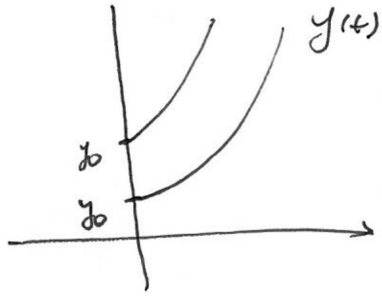
2.5 Autonomous Equations and Population Dynamics

1) Exp. growth

$$\begin{cases} \dfrac{dy}{dt} = ry \\ y(0) = y_0 \end{cases}$$



$y(t) = y_0 e^{rt}$

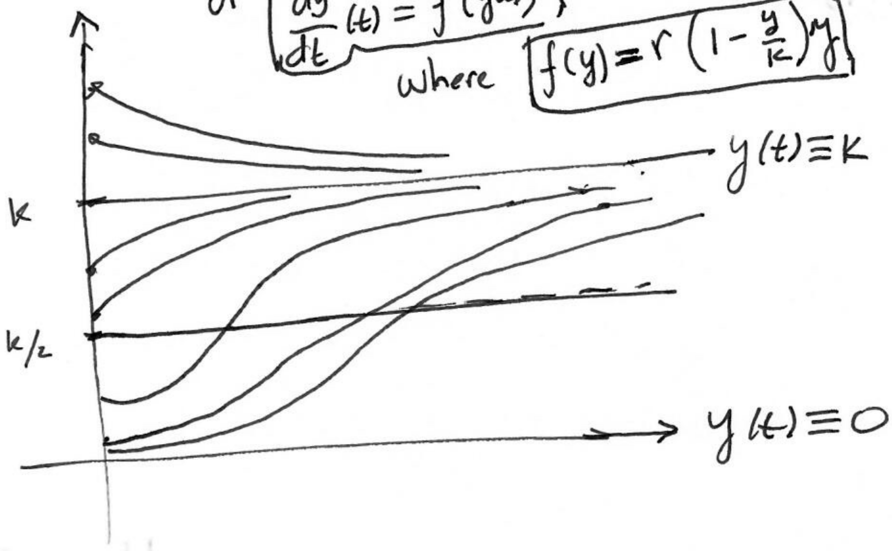
Exp. growth.
Not realistic.

2) Logistic growth

$$\boxed{\frac{dy}{dt} = (r - ay)y = r\left(1 - \frac{y}{K}\right)y} \qquad K = \frac{r}{a} \qquad (1.1)$$

or $\boxed{\dfrac{dy}{dt}(t) = f(y(t))}$ where $\boxed{f(y) = r\left(1 - \dfrac{y}{K}\right)y}$

What are the conclusions from applying thm 2.4.2 to an IVP for the Logistic equ.?



1) Obviously, $y(t) \equiv K$ and $y(t) \equiv 0$ are two solns. of (1.1) for the IVP $y(0) = K$, $y(0) = 0$, respectively.

2) a) $0<y<K \Rightarrow y'(t) = r\left(1-\overset{>0}{\frac{y}{K}}\right)\overset{>0}{y} > 0$

$\Rightarrow y(t) \nearrow$.

b) $y>K \Rightarrow y'(t) = r\left(1-\overset{<0}{\frac{y}{K}}\right)\overset{>0}{y} < 0$

$\Rightarrow y(t) \searrow$

3) The growth rate $\frac{dy}{dt}(t)$ changes according to the values of $y(t)$.

a) $y'(t)$ is increasing when its slope $y''(t)>0$ and is decreasing when its slope $y''(t)<0$.

So we need to analize

$$y''(t) = \left[r\left(1-\frac{y(t)}{k}\right)y(t)\right]'$$

$$= \frac{d}{dt}\left[f(y(t))\right] = f'(y(t))\frac{dy}{dt} =$$

$$= f'(y)\,f(y).$$

So $\frac{d^2y}{dt^2}(t) = f'(y(t))\,f(y(t))$

a) $y''(t)>0$, when $f'(y)$ and $f(y)$ are both positive or both negative

b) $y''(t)<0$, when $f'(y)$ and $f(y)$ have different sign.

Now, $$f(y) = r\left(1 - \frac{y}{k}\right) y$$

then $$f'(y) = r\left(1 - \frac{y}{k}\right) - \frac{r}{k} y$$

$$= r - 2\frac{r}{k} y = r\left(1 - \frac{2}{k} y\right)$$

or $$f'(y) = r\left(1 - \frac{y}{\frac{k}{2}}\right)$$

therefore,

i) If $y = k/2 \Rightarrow f'(y) = 0 \Rightarrow y''(t) = 0$.

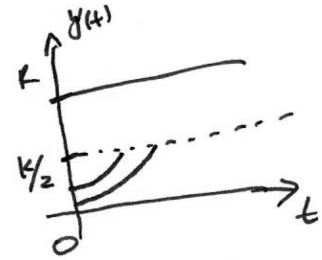
Candidate to inflection points.

ii) If $0 < y < k/2 \Rightarrow f'(y) > 0$ and $f(y) > 0$

$\Rightarrow y''(t) = f(y) f'(y) > 0$

and the growth is accelerated

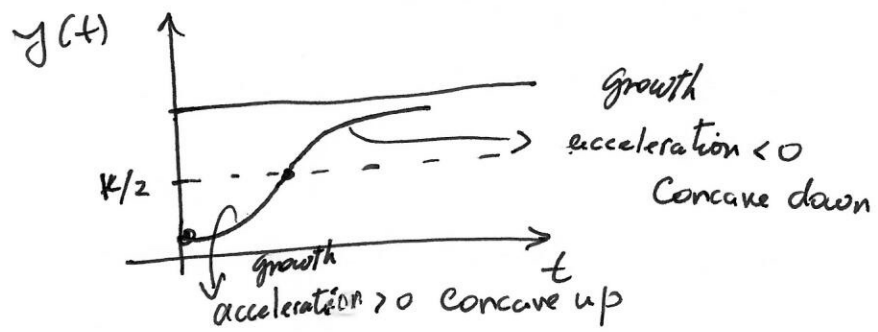
graph is concave up.



iii) If $k/2 < y < k \Rightarrow f'(y) < 0$ and $f(y) > 0$

then $y''(t) = f(y) f'(y) < 0$

and the growth acceleration is decreasing.



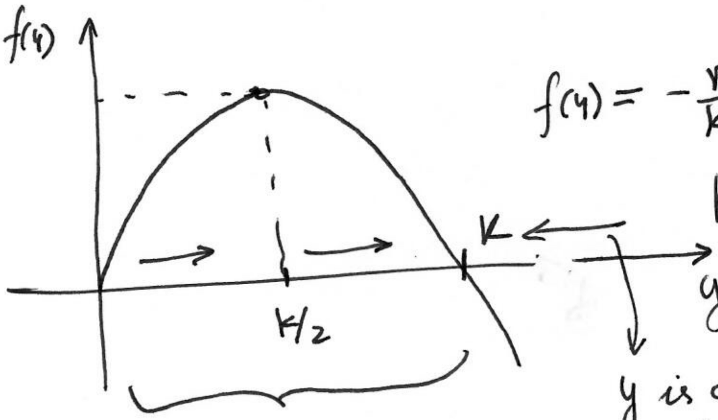
4) The Solns. below or above $y(t) = K$ never cross the line $y = K$. This is due to thm 2.4.2 <u>Uniqueness</u> property.

So they converge asymptotically to $y(t) \equiv K$.

and they remain in the interval $(0, K)$.

or $(K, +\infty)$, respectively.

To help in the visualization of the Solns, it is Convenient to consider the graph of $f(y)$



$f(y) = -\frac{r}{K} y^2 + ry$

parabola

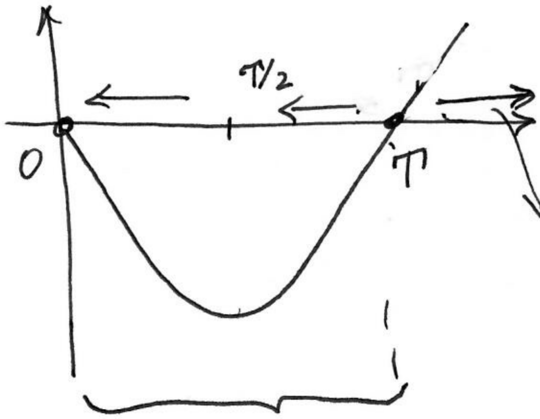
y is increasing because $\frac{dy}{dt} = f(y) > 0$

y is decreasing because $\frac{dy}{dt} = f(y) < 0$.

3) Critical Treshold

$$\frac{dy}{dt}(t) = -r\left(1 - \frac{y}{T}\right)y = f(y)$$

Geometric Analysis



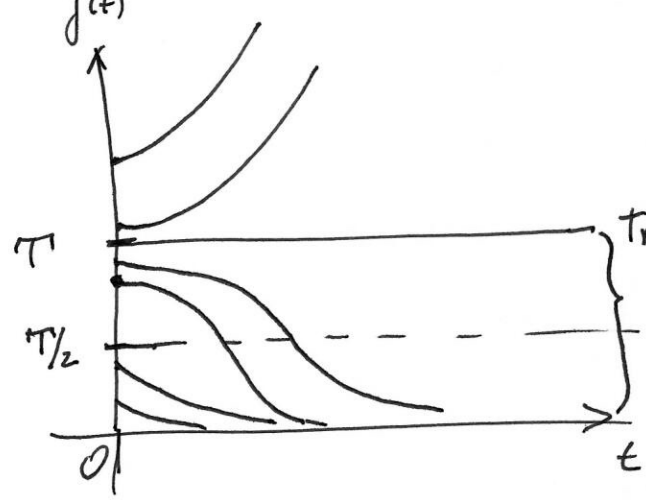
y increases

$\frac{dy}{dt} = f(y) > 0$.

y is decreasing

$\frac{dy}{dt} = f(y) < 0$

then



treshold

population becomes extinct.
below T ($y < T$).

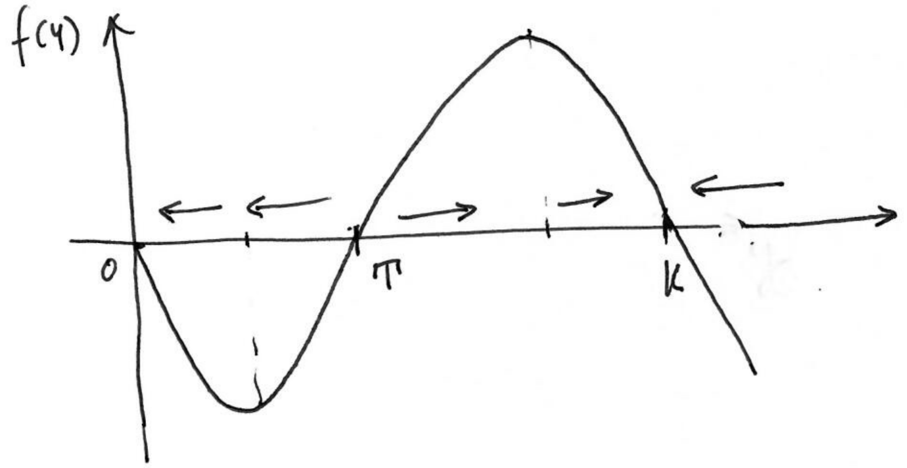
4) Logistic Growth with a Treshold.

$$\frac{dy}{dt}(t) = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y$$

$r>0$

$0<T<K.$

Geometric Analysis



Then,