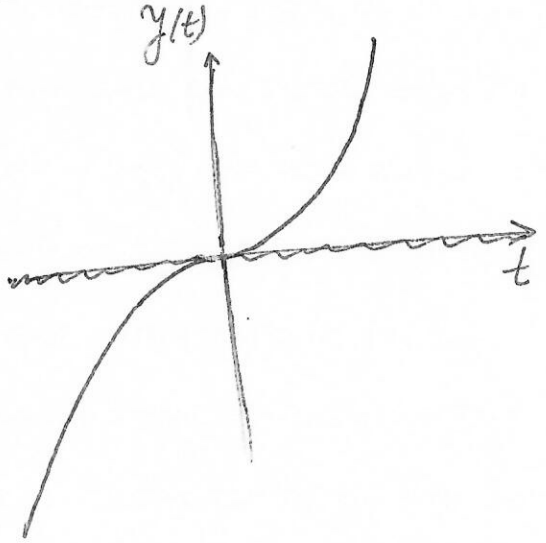
Diff. form : $\dfrac{dy}{y^{2/3}} = dt$ integrating $\int y^{-2/3}\,dy = t + C$

or $3y^{1/3} = t + C$

$\Rightarrow \quad y^{1/3} = \dfrac{t+C}{3} \Rightarrow y(t) = \dfrac{(t+C)^3}{3^3}$

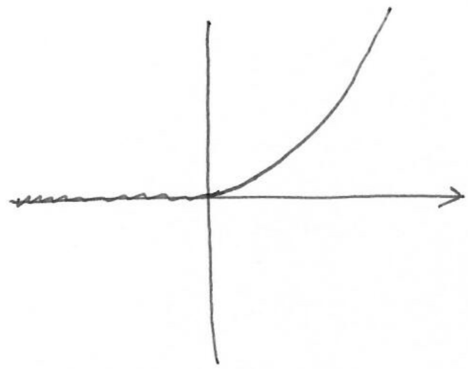
I.C. $0 = y(0) = \dfrac{C^3}{27} \Rightarrow \underline{C = 0}$

or $\boxed{y_2(t) = \dfrac{t^3}{27}}$



$y_1(t) \equiv 0$ and $y_2(t) \equiv \frac{t^3}{27}$ are not the only two Solutions of the IVP.

In fact,



$$y_3(t) = \begin{cases} 0, & -\infty < t \le 0 \\ \frac{t^3}{3^3}, & t > 0 \end{cases}$$

is also a solution.

Since,

1) $y_3(t)$ is differentiable everywhere

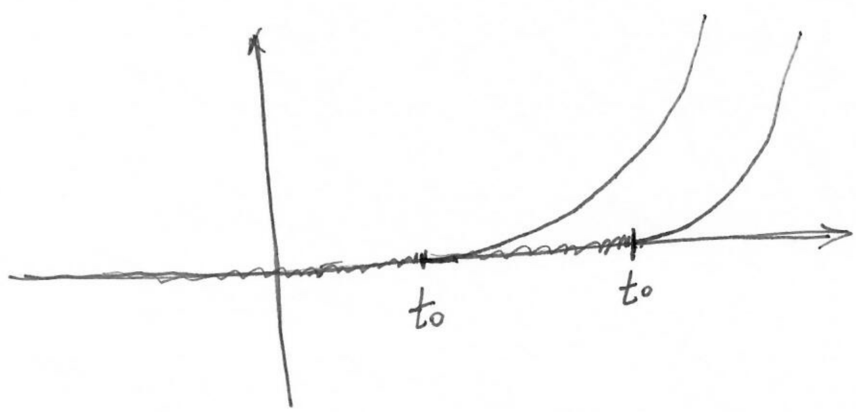
2) $y_3'(t) = y_3^{2/3}(t)$ Satisfies the equation.

3) Also, $y(0) = 0$. " I.C.

Any function

$$y(t) = \begin{cases} 0, & -\infty < t \le t_0 \\ \left(\frac{t - t_0}{3}\right)^3, & t > 0, \ t_0 > 0 \end{cases}$$

is also a solution. So IVP has infinitely many solns.



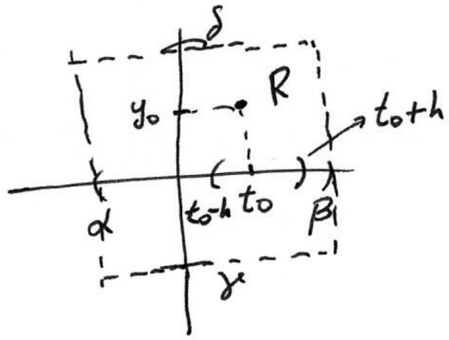
How about an IVP for the general 1^{st} order (in general nonlinear):

$$\begin{cases} y'(t) = f(t, y(t)) & (6.1) \\ y(t_0) = y_0 & (6.2) \end{cases}$$

Is it possible to have condition to guarantee
1) Existence of solns on certain interval (a,b)
2) In case there is a soln., is that soln. unique?

Theo 2.4.2

Hip: 1) If $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are conts on $R: (\alpha,\beta)\times(\gamma,\delta)$
2) (t_0, y_0) is in R



Then,
there exists an interval:
$(t_0-h, t_0+h) \subset (\alpha,\beta)$

where
1) There is a soln. $y(t) = \phi(t)$
2) The soln. $y = \phi(t)$ is unique.

Back to our previous example:

$$f(t,y) = y^{2/3}$$

$$\frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3} = \frac{2}{3\sqrt[3]{y}}$$, clearly not defined at $y=0$

So $\frac{\partial f}{\partial y}$ is not conts in any rectangle in ty-plane containing the point $(0,0)$.

As a consequence, there is no guarantee that a soln. for the IVP

$$y' = y^{2/3}, \quad y(0) = 0$$

will exist. Also, in case there is one, there is no guarantee that this soln. will be unique.

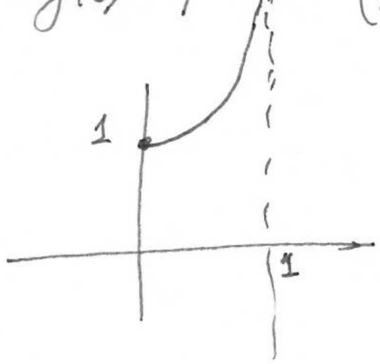
Singularities of the differential equ. ($f(t,y)$ is not defined) are not the only source to determine the interval of validity ODE (nonlinear), with $f, \frac{\partial f}{\partial y}$ satisfying thm 2.4.2 in $R: (\alpha, \beta) \times (\gamma, \delta)$

and $(t_0, y_0) \in R$, may have solutions singular at a point completely unrelated to the singularities of the $f(t,y)$.

<u>Example:</u> $y' = y^2, \quad y(0) = 1$ (8.1)

$$y(t) = \frac{1}{1-t}.$$

Int. Validity $(-\infty, 1)$.



Notice: $f(t,y) = y^2$

$$\frac{\partial f}{\partial y} = 2y$$

Both are conts for any pair (t_0, y_0)

So the rectangle R can be the whole plane $\mathbb{R}^2$

However, the interval of validity is not $(-\infty, \infty)$.

for the above IVP (8.1).