2.6 Exact Equations.

Consider problem 22 Sect 2.2 (Separable equs.)

$$y'(x) = \frac{3x^2}{3y^2(x) - 4}, \quad y(1) = 0.$$

or

$$\boxed{-3x^2 + (3y^2 - 4)y' = 0,} \tag{1.1}$$

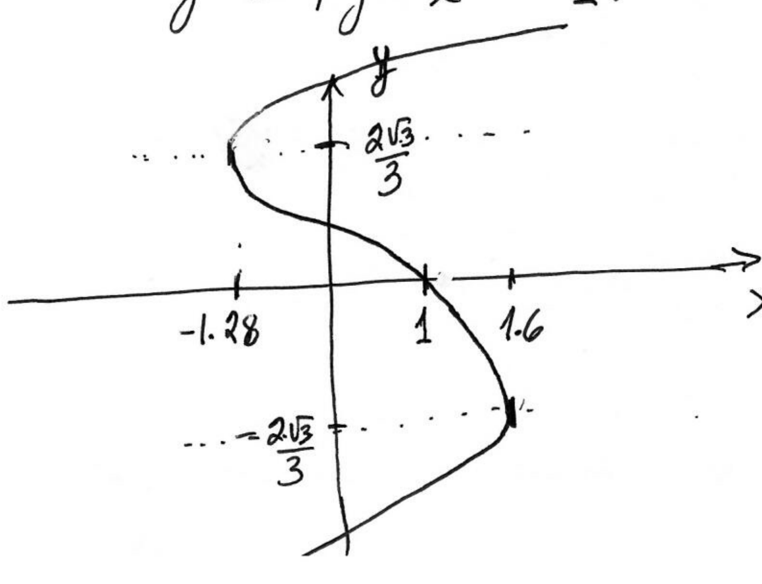
we found by separating variables that

$$\boxed{y^3 - 4y - x^3 = C} \tag{1.2}$$

is the family of integral curves of (1)

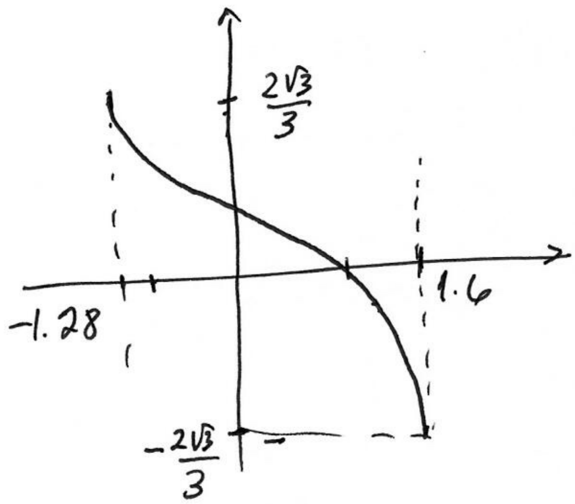
Using the I.C., we also determine the integral curve of the IVP:

$$y^3 - 4y - x^3 = -1. \tag{1.3}$$



Notice :

a) Part of the integral curve is a Soln. of this IVP. In fact,



The curve $y^3 - 4y - x^3 = -1$,

with the interval of Validity : $(-1.28, 1.6)$

and $-\frac{2\sqrt{3}}{3} < y < \frac{2\sqrt{3}}{3}$. (The I.C. helps to determine this range of y)

is a solution of our IVP.

We will say that $y^3 - 4y - x^3 = -1$ defines the Soln. of our IVP implicitly.

T

b) Using the uniqueness and existence theorem, we can prove that this (soln. in (a)) is the unique soln.

Notice that

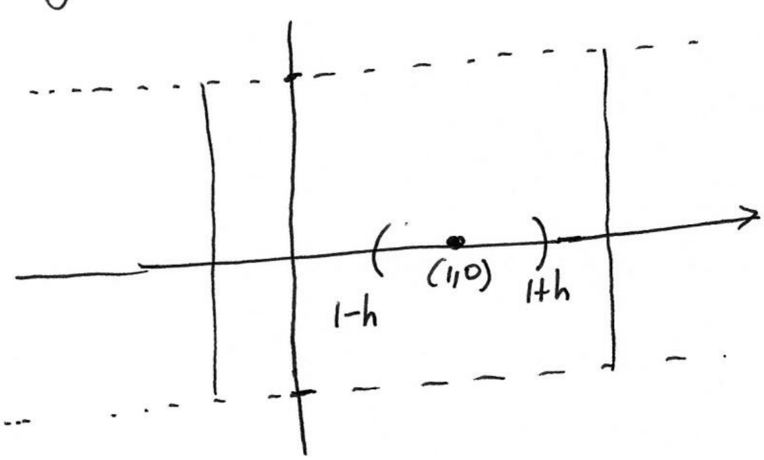
$$f(x,y) = \frac{3x^2}{3y^2-4},$$

Singular at $y = \pm \frac{2\sqrt{3}}{3}$

and

$$\frac{\partial f}{\partial y}(x,y) = \frac{-18x^2y}{(3y^2-4)^2},$$

are both conts. in a rectangular region containing the initial point $(1,0)$.



Therefore, there is a unique soln for the IVP in an interval $(1-h, 1+h)$, $h>0$

In part (a), we found the max. interval for this unique soln: $(-1.28, 1.6)$.

Alternative procedure to obtain integral curves of (1.1)

We will try to find a function of two variables

$$\varphi = \varphi(x,y)$$

Such that
$$\varphi_x(x,y) = -3x^2 \qquad (4.1)$$
$$\varphi_y(x,y) = 3y^2-4 \qquad (4.2)$$

1) We start by integrating (4.1) w.r.t. x. $\int dx$.

So,
$$\varphi(x,y) = \int \varphi_x(x,y)\,dx = -\int 3x^2 dx = -x^3 + C(y) \quad (4.3)$$

2) Now, we use (4.2) to determine $C(y)$

In fact,
$$3y^2-4 \overset{(4.2)}{=} \varphi_y(x,y) \overset{(4.3)}{=} -3x^2 + c'(y)$$

Then,
$$c'(y) = 3y^2-4$$

$\int dy$:
$$C(y) = y^3-4y+d$$

Therefore,

$$\psi(x,y) = -x^3 + y^3 - 4y + d$$

and

$$\psi(x,y) = \tilde{C}$$

or $$-x^3 + y^3 - 4y = \tilde{C} - d = c$$

is the same family of integral curves (1.2) obtained by separation of variables.

The previous computation suggests a procedure to apply to obtain Solutions of ODEs given by

$$M(x,y) + N(x,y)\, y' = 0. \qquad (5.1)$$

If we can find $\psi(x,y)$ Such that

$$\psi_x(x,y) = M(x,y) \quad \text{and} \quad \psi_y(x,y) = N(x,y)$$

and $\psi(x,y) = C$ defines $y = y(x)$ implicitly as a differentiable function of x.

Then $\psi(x,y) = C$ defines (implicitly) the solns. of (5.1).

In fact, $M(x,y) + N(x,y)\, y' = \psi_x(x,y) + \psi_y(x,y)\, y' =$

$$= \frac{d}{dx}\left[\psi(x,y(x))\right] = 0$$

and the Solutions for the equation

$$\frac{d}{dx}\left[\psi(x,y(x))\right] = 0$$

are $\psi(x,y) = C$ integral curves of (5.1)

Which also Contains the implicit Solns. of (5.1).

Given an equation like (5.1)

Is it always possible to find a function $\psi(x,y)$ such that

$$\psi(x, y(x)) = C$$

defines an implicit soln. of (5.1) ?

The answer is no

Def – If a function $\psi(x,y)$ defined on certain two-dimensional region R can be found such that

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x,y)$$

then equ. (5.1) is called an exact equation.

Questions: ① Is it possible to know that a given ODE in the form (5.1) is exact?

② In the case (5.1) is exact, How can get obtain $\psi(x,y)$ s.t. $\psi(x,y) = c$ are integral curves of (5.1)

Let us Consider two more examples:

Exercise 14 (Sect. 2.6)

$$\begin{cases} 9x^2+y-1-(4y+x)\,y'=0 & (6.1) \\ y(1)=0 & (6.2) \end{cases}$$

Equ. (6.1) can also be written as

$$y' = \frac{9x^2+y-1}{4y-x}$$

Not linear
Not Separable

Let us follow the previous procedure

We will try to find

$$\psi = \psi(x,y)$$

Such that

$$\psi_x(x,y) = 9x^2+y-1 \qquad (6.3)$$

$$\psi_y(x,y) = -4y+x \qquad (6.4)$$

$\int (6.3)\,dx$: $\psi(x,y) = \int (9x^2+y-1)\,dx = 3x^3+xy-x+C(y)$

Using (6.4) to determine $C(y)$.

$$-4y + \cancel{x} = \psi_y(x,y) = \cancel{x} + C'(y)$$

Then,

$$C'(y) = -4y$$

and $\int dy$: $C(y) = -2y^2 + d$

Finally,

$$\psi(x,y) = 3x^3 + xy - x - 2y^2 + d.$$

Therefore, integral curves of (6.1) are given by

$$\psi(x,y) = \tilde{C}$$

or $\boxed{3x^3 + xy - x - 2y^2 = C}$, where $C = \tilde{C} - d$.

This equation defines the soln. $y = y(x)$ of (6.1) implicitly.

Also, according to our previous definition Equ. (6.1) is an exact equation.

Determine C using I.C. (6.2).

Exercise 21 (Sect. 2.6) (Not all D.E. are exact)

$$y + (2x - ye^y)y' = 0 \qquad (8.1).$$

Let's try to find $\psi = \psi(x,y)$

Such that

$$\psi_x(x,y) = y \qquad (8.2)$$

$$\psi_y(x,y) = 2x - ye^y \qquad (8.3)$$

$\int dx$: $\quad \psi(x,y) = xy + C(y)$

using (8.3)

$$2x - ye^y = \psi_y(x,y) = x + C'(y)$$

or $\quad C'(y) = x - ye^y \quad$ Imposible to obtain Such a $C(y)$!

Theorem 2.6.1 in the book give us a Condition on $M(x,y)$ and $N(x,y)$ that will guarantee the existence of

$$\psi = \psi(x,y)$$

Such that $\quad \psi_x = M, \quad$ and $\psi_y = N.$

Thm 2.6.1

1) $M(x,y),\ N(x,y),\ M_y(x,y),\ N_x(x,y)$

Conts in $R:\ \alpha < x < \beta,\ \ \gamma < y < \delta$

Then

$$M(x,y) + N(x,y)\,y' = 0$$

is an exact differential equ.

if and only if

$M_y(x,y) = N_x(x,y)$, at each point in R.

Discuss meaning of exact.

Back to Exercise 21

Exercise 21

$$y + (2x - ye^{y})y' = 0 \qquad (10.1)$$

Notice that

$$M_y = 1 \neq 2 = N_x$$

So (10.1) is not exact.

However, if we multiply it by $\mu(y) = y$

$$y^2 + (2xy - y^2e^y) = 0 \qquad (10.2)$$

and check again M_y, N_x

$M_y = 2y = N_x$ ✓ So, now the new equation (10.2) is exact.

Let's solve it

$$\begin{cases} \Psi_x = y^2 & (10.3) \\ \Psi_y = 2xy - y^2e^y & (10.4) \end{cases}$$

Int. $\int dx$ (10.3)

$$\Psi(x,y) = xy^2 + C(y)$$

Using (10.4)

$$\cancel{2xy} + C'(y) = \cancel{2xy} - y^2 e^y$$

then $\quad C'(y) = -y^2 e^y \quad$ and $\quad C(y) = -\int y^2 e^y dy$

Therefore, $\quad \psi(x,y) = C \quad$ integral curves are

$$\boxed{x y^2 - e^y\left(y^2 - 2y + 2\right) = C} \qquad (11.1)$$

In the next page, there is a procedure on how to obtain

$$\mu(y) = y \quad ?$$

Rmk: $\quad -\int \underset{f}{y^2}\, \underset{g}{e^y} dy = -\left[y^2 e^y - 2\int y e^y dy\right] = -\left[y^2 e^y - 2(ye^y - e^y)\right]$

$$= -\left[y^2 e^y - 2ye^y + 2e^y\right] = -y^2 e^y + 2e^y - 2ye^y$$

How to obtain $\mu(y) = y$?

Not a definite answer. Actually, in most cases it will be very hard or impossible to obtain a $\mu = \mu(x,y)$ that makes the non-exact equation

$$M^{(x,y)} + N^{(x,y)} y' = 0 \qquad (12.1)$$

exact.

In some cases as those found in the homework problems it is possible.

This is a possible way to accomplish it.

After multiplying (12.1) by $\mu(x,y)$

$$(\mu M)(x,y) + (\mu N)(x,y)\, y' = 0 \qquad (12.2)$$

For (12.2) to be exact

$$(\mu M)_y = (\mu N)_x$$

or $\mu_y M + \mu M_y = \mu_x N + \mu N_x$

Now, if $\mu_x = 0$. It means μ is only a function of y or $\mu = \mu(y)$.

then,

$$\mu_y M = \mu (N_x - M_y)$$

or

$$\boxed{\mu_y(y) = \left(\frac{N_x - M_y}{M}\right)\mu(y)} \quad (13.1)$$

For (13.1) to make sense

$$\frac{N_x - M_y}{M} = f(y)$$

Otherwise, we will not be able to find $\mu(y)$.

In our exercise #21.,

$$\frac{N_x - M_y}{M} = \frac{2-1}{y} = \frac{1}{y} \quad ✓$$

Thus, $\mu'(y) = \frac{1}{y}\mu(y)$ or $\boxed{\mu'(y) - \frac{1}{y}\mu(y) = 0}$

and using int. factor for linear eqns. $\alpha(y) = e^{-\int \frac{1}{y} dy} = e^{-\ln y} = \frac{1}{y}$

Therefore, $\mu(y) = 1/y$

and $\frac{d}{dy}\left[\mu(y)\frac{1}{y}\right] = 0 \Rightarrow \mu(y)\frac{1}{y} = C \Rightarrow \mu(y) = cy$

we choose $\boxed{\mu(y) = y}$